Matching of orbits of certain $N$-expansions with a finite set of digits

Yufei Chen
TU Delft & East China Normal University

Joint work with Cor Kraaikamp (TU Delft)
The regular continued fraction

It is well known that every real number \( x \) can be written as a finite (in case \( x \in \mathbb{Q} \)) or infinite (regular) continued fraction of the form:

\[
x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots} \cfrac{1}{a_n + \cdots}}}
\]

\[= [a_0; a_1, a_2, \ldots, a_n, \ldots], \tag{1}\]

where \( a_0 \in \mathbb{R} \) such that \( x - a_0 \in [0, 1) \), and \( a_n \in \mathbb{Z} \) for \( n \geq 1 \).
$N$-expansions

In 2008, Ed Burger and his co-authors introduced new continued fraction expansions, which are a nice variation on the RCF-expansion from (1).

Let $N \in \mathbb{N}_{\geq 2}$ be a fixed positive integer, and define the map $T_N : [0, 1) \rightarrow [0, 1)$ by:

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \quad x \neq 0; \quad T_N(0) = 0.$$

Setting $d_1 = d_1(x) = \lfloor N/x \rfloor$, and $d_n = d_n(x) = d_1 \left( T_n^{n-1}(x) \right)$, whenever $T_n^{n-1}(x) \neq 0$, we find:

$$x = \frac{N}{d_1 + \frac{N}{d_2 + \cdots + \frac{N}{d_n + T_n^m(x)}}}. \quad (2)$$

Taking finite truncations yield the convergents, which converge to $x$.

A number of properties resembling those of the RCF-expansion, although in some cases there is (or seems to be) an unexpected variation to the standard case.
$N$-expansions

Figure: The map $T_N$ for $N = 2$. 
$N$-expansions with a finite digit set

In his MSc-thesis from 2015, Niels Langeveld considered $N$-expansions on an interval not containing 0.

To be more precise: let $N \in \mathbb{N}$ such that $0 < \alpha \leq \sqrt{N} - 1$, then we define $I_\alpha := [\alpha, \alpha + 1]$ and $I^-_\alpha := [\alpha, \alpha + 1)$ and investigate the continued fraction map $T_\alpha : I_\alpha \to I^-_\alpha$, defined as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d(x) := \lfloor \frac{N}{x} - \alpha \rfloor$.

Note that due to the fact that $\alpha > 0$ there are only finitely many values of partial quotients $d$ possible. Furthermore, all expansions are infinite.
As an example, $N = 2$, $\alpha = \alpha_{\text{max}} = \sqrt{2} - 1$: 

\[
\sqrt{2} \\
\sqrt{2} - 1 \quad 2 - \sqrt{2} \quad 2(\sqrt{2} - 1) \\
\sqrt{2}
\]
This new $N$-expansion (with a finite digit set) could be viewed as a small and insignificant variation of the $N$-expansions with infinitely many digits ... but actually the situation is suddenly dramatically different and more difficult!

Suddenly, for certain values of $N$ and $\alpha$ “gaps” in the interval $I_\alpha$ appear. As an example, take $N = 51$, $\alpha = 6$. In this case there are only 2 digits (viz. 1 and 2), and setting for $n \geq 0$: $r_n = T^n_\alpha (\alpha + 1)$, $\ell_n = T^n_\alpha (\alpha)$, and in general for a digit $i$:

$$f_i = f_i(N) = \frac{\sqrt{4N + i^2} - i}{2},$$

as the fixed point of $T_\alpha$ with digit $i$, then we immediately see two gaps popping up:
$N$-expansions with a finite digit set

Definition

A maximal open interval $(a, b) \subset I_\alpha$ is called a gap of $I_\alpha$ if for almost every $x \in I_\alpha$ there is an $n_0 \in \mathbb{N}$ for which $T^n_\alpha(x) \notin (a, b)$ for all $n \geq n_0$. 

Figure: $N = 51$, $\alpha = 6$
Figure: Orbit simulation for $N = 9$, $\alpha = 1.99$

TU Delft, DIAM (2022)
Some ergodic properties

Since $\inf |T'_\alpha| > 1$, applying Theorem 1 from the classical 1973 paper by Lasota and Yorke immediately yields the following assertion:

Lemma

If $\mu$ is an absolutely continuous invariant probability measure for $T_\alpha$, then there exists a function $h$ of bounded variation such that

$$\mu(A) = \int_A h \, d\lambda, \quad \lambda - \text{a.e.},$$

with $\lambda$ the Lebesgue measure,

i.e. any absolutely continuous invariant probability measure has a version of its density function of bounded variation.

