Matching of orbits of certain N-expansions with a finite set of digits

Yufei Chen
TU Delft &East China Normal University

Joint work with Cor Kraaikamp (TU Delft)

The regular continued fraction

It is well known that every real number x can be written as a finite (in case $x \in \mathbb{Q}$) or infinite (regular) continued fraction of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} = [a_0; a_1, a_2, \ldots, a_n, \ldots], \tag{1}$$

where $a_0 \in$ such that $x - a_0 \in [0, 1)$, and $a_n \in$ for $n \geq 1$.

N-expansions

In 2008, Ed Burger and his co-authors introduced new continued fraction expansions, which are a nice variation on the RCF-expansion from (1).

Let $N \in \mathbb{N}_{\geq 2}$ be a fixed positive integer, and define the map $T_N : [0,1) \to [0,1)$ by:

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \ x \neq 0; \quad T_N(0) = 0.$$

Setting $d_1 = d_1(x) = \lfloor N/x \rfloor$, and $d_n = d_n(x) = d_1\left(T_N^{n-1}(x)\right)$, whenever $T_N^{n-1}(x) \neq 0$, we find:

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_n + T_N^n(x)}}}. \tag{2}$$

Taking finite truncations yield the convergents, which converge to x.

A number of properties resembling those of the RCF-expansion, although in some cases there is (or seems to be) an unexpected variation to the standard case.

N-expansions

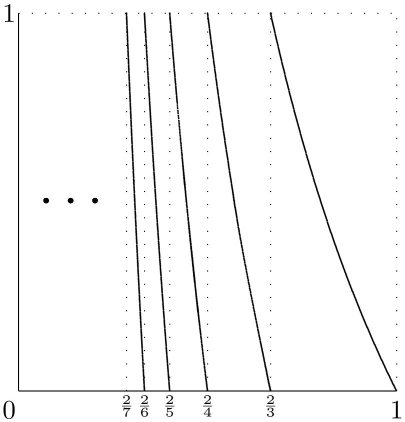


Figure: The map T_N for $N = 2$.

N-expansions with a finite digit set

In his MSc-thesis from 2015, Niels Langeveld considered N-expansions on an interval **not** containing 0.

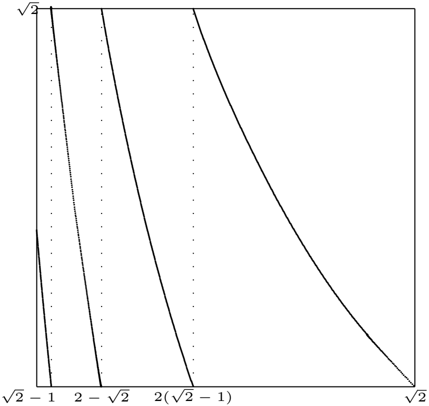
To be more precise: let $N \in_{\geq 2}$ and $\alpha \in$ such that $0 < \alpha \leq \sqrt{N} - 1$, then we define $I_\alpha := [\alpha, \alpha + 1]$ and $I_\alpha^- := [\alpha, \alpha + 1)$ and investigate the continued fraction map $T_\alpha : I_\alpha \to I_\alpha^-$, defined as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor$.

Note that due to the fact that $\alpha > 0$ there are only finitely many values of partial quotients d possible. Furthermore, *all* expansions are infinite.

As an example, $N = 2$, $\alpha = \alpha_{\text{max}} = \sqrt{2} - 1$:



N-expansions with a finite digit set

This new N-expansion (with a finite digit set) could be viewed as a small and insignificant variation of the N-expansions with infinitely many digits ... but actually the situation is suddenly dramatically different and more difficult!

Suddenly, for certain values of N and α "*gaps*" in the interval I_α appear. As an example, take $N = 51$, $\alpha = 6$. In this case there are only 2 digits (viz. 1 and 2), and setting for $n \geq 0$: $r_n = T_\alpha^n(\alpha + 1)$, $\ell_n = T_\alpha^n(\alpha)$, and in general for a digit i:

$$f_i = f_i(N) = \frac{\sqrt{4N + i^2} - i}{2},$$

as the fixed point of T_α with digit i, then we immediately see two *gaps* popping up:

