Lochs-type theorems beyond positive entropy

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OWNS 2022

From decimals to continued fractions

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of $x \in [0,1]$,

$$x = 0.d_1 d_2 \ldots \in [0,1]$$

- determine the number $L_n(x)$ of CFE-digits (partial quotients)
deduced without error

$$x = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

Natural to consider the quotient $L_n(x)/n$:

- rate of CFE digits per decimal digit,
- compares relative information/redundancy of expansions.

Very small rates

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of $x \in [0,1]$,

$$x = 0.d_1 d_2 \ldots \in [0,1]$$

- determine the number $L_n(x)$ of CFE-digits (partial quotients) deduced without error

$$x = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

Theorem (Faivre, 2001. Wu, 2006)

For $x \in (0,1)$ having Lévy constant $\beta(x) := \lim_{n\to\infty} \log q_n(x)/n$,

$$\lim_{n\to\infty} \frac{L_n(x)}{n} = \frac{\ln 10}{2\beta(x)}.$$

As $\beta(x)$ takes arbitrarily large values, the rate $L_n(x)/n$ takes arbitrarily small values.

Lochs' Theorem

- Given n decimal digits $d_1, d_2, \ldots, d_n$ of x,

$$x = 0.d_1 d_2 \ldots \in [0, 1],$$

- $L_n(x)$ continued fraction digits (partial quotients)

$$x = [0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

Lochs' theorem, 1964

$$\frac{L_n(x)}{n} \to \frac{6 \ln 10 \ln 2}{\pi^2} \approx 0,97 \quad \text{a.e. } x \quad \text{(Lebesgue measure)}$$

when $n \to \infty$.

"Lochs' example". The first 1000 decimals of π determine exactly 968 partial quotients of π.

Outline

Natural question

Given $x \in (0,1)$ and $n \in \mathbb{N}$.
How large is the number $L_n(x)$ of digits determined in one expansion of the real number $x \in (0,1)$ when a number n of digits of x are given in some other expansion?

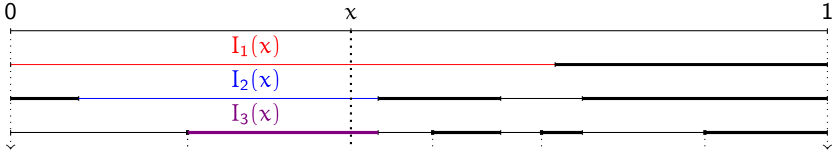
- Partitions, Lochs' index and entropy.
- Dajani and Fieldsteel's results a.e./in measure for positive entropy.
- Our extension to zero/infinite entropy.
  - The notion of weight function.
  - Our main general result.
- Zero entropy: from binary digits to (characteristic) Sturmian words.
- Three instances: Farey, Stern-Brocot and "three-distance" inspired.

Partitions

Definitions and notations

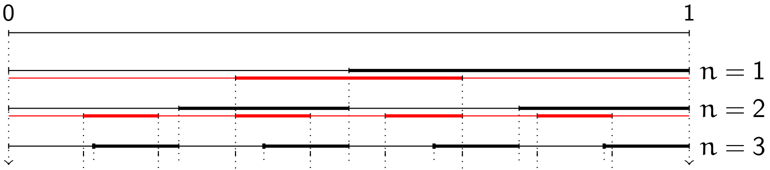
- A topological partition of $[0, 1]$ is a set P of intervals:
  - open (nonempty),
  - disjoints
  - the union of their closures equals $[0, 1]$.
- A sequence of partitions $\mathcal{P} = \{P_n\}_{n \in \mathbb{N}_0}$ is a sequence of topological partitions
- E is the set of endpoints of the intervals of $\mathcal{P}$.
- $I_n(x)$ is the interval of P_n that contains x (if $x \notin E$).