One have the following result:

Theorem

Let $N \geq 2$. Then there is a unique absolutely continuous invariant probability measure $\mu_\alpha$ such that $T_\alpha$ is ergodic with respect to $\mu_\alpha$. 
Consider $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, define $I_\alpha := [\alpha, \alpha + 1]$ and $I^-_\alpha := [\alpha, \alpha + 1)$, and the continued fraction map $T_\alpha : I_\alpha \rightarrow I^-_\alpha$, defined as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d : I_\alpha \rightarrow \mathbb{R}$ is defined by $d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor$.

In 2017, by investigating the above continued fraction map $T_\alpha : I_\alpha \rightarrow I^-_\alpha$, Cor and Niels found that there exists a plateau in entropy simulation graph in case $N = 2$, (in fact, with $\alpha \in \left(\frac{33 - 5}{2}, \sqrt{2} - 1\right)$), and also gives many important results for it.
Matching of orbits of certain $N$-expansions with a finite set of digits.
Matching

**Definition**

We say matching holds for $\alpha$ if there are $K, M \in \mathbb{N}_{\geq 2}$ such that

$$T_{N, \alpha}^K(\alpha) = T_{N, \alpha}^M(\alpha + 1).$$

The numbers $K, M$ are called the matching exponents, $KM$ is called the matching index and an interval $(c, d)$ such that for all $\alpha(c, d)$ we have the same matching exponents is called a matching interval.

Cor and Niels get

**Theorem**

*Let $N = 2$ and $\alpha \in \left(\frac{\sqrt{33} - 5}{2}, \sqrt{2} - 1\right)$, then $T^3(\alpha) = T^3(\alpha + 1)$.***
Natural extensions

A way to find the invariant density of the absolutely continuous invariant measure of \( T(\alpha) \), is:

1. Constructing a natural extension domain such that \( \mathcal{T}(x, y) \) is almost bijective and minimal from a measure theoretic point of view,

2. Simply projecting this map onto the first coordinate.

Here they use the ‘standard’ natural extension map \( \mathcal{T}(x, y) = \left( T(x), \frac{N}{d_1(x)+y} \right) \), where \( N = 2 \).
Natural extensions

A way to find the invariant density of the absolutely continuous invariant measure of $T(\alpha)$, is:

1. Constructing a natural extension domain such that $\mathcal{T}(x, y)$ is almost bijective and minimal from a measure theoretic point of view,

2. Simply projecting this map onto the first coordinate.

Here they use the ‘standard’ natural extension map $\mathcal{T}(x, y) = \left( T(x), \frac{N}{d_1(x) + y} \right)$, where $N = 2$. 
Cor and Niels guessed the shape of the domain of natural extension from simulation.
By projecting $T(x, y)$ onto the first coordinate, Cor and Niels obtained the following result:

**Theorem**

For $N = 2$ and $\alpha \in \left(\frac{\sqrt{33} - 5}{2}, \sqrt{2} - 1\right)$, the natural extension can be build (see the Figure on the above page). Moreover the invariant density is given by:

$$
f(x) = H \left( \frac{D}{2 + Dx} \mathbf{1}(\alpha, T(\alpha+1)) + \frac{E}{2 + Ex} \mathbf{1}(T(\alpha+1), T^2(\alpha)) 
+ \frac{F}{2 + Fx} \mathbf{1}(T^2(\alpha), \alpha+1) - \frac{A}{2 + Ax} \mathbf{1}(\alpha, T^2(\alpha+1)) 
- \frac{B}{2 + Bx} \mathbf{1}(T^2(\alpha+1), T(\alpha)) - \frac{C}{2 + Cx} \mathbf{1}(T(\alpha), \alpha+1) \right),
$$

with $A = \frac{\sqrt{33} - 5}{2}$, $B = \sqrt{2} - 1$, $C = \frac{\sqrt{33} - 3}{6}$, $D = 2\sqrt{2} - 2$, $E = \frac{\sqrt{33} - 3}{2}$, $F = \sqrt{2}$ and $H^{-1} = \log \left( \frac{1}{32} (3 + 2\sqrt{2})(7 + \sqrt{33})(\sqrt{33} - 5)^2 \right) \approx 0.25$ the normalizing constant.
$N$-expansions with a finite digit set

Cor and Niels provide a more elegant proof to show that the entropy is constant based on the so-called quilting (Figure in next page) of natural extensions.