N-expansions with a finite digit set

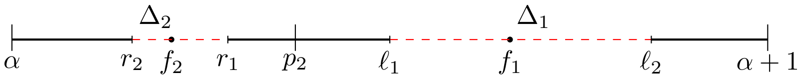
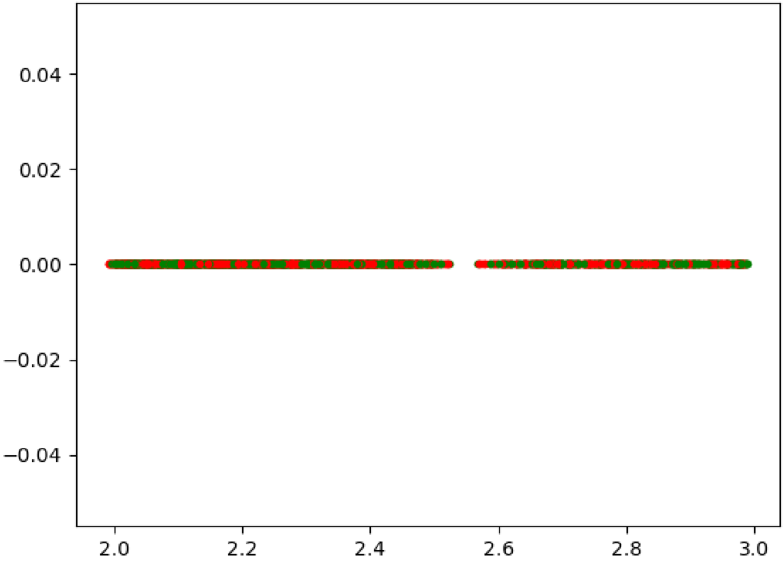


Figure: $N = 51$, $\alpha = 6$

Definition

A maximal open interval $(a,b) \subset I_\alpha$ is called a *gap* of I_α if for almost every $x \in I_\alpha$ there is an $n_0 \in \mathbb{N}$ for which $T_\alpha^n(x) \notin (a,b)$ for all $n \geq n_0$.


0.04
0.02
0.00
−0.02
−0.04
2.0
2.2
2.4
2.6
2.8
3.0

Some ergodic properties

Since $\inf |T'_\alpha| > 1$, applying Theorem 1 from the classical 1973 paper by Lasota and Yorke immediately yields the following assertion:

Lemma

If μ is an absolutely continuous invariant probability measure for T_α, then there exists a function h of bounded variation such that

$$\mu(A) = \int_A h\, d\lambda,\ \lambda - \textit{a.e.},\ \textit{with } \lambda \textit{ the Lebesgue measure},$$

i.e. any absolutely continuous invariant probability measure has a version of its density function of bounded variation.

One have the following result:

Theorem

Let $N \in_{\geq 2}$. Then there is a unique absolutely continuous invariant probability measure μ_α such that T_α is ergodic with respect to μ_α.

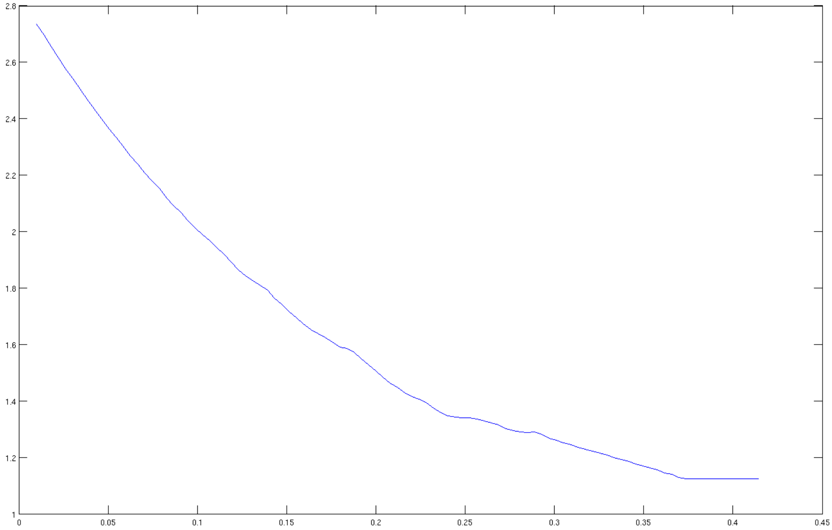
Motivation

Consider $N \in \mathbb{N}_{\geq 2}$ and $\alpha \in$ such that $0 < \alpha \leq \sqrt{N} - 1$, define $I_\alpha := [\alpha, \alpha + 1]$ and $I_\alpha^- := [\alpha, \alpha + 1)$ and the continued fraction map $T_\alpha : I_\alpha \to I_\alpha^-$, defined as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d : I_\alpha \to$ is defined by $d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor$.

In 2017, by investigating the above continued fraction map $T_\alpha : I_\alpha \to I_\alpha^-$, Cor and Niels found that there exists a plateau in entropy simulation graph in case $N = 2$, (in fact, with $\alpha \in (\frac{\sqrt{33}-5}{2}, \sqrt{2} - 1)$), and also gives many important results for it.


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45

Matching

Definition

We say matching holds for α if there are $K, M \in \mathbb{N}_{\geq 2}$ such that $T^K_{N,\alpha}(\alpha) = T^M_{N,\alpha}(\alpha+1)$. The numbers K, M are called the matching exponents, KM is called the matching index and an interval (c,d) such that for all $\alpha(c,d)$ we have the same matching exponents is called a matching interval.