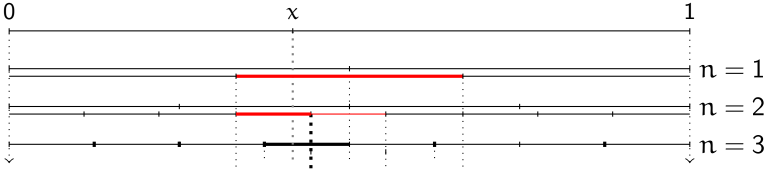
The partitions are not necessarily self-refining.



Lochs' index: definition

In black, the sequence of partitions associated with base 2.
In red, the sequence of partitions associated with base 3.





$\mathcal{B}$: binary, $\mathcal{T}$: ternary

$$I_3^{\mathcal{B}}(x) \subseteq I_1^{\mathcal{T}}(x) \quad \text{but} \quad I_3^{\mathcal{B}}(x) \not\subseteq I_2^{\mathcal{T}}(x)$$

The first **3 binary digits** of x provide only 1 ternary digit.

Lochs' index: definition

Consider

- $\mathcal{P}^1$ and $\mathcal{P}^2$ two sequences of partitions.
- $I_n^1(x)$: the interval of depth n of $\mathcal{P}^1$ that contains x.
- $I_n^2(x)$: the interval of depth n of $\mathcal{P}^2$ that contains x.

Following Bosma, Dajani & Kraaincamp and Dajani & Fieldsteel:

Lochs' index

For $x \in [0, 1]$ (not an endpoint) and each $n \in \mathbb{N}$, the Lochs' index is defined as

$$L_n(x, \mathcal{P}^1, \mathcal{P}^2) = \sup\{\ell \geqslant 0 : I_n^1(x) \subseteq I_\ell^2(x)\}.$$

Informally: n digits of x in $\mathcal{P}^1$ provide $L_n(x, \mathcal{P}^1, \mathcal{P}^2)$ digits of x in $\mathcal{P}^2$

Measure and entropy

Entropy of a sequence of partitions $\mathcal{P}$

- λ is a Borel probability measure on $[0,1]$.
- $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is a sequence of partitions.

Assume that $\lambda(\text{endpoints}) = 0$,

$$h_\lambda(\mathcal{P}) = \lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{n} \quad \text{a.e. (resp. in measure)} \quad (\lambda),$$

if the limit exists.

Lochs' index for positive entropy

Theorem (Dajani and Fieldsteel, 2001)

- $\mathcal{P}^1$ and $\mathcal{P}^2$ are two sequences of partitions of $[0,1]$,
- λ is a Borel probability measure on $[0,1]$.

The following limit holds

$$\lim_{n\to\infty} \frac{1}{n} L_n(x, \mathcal{P}^1, \mathcal{P}^2) = \frac{h_\lambda(\mathcal{P}^1)}{h_\lambda(\mathcal{P}^2)}$$

almost everywhere with respect to λ, if

- $h_\lambda(\mathcal{P}^1)$ and $h_\lambda(\mathcal{P}^2)$ are their entropies a.e. (λ) and they are positive.

In measure λ, if

- $h_\lambda(\mathcal{P}^1)$ and $h_\lambda(\mathcal{P}^2)$ are their entropies in measure (λ) and they are positive,
- $\mathcal{P}^2$ is self-refining.

Dajani and Fieldsteel implies Lochs

Theorem (Dajani and Fieldsteel, 2001)

- $\mathcal{P}^1$ and $\mathcal{P}^2$ are two sequences of partitions of $[0, 1]$,
- λ is a Borel probability measure on $[0, 1]$.

The following limit holds

$$\lim_{n\to\infty} \frac{1}{n} L_n(x, \mathcal{P}^1, \mathcal{P}^2) = \frac{h_\lambda(\mathcal{P}^1)}{h_\lambda(\mathcal{P}^2)}$$

almost everywhere with respect to λ, if

- $h_\lambda(\mathcal{P}^1)$ and $h_\lambda(\mathcal{P}^2)$ are their entropies a.e. (λ) and they are positive.