**Theorem**

Let $(T_\alpha, \Omega_\alpha, B_\alpha, \mu)$ and $(T_\beta, \Omega_\beta, B_\beta, \mu)$ be two dynamical systems as in our setting. Furthermore let $D_1 = \Omega_\alpha \setminus \Omega_\beta$ and $A_1 = \Omega_\beta \setminus \Omega_\alpha$. If there is a $k \in \mathbb{N}$ such that $T_\alpha^k(D_1) = T_\beta^k(A_1)$ then the dynamical systems are isomorphic.

Since isomorphic systems have the same entropy, by using Rohlin’s formula to calculate the entropy for $\alpha = \sqrt{2} - 1$, it will give the following corollary.

**Corollary**

For $N = 2$, the entropy function is constant on $\left(\frac{\sqrt{33} - 5}{2}, \sqrt{2} - 1\right)$ and the value is approximately 1.14.
Matching of orbits of certain $N$-expansions with a finite set of digits
The following questions about entropy arise:

i. For every integer $N \geq 2$ such plateaux exist, is there an interval in $(0, \sqrt{N} - 1)$ for which the entropy function is constant?

ii. For every integer $N \geq 2$ for which $\alpha \in (0, \sqrt{N} - 1)$ do we have matching?
In this paper we proved, that for every integer $N \geq 2$ such matching and plateaux exist, and give them explicitly.

i. The number of such plateaux will be a function of $N$.

ii. The matching

$$T_3^3(\alpha) = T_3^3(\alpha + 1),$$

always exists in $\alpha \in (0, \sqrt{N} - 1)$. 


Matching of orbits of certain $N$-expansions with a finite set of digits
$N$-expansions with a finite digit set

Figure: Number of plataux for $N = 2, \ldots, 200$ (left) and $N = 2, \ldots, 10,000$ (right)
$N$-expansions with a finite digit set

**Map:** Let $T_{\alpha} : [\alpha, \alpha + 1] \rightarrow [\alpha, \alpha + 1)$ be the Gauss map defined as:

$$T_{\alpha}(x) := \frac{N}{x} - d(x),$$

where $d : I_{\alpha} \rightarrow \text{is defined by } d(x) := \lfloor \frac{N}{x} - \alpha \rfloor$.

**Digit set:** Here the (finite) set of partial quotients (i.e. digits) for $T_{\alpha}$ is denoted by

$$\{d, d + 1, \ldots, d + i\}.$$

**Partition:** $\mathcal{P} = \bigcup I_k$ of $[\alpha, \alpha + 1]$, where $I_k = \{x \mid d_1(x) = k\}$, $k = d, \ldots, \cdot \cdot \cdot$.
Lemma

Let $N \in \mathbb{N}$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, we have that $d \in \{1, 2, \ldots, N - 1\}$ and $\lim_{\alpha \downarrow 0} d = N - 1$.

Proof: From $\alpha < N/(\alpha + 1) - d$, it follows that $\alpha^2 + (d + 1)\alpha + d - N < 0$. Since $\alpha, d > 0$ it follows that $d < N$. Furthermore, if $\alpha$ tends to 0 it follows that $d$ tends to $N - 1$. Note that if $\alpha = 0$, we have that $d = N$.

The following result gives bounds on the number $i + 1$ of possible digits.

Lemma

For all $N \in \mathbb{N}$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, $d \geq 1$, one has $\frac{d}{\alpha} \leq i < \frac{d+1}{\alpha} + 2$, where $i + 1$ is the number of possible digits. Furthermore, $\lim_{\alpha \downarrow 0} i = +\infty$. 
Now define $A_{N,d,i}$ be the set of all $\alpha \in (0, \sqrt{N} - 1]$ with digit set \{d, d + 1, \ldots, d + i\}. Furthermore, we define the sets $X_{N,d,i}$ and $X_{N,d,i,k}$ as follows, for $k = d, \ldots, d + i - 1$:

\[
X_{N,d,i} = \left\{ \alpha \in A_{N,d,i} \mid T_\alpha(\alpha) \in I_d^0, T_\alpha(\alpha + 1) \in I_{d+i}^0 \right\}; \tag{3}
\]

\[
X_{N,d,i,k} = \left\{ \alpha \in X_{N,d,i} \mid T_\alpha^2(\alpha) \in I_k^0, T_\alpha^2(\alpha + 1) \in I_{k+1}^0 \right\}. \tag{4}
\]

where

\[
X_{N,d,i} = \left\{ \frac{N}{d + 1 + \alpha} < \frac{N}{\alpha} - (d + i) < \alpha + 1, \alpha < \frac{N}{\alpha + 1} - d < \frac{N}{d + i + \alpha} \right\}.
\]