Lochs'theorem:

- λ is the Lebesgue measure;
- Decimals have a.e. entropy equal to $\ln 10$;
- Continued fractions have a.e. entropy equal to $\pi^2/(6 \ln 2)$.

Beyond of positive entropy

Remark (Dajani and Fieldsteel, 2001)

$$\lim_{n\to\infty} \frac{1}{n} L_n(x, \mathcal{P}^1, \mathcal{P}^2) = \begin{cases} 0, & h_\lambda(\mathcal{P}^1) = 0 \text{ and } h_\lambda(\mathcal{P}^2) \neq 0, \\ \infty, & h_\lambda(\mathcal{P}^2) = 0 \text{ and } h_\lambda(\mathcal{P}^1) \neq 0. \end{cases}$$

Is it possible to be more precise?

Weight functions and log-balanced sequences of partitions

- λ is a Borel probability measure on $[0, 1]$.
- $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is a sequence of partitions.

Entropy $h_\lambda(\mathcal{P})$

$$h_\lambda(\mathcal{P}) = \lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{n} \quad \text{a.e. (in measure)} \quad (\lambda),$$

if the limit exists.

Weight function f

A map $f : \mathbb{N} \mapsto \mathbb{R}$, so that

$$\lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{f(n)} = 1 \quad \text{a.e. (in measure)} \quad (\lambda),$$

Positive entropy

Almost everywhere or in measure λ,

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{P}^1, \mathcal{P}^2)}{n} = \frac{h_\lambda(\mathcal{P}^1)}{h_\lambda(\mathcal{P}^2)} \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{h_\lambda(\mathcal{P}^2) L_n(x, \mathcal{P}^1, \mathcal{P}^2)}{h_\lambda(\mathcal{P}^1) n} = 1$$

Weight functions and log-balanced sequences of partitions

- λ is a Borel probability measure on $[0,1]$.
- $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is a sequence of partitions.

Definition

$\mathcal{P}$ is log-balanced a.e. (resp. *in measure*) with respect to λ if $\lambda(\text{endpoints}) = 0$ and there is some function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{f(n)} = 1 \quad a.e. \text{ (resp. in measure)} \quad (\lambda).$$

If so, f is called a weight function of $\mathcal{P}$ a.e. (resp. *in measure*) with respect to λ.

Simple facts

What does a log-balanced sequence of partitions look like?

If $\mathcal{P}$ is a log-balanced sequence of partitions with respect to λ:

- λ has no atoms: $\lambda(\{x\}) = 0$,
- the norms of the partitions tend to zero:

$$\sup\{\lambda(I) : I \in P_n\} \to 0 \text{ as } n \to \infty.$$

A realization result

Given any $f : \mathbb{N} \mapsto \mathbb{R}$, $f(n) \to \infty$ as $n \to \infty$, there exists a sequence of partitions that has f as an a.e. weight function with respect to the Lebesgue measure.

Our main theorem: beyond positive entropy

- $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions.
- λ is a Borel probability measure on $[0,1]$.

The following limit holds

$$\lim_{n\to\infty} \frac{f_2(L_n(x,\mathcal{P}^1,\mathcal{P}^2))}{f_1(n)} = 1$$

almost everywhere with respect to λ, if

- f_1 and f_2 are the corresponding weight functions a.e. (λ),
- $\lim_{n\to\infty} f_1(n)/\ln n = +\infty$;
- f_2 is nondecreasing;
- $\sqrt[n]{|f_2(n)|} \to 1$ as $n \to \infty$.

In measure λ, if

- f_1 and f_2 are the corresponding weight functions in measure (λ),
- f_2 is nondecreasing;
- $\sqrt[n]{|f_2(n)|} \to 1$ as $n \to \infty$;
- $\mathcal{P}^2$ is self-refining.

Ideas of the proof

For a log-balanced sequence of partitions with weight function f,

$$\lambda(I_n(x)) \approx e^{-f(n)}$$

Roughly,

$$L_n(x, \mathcal{P}_1, \mathcal{P}_2) = m \quad \text{means} \quad \lambda(I^1_n(x)) \approx \lambda(I^2_m(x))$$

Then,

$$e^{-f_1(n)} \approx \lambda(I^1_n(x)) \approx \lambda(I^2_m(x)) \approx e^{-f_2(m)}.$$

So,

$$L_n(x, \mathcal{P}_1, \mathcal{P}_2) = m \approx f_2^{-1}(f_1(n))$$

Finally,

$$\frac{f_2(L_n(x, \mathcal{P}_1, \mathcal{P}_2))}{f_1(n)} \approx 1$$

Ideas of the proof

An interval is ϵ-good for $\mathcal{P}$ and its weight function f, if

$$e^{-(1+\epsilon)f(n)} < \lambda(I_n(x)) < e^{-(1-\epsilon)f(n)}.$$

Deal with the set

$$D_{n,\epsilon} := \{x : \ I_n^1(x) \text{ and } I_{m(n)}^2(x) \ \text{ are both } \epsilon-\text{good}\}$$

with $m_n(x) = \text{``}f_2^{-1}\text{''}((1-\eta)f_1(n)) \ll \text{``}f_2^{-1}\text{''}(f_1(n))$.

- Almost everywhere:
  Borel-Cantelli + $\lim_{n\to\infty} f_1(n)/\ln n = +\infty$ implies

$$\lambda(\{x : x \in D_{n,\epsilon} \text{ i.o. }\}) = 0.$$

- In measure:
  It suffices that $f_1(n) \to \infty$ to ensure that $\lambda(D_{n,\epsilon}) \to 0$.

For $f_2 : \mathbb{N} \to \mathbb{R}$, define $f_2^{[-1]}(y) = \min\{n \in \mathbb{N} : f_2(n) \geqslant y\}$

$$\sqrt[n]{f_2(n)} \to 1 \Longrightarrow f_2(f_2^{[-1]}(y)) \to 1 \quad \text{as} \quad y \to \infty.$$

Characteristic Sturmian words

Let $x \in (0,1) \setminus \mathbb{Q}$. Consider the sequence of fractional parts of the multiples nx of x with $n \geqslant 1$:

$$n \mapsto \{nx\}.$$

Consider the intervals

$$J_0 = [0, 1-x) \text{ and } J_1 = (1-x, 1].$$

Define the word ω as follows

$$\omega[n] = \begin{cases} 0, & \text{if } \{nx\} \in J_0, \\ 1, & \text{if } \{nx\} \in J_1. \end{cases}$$

The letters of ω are the characteristic Sturmian digits.

Each irrational x produces an infinite word of 0's and 1's.

From binary to characteristic Sturmian words

- Given n binary digits $b_1, b_2, \ldots, b_n$ of $x \in [0, 1]$,

$$x = (0.b_1 b_2 \ldots)_2 \in [0, 1] \,.$$

- Estimate the number $L_n(x)$ of characteristic Sturmian-digits deduced without error.

The partition associated with characteristic Sturmian words is the Farey partition.

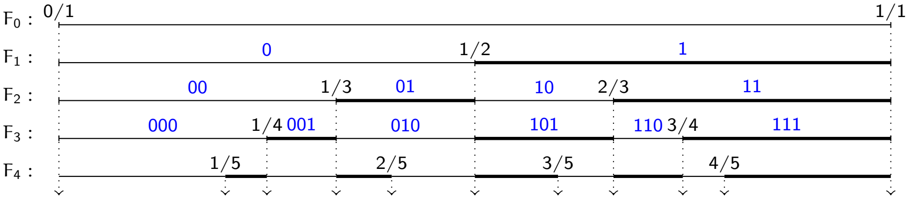
Farey

The Farey partition $\mathcal{F}_n$

$\mathcal{F}_0 := [0/1, 1/1]$

$\mathcal{F}_n$ is built from $\mathcal{F}_{n-1}$

each interval $[a/c, b/d]$ is split by its mediant $(a+b)/(c+d)$
if and only if $c + d \leqslant n + 1$



Each interval gathers the irrational numbers whose characteristic Sturmian words begin with the blue prefix.