Actually, it means that we let the first second digit of $\alpha$ and $\alpha + 1$ are, respectively, $\alpha : d + i, d$, and $\alpha + 1 : d, d + i$.
We obtained the following result (Matching):

**Theorem**

Let $N \geq 2$ be an integer, and let $d, i \in, i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in X_{N,d,i}$, one has that $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha + 1) \in I_{k+1}$ for some $k \in \{d, \ldots, d + i - 1\}$. Moreover, $T_\alpha^3(\alpha) = T_\alpha^3(\alpha + 1)$. 
Proof: By definition of $X_{N,d,i}$ and $T_\alpha$, one has for $\alpha \in X_{N,d,i}$ that $T_\alpha^2(\alpha) = \frac{N}{\frac{N}{\alpha}-(d+i)} - d$, and that $T_\alpha^2(\alpha + 1) = \frac{N}{\frac{N}{\alpha+1} - d} - (d + i)$. Then,

$$\frac{N}{T_\alpha^2(\alpha)} = \frac{N}{\frac{N}{\alpha}-(d+i)} - d = -\frac{N(N-(d+i)\alpha)}{Nd-(d^2+di+N)\alpha},$$
$$\frac{N}{T_\alpha^2(\alpha + 1)} = \frac{N}{\frac{N}{\alpha+1} - (d+i)} = -\frac{N(N-d(\alpha + 1))}{(d+i-\alpha-1)N-d(d+i)(\alpha + 1)},$$

Then one easily finds that:
$N$-expansions with a finite digit set

\[
\frac{N}{T^2(\alpha)} - \left( \frac{N}{T^2(\alpha + 1)} - 1 \right) = (d^2 + di - N(i - 1)) \cdot R_{N,d,i,\alpha},
\]

where $R_{N,d,i,\alpha}$ satisfies:

\[
R_{N,d,i,\alpha} = \frac{((\alpha^2 + \alpha)d^2 + ((-2\alpha - 1)N + di\alpha(\alpha + 1)) + (N - \alpha(i - \alpha - 1))N)}{(-d^2\alpha + (-i\alpha + N)d - N\alpha)((-\alpha - 1)d^2 + (-i\alpha + N - i)d + N(i - \alpha - 1))}.
\]

Note that if $d^2 + di - N(i - 1) = 0$, so if $N = \frac{d(d + i)}{i - 1}$, we have that:

\[
\frac{N}{T^2_\alpha(\alpha)} = \frac{N}{T^2_\alpha(\alpha + 1)} - 1.
\]
$N$-expansions with a finite digit set

Since the length of the interval $[\alpha, \alpha + 1)$ is 1, we see that for $N = \frac{d(d+i)}{i-1}$ we have matching in 3 steps: $T_\alpha^3(\alpha) = T_\alpha^3(\alpha + 1)$. Furthermore, $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha + 1) \in I_{k+1}$ for some $k \in \{d, \ldots, d + i - 1\}$.

Theorem is proved.

Note that an immediate consequence of the proof of Theorem is that for $N = \frac{d(d+i)}{i-1}$,

$$X_{N,d,i} = \bigcup_{k=d}^{d+i-1} X_{N,d,i,k}.$$ 

By simulation (Figure on the following page) to extension map $T(x, y) = \left(T(x), \frac{N}{d_1(x)+y}\right)$, we can also figure the shape of the domain of natural extension.
Matching of orbits of certain $N$-expansions with a finite set of digits
$N$-expansions with a finite digit set

Then, we proved and obtained:

**Theorem**

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha \in X_{N,d,i}$ arbitrary, we get the natural extension graph for the planar domain $\Omega_{\alpha}$, which is a polygon bounded by the straight line segments between the vertices (in clockwise order) $(\alpha, A), \left(T_{\alpha}^{2}(\alpha + 1), A\right), \left(T_{\alpha}^{2}(\alpha + 1), B\right), \left(T_{\alpha}(\alpha), B\right), \left(T_{\alpha}(\alpha), C\right), \left(\alpha + 1, C\right), \left(\alpha + 1, F\right), \left(T_{\alpha}^{2}(\alpha), F\right), \left(T_{\alpha}^{2}(\alpha), E\right), \left(T_{\alpha}(\alpha + 1), E\right), \left(T_{\alpha}(\alpha + 1), D\right), (\alpha, D)$, and finally ‘back’ to $(\alpha, A)$, where $\Omega_{\alpha}$ is illustrated for various $\alpha$, where $0 < A < B < C < D < E < F$.

**Theorem**

$X_{N,d,i} = (A, B)$, where

$$X_{N,d,i} = \left\{ \frac{N}{d + 1 + \alpha} < \frac{N}{\alpha} - (d + i) < \alpha + 1, \; \alpha < \frac{N}{\alpha + 1} - d < \frac{N}{d + i + \alpha} \right\}.$$
$N$-expansions with a finite digit set

$\Omega_\alpha$ and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$; (b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$. 

\( N \)-expansions with a finite digit set

\[ \Omega_\alpha \text{ and } T(\Omega_\alpha) \text{ with (a): } \alpha \in X_{N,d,i,1}; (b): \alpha \in X_{N,d,i,2}; \text{ (c): } \alpha \in X_{N,d,i,3}, \text{ for } N = 2, d = 1, i = 3. \]
$N$-expansions with a finite digit set

$\Omega_\alpha$ and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$; (b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$. 
$N$-expansions with a finite digit set

\[ \Omega_\alpha \text{ and } \mathcal{T}_\alpha(\Omega_\alpha) \text{ with (a): } \alpha \in X_{N,d,i,1}, \ d = 1, \ i = 3; \ (b): \ \alpha \in X_{N,d,i,1}, \ d = 2, \ i = 7 \text{ for } N = 3. \]
$N$-expansions with a finite digit set

$\Omega_{\alpha}$ and $T_{\alpha}(\Omega_{\alpha})$ with (a): $\alpha \in X_{N,d,i,1}$, $d = 1, i = 3$; (b): $\alpha \in X_{N,d,i,1}$, $d = 2, i = 7$ for $N = 3$. 
**Theorem**

Let \( N \geq 2 \) be an integer, and let \( d \geq 1 \) and \( i \geq 2 \) be integers, such that \( N = \frac{d(d+i)}{i-1} \). Let \( \alpha \in \overline{X}_{N,d,i} \) arbitrary, the density \( f_\alpha(x) \) of the \( T_\alpha \)-invariant measure \( \mu_\alpha \) is given by

\[
\begin{align*}
    f_\alpha(x) &= H\left( \frac{D}{N + Dx} \mathbf{1}_{\alpha,T(\alpha)+1}(x) + \frac{E}{N + Ex} \mathbf{1}_{T(\alpha)+1},T^2(\alpha)(x) + \frac{F}{N + Fx} \mathbf{1}_{T^2(\alpha),\alpha+1}(x) \\
    &\quad - \frac{A}{N + Ax} \mathbf{1}_{\alpha,T^2(\alpha)+1}(x) - \frac{B}{N + Bx} \mathbf{1}_{T^2(\alpha)+1},T(\alpha)(x) - \frac{C}{N + Cx} \mathbf{1}_{T(\alpha),\alpha+1}(x) \right).
\end{align*}
\]

where

\[
    H^{-1} = 2 \log A + 2 \log(B + 1) - \log(N - (A + 1)d) - \log(N - (d + i)B),
\]
Theorem

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha, \beta \in [A, B] = \overline{X}_{N,d,i}$, $\alpha < \beta$ arbitrary. Then the dynamical systems $(\Omega_\alpha, \bar{B}_\alpha, \bar{\mu}_\alpha, T_\alpha)$ and $(\Omega_\beta, \bar{B}_\beta, \bar{\mu}_\beta, T_\beta)$ are metrically isomorphic.
$N$-expansions with a finite digit set

Figure: $\Omega_\alpha$ and $\Omega_\beta$ for $\alpha, \beta \in X_{N,d,i,k}$, where $\alpha < \beta$. 