Farey partition

Weight for Farey

The Farey partition is a. e. log-balanced for the Leb. measure,

$$\text{Weight function:} \quad f_{\mathcal{F}}(n) = 2\ln n, \quad n \geqslant 2.$$

Idea of the proof

Fix $x \in (0,1) \setminus \mathbb{Q}$ and n. Let $m := m(x,n)$ and $r := r(x,n)$ be such that

$$(r+1)q_m + q_{m-1} \leqslant n+1 < (r+2)q_m + q_{m-1}, \; m \geqslant 0, \text{ and } 0 \leqslant r < a_{m+1},$$

where $q_m = q_m(x)$ is the continuant associated with x.
The only interval of F_n that contains x measures

$$|I_n^{\mathcal{F}}(x)| = (((r+1)q_m + q_{m-1})q_m)^{-1}.$$

Then,

$$\frac{1}{(n+1)^2} \leqslant \left|I_n^{\mathcal{F}}(x)\right| \leqslant \frac{2(r+3)}{(n+1)^2}.$$

Take "logs", recall Borel-Berstein to bound $\log a_m/m$ with $m = O(\ln n)$.

From binary to Farey (or Sturm)

$\lambda =$ Lebesgue measure.

- The sequence $\mathcal{B}$ of binary intervals is log-balanced a.e.

  Weight: $f_{\mathcal{B}}(n) = (\ln 2)n.$

- The Farey sequence of partitions, $\mathcal{F}$, is log-balanced a.e.

  Weight: $f_{\mathcal{F}}(n) = 2 \ln n.$

- Our result implies

$$\lim_{n\to\infty} \frac{2\ln(L_n(x, \mathcal{B}, \mathcal{F}))}{(\ln 2)n} = 1 \quad \text{a.e.}$$

- Informally: $\ln(L_n(x, \mathcal{B}, \mathcal{F})) \sim (\ln(2)/2)n$, we can say

  n binary digits provide about $(\sqrt{2})^n$ Farey digits.

Farey and Stern-Brocot sequences of partitions

Farey partition $\mathcal{F}_n$

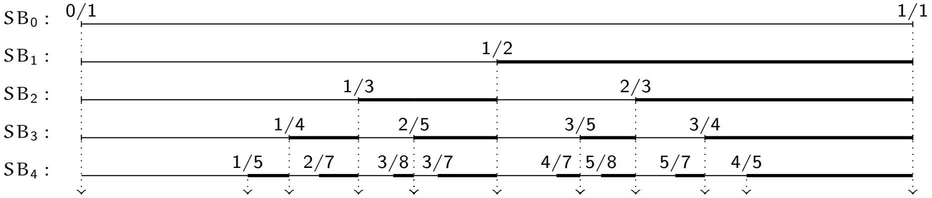
$F_0 := [0/1, 1/1]$

F_n arises from F_{n-1} by
dividing each interval $[a/c, b/d]$ by
its mediant $(a+b)/(c+d)$
only if $c + d \leqslant n + 1$

Stern-Brocot partitions $\mathcal{SB}_n$

$SB_0 := [0/1, 1/1]$

SB_n arises from SB_{n-1} by
dividing each interval $[a/c, b/d]$ by
its mediant $(a+b)/(c+d)$
(always)



Stern-Brocot partition

Weight function for Stern-Brocot

With respect to the Lebesgue measure, the Stern-Brocot sequence of partitions, $\mathcal{SB}$,

- has zero entropy,
- is log-balanced in measure with weight $f_{\mathcal{SB}}(n) = \dfrac{\pi^2}{6}\dfrac{n}{\log n}$, $n \geqslant 2$,
- is not log-balanced a.e.

Idea of the proof

Fix $x = [a_1, a_2, \dots]$ and $n \in \mathbb{N}$. Then,

$$\left|I_n{}^{\mathcal{SB}}(x)\right| = (((r+1)q_m + q_{m-1})q_m)^{-1}$$

where $\sum_{i=1}^{m} a_i \leqslant n < \sum_{i=1}^{m+1} a_i$ and $r = n - \sum_{i=1}^{m} a_i$.

Classical: $\log q_m/m \sim \pi^2/(12\ln 2))$ (as $m \to \infty$) a.e.,
and
$n \approx \sum_{i=1}^{m} a_i \approx m \log m/(\ln 2)$ in measure but not a.e.

The family $3\mathcal{D}(\alpha)$

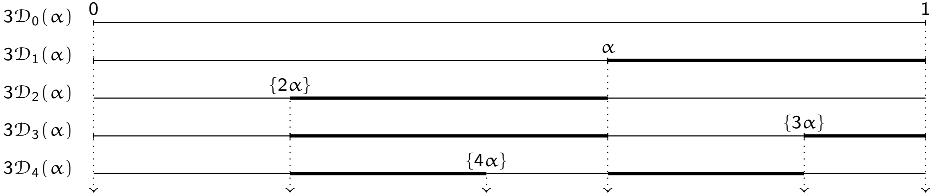
The three distance partition

Fix an irrational $\alpha \in (0, 1)$.

Consider the sequence of fractionals parts of the multiples of $k\alpha$:

$$k \mapsto \{k\alpha\}.$$

The intervals of $3\mathcal{D}_\alpha(n)$ have the points $\{k\alpha\}_{1 \leqslant k \leqslant n}$ as endpoints.



The family $3\mathcal{D}(\alpha)$

Three-distance sequence of partitions

The sequence of partitions $3\mathcal{D}(\alpha)$ is

- log-balanced a.e. with respect to the Leb. measure with weight function

$$f_{\mathcal{SB}}(n) = \ln n$$

  for α in a set of measure 1.

- There exists an uncountable set of α's so that the sequence of partitions $3\mathcal{D}(\alpha)$ are not log-balanced even in measure.

The proof is based on the three-distance theorem.

Conclusions

- We introduce the notions of log-balanced sequence of partitions and weight function.
- There are natural zero entropy instances with weight function
  - a.e.
  - in measure but not a.e.
  - not log-balanced at all.
- For any function $f(n)$ that goes to infinite with n, there exists a sequence of partitions that realizes f as weight function.
- Our main results are Lochs-type theorems for log-balanced sequences of partitions beyond positive entropy: zero and infinite entropy.
- Our results holds for a large class of sequences of partitions (even not self-refining).
- However, our results do not hold when the weight of the "source" sequence of partitions grows very slowly or when the weight of the "target" sequence of partitions grows exponentially fast.

Open questions/further work

- Are there any other "natural" sequences of partitions of zero or infinite entropy?
  Some candidates: instances of Variable Length Markov Chains sources, fibred maps with indifferent fixed points.
- Is it possible to relax our assumptions on the growth of the weight function?
  Work in progress by Brigitte Vallée for the self-refining case.
- In our work, we prove a Lochs-type theorem in distribution from continued fractions to Farey.
  Is it possible to obtain such results in distribution for others sequences of partitions?
- Is the notion of weight function relevant in other contexts beyond the Lochs' index?

$$T + \cfrac{1}{H + \cfrac{1}{A + \cfrac{1}{N + \cfrac{1}{K + \cfrac{1}{Y + \cfrac{1}{O + \cfrac{1}{U}}}}}}}$$