TU Delft, DIAM (2022)
$N$-expansions with a finite digit set

\[ T(\alpha) = \frac{N}{\alpha + d + 1} \]

\[ T(\alpha + 1) = \frac{N}{\alpha + d + 1} \]

Figure: $\Omega_B$ (left) and $\Omega_A$ (right).
N-expansions with a finite digit set

By using Rohlin’s formula to calculate the entropy for $\alpha = B$, we get the entropy is constant.

**Theorem**

Let $N \geq 2$ be an integer, and let $d, i \in \mathbb{N}, i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in [A, B] = X_{N,d,i}$, one has that the entropy function $h(T_\alpha)$ is constant on $[A, B] = X_{N,d,i}$, and is given by:

$$h(T_\alpha) = \log N - 2H\left( \left. \left( \text{Li}_2\left( -\frac{Ex}{N} \right) + \log x \log\left( \frac{Ex}{N} + 1 \right) \right) \right|_B^{B+1} 
- \left( \text{Li}_2\left( -\frac{Ax}{N} \right) + \log x \log\left( \frac{Ax}{N} + 1 \right) \right) \right|_B^D 
- \left( \text{Li}_2\left( -\frac{Cx}{N} \right) + \log x \log\left( \frac{Cx}{N} + 1 \right) \right) \right|_D^{B+1} \right),$$

where $H^{-1} = 2 \log A + \log(A+1) + \log(B+1) - \log\left( N - (A+1)d \right) - \log\left( N - (d+i)B \right)$ is the normalising constant for the $T_\alpha$-invariant measure $\mu_\alpha$ for $\alpha \in X_{N,d,i}$. 

TU Delft, DIAM (2022)
In case $N = 2$, our method yields only one plateau with equal entropy which follows from our method. This is the interval
\[ [A, B] = \left[ \frac{\sqrt{33} - 5}{2}, \sqrt{2} - 1 \right] = [0.3722813 \ldots, 0.4142136 \ldots], \]
which was already found by Cor and Niels, where it was also determined that for $\alpha \in [A, B]$ we have that $h(T_\alpha) = 1.137779584292255 \ldots$ and $H = 3.965116120651161 \ldots$. 
Matching of orbits of certain $N$-expansions with a finite set of digits.
Example

In case $N = 8$ it follows from our method that there are five plateaux of equal entropy; see Table 1.

<table>
<thead>
<tr>
<th>$(d, i)$</th>
<th>Plateau intervals</th>
<th>Approximation of interval</th>
<th>$H_\alpha$</th>
<th>$h(T_\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 2)</td>
<td>$\left[\frac{\sqrt{57} - 5}{2}, \frac{\sqrt{33} - 3}{2}\right]$</td>
<td>[1.2749,1.3723]</td>
<td>18.377877038370</td>
<td>0.9212748062044</td>
</tr>
<tr>
<td>(4, 6)</td>
<td>$\left[\frac{3\sqrt{17} - 11}{2}, \frac{\sqrt{41} - 5}{2}\right]$</td>
<td>[0.6847,0.7016]</td>
<td>11.239480662654</td>
<td>1.8212263472923</td>
</tr>
<tr>
<td>(5, 11)</td>
<td>$\left[\frac{3\sqrt{97} - 29}{2}, \frac{\sqrt{57} - 7}{2}\right]$</td>
<td>[0.2733,0.2749]</td>
<td>9.9626774452815</td>
<td>2.7933207303296</td>
</tr>
<tr>
<td>(6, 22)</td>
<td>$\left[\frac{\sqrt{321} - 17}{2}, \frac{2\sqrt{3} - 3}{2}\right]$</td>
<td>[0.4582,0.4641]</td>
<td>9.2212359716540</td>
<td>2.2547418855378</td>
</tr>
<tr>
<td>(7, 57)</td>
<td>$\left[\frac{3\sqrt{473} - 65}{2}, \frac{\sqrt{17} - 4}{2}\right]$</td>
<td>[0.1228,0.1231]</td>
<td>8.7715446381451</td>
<td>3.3495778601659</td>
</tr>
</tbody>
</table>

Table: The pairs of integers $d \geq 1, i \geq 2$, the related plateau intervals $[A, B]$ and constant entropy $h(T_\alpha)$ for $\alpha \in [A, B]$. Here $N = 8.$
Matching of orbits of certain $N$-expansions with a finite set of digits.
Thank you for your attention!