Cor and Niels get

Theorem

Let $N = 2$ *and* $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$, *then* $T^3(\alpha) = T^3(\alpha+1)$.

Natural extensions

A way to find the invariant density of the absolutely continuous invariant measure of $T(\alpha)$, is:

1. Constructing a natural extension domain such that $\mathcal{T}(x, y)$ is almost bijective and minimal from a measure theoretic point of view,

2. Simply projecting this map onto the first coordinate.

Here they use the 'standard' natural extension map $\mathcal{T}(x, y) = \left(T(x), \frac{N}{d_1(x)+y}\right)$, where $N = 2$.

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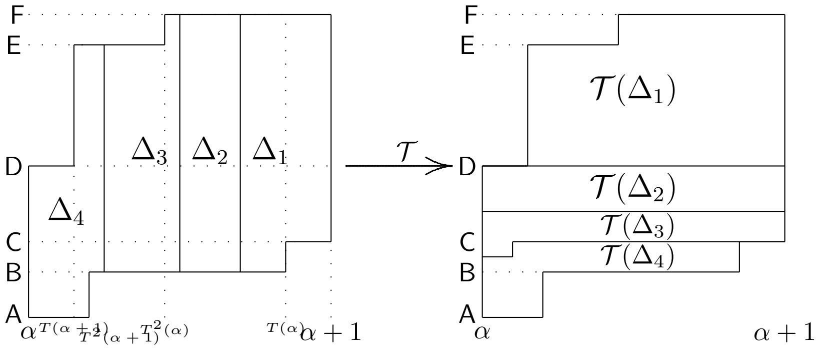
N-expansions with a finite digit set

Cor and Niels guessed the shape of the domain of natural extension from simulation.



N-expansions with a finite digit set

By projecting $\mathcal{T}(x,y)$ onto the first coordinate, Cor and Niels obtained the following result:

Theorem

For $N=2$ and $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$, the natural extension can be build (see the Figure on the above page). Moreover the invariant density is given by:

$$
\begin{aligned}
f(x) \quad = \quad & H\Big(\frac{D}{2+Dx}\mathbf{1}_{(\alpha,T(\alpha+1))} + \frac{E}{2+Ex}\mathbf{1}_{(T(\alpha+1),T^2(\alpha))} \\
& + \frac{F}{2+Fx}\mathbf{1}_{(T^2(\alpha),\alpha+1)} - \frac{A}{2+Ax}\mathbf{1}_{(\alpha,T^2(\alpha+1))} \\
& - \frac{B}{2+Bx}\mathbf{1}_{(T^2(\alpha+1),T(\alpha))} - \frac{C}{2+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}\Big),
\end{aligned}
$$

with $A=\frac{\sqrt{33}-5}{2}, B=\sqrt{2}-1, C=\frac{\sqrt{33}-3}{6}, D=2\sqrt{2}-2, E=\frac{\sqrt{33}-3}{2}, F=\sqrt{2}$ and $H^{-1}=\log\left(\frac{1}{32}(3+2\sqrt{2})(7+\sqrt{33})(\sqrt{33}-5)^2\right)\approx 0.25$ the normalizing constant.

N-expansions with a finite digit set

Cor and Niels provide a more elegant proof to show that the entropy is constant based on the so-called quilting (Figure in next page) of natural extensions.

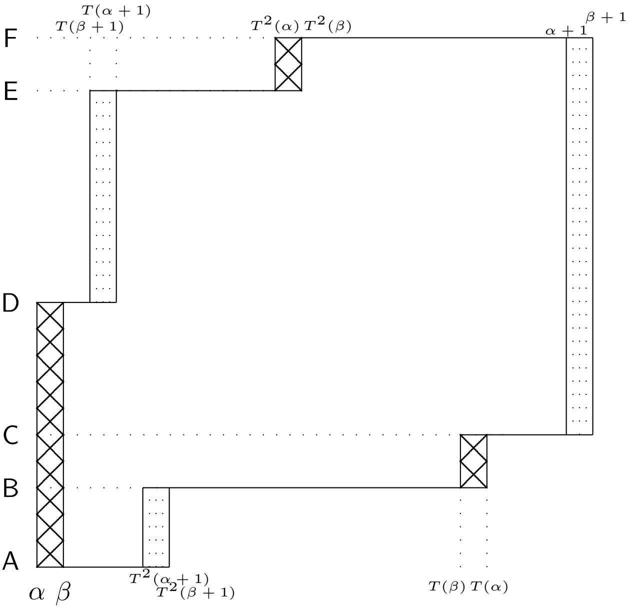
Theorem

Let $(\mathcal{T}_\alpha, \Omega_\alpha, \mathcal{B}_\alpha, \mu)$ *and* $(\mathcal{T}_\beta, \Omega_\beta, \mathcal{B}_\beta, \mu)$ *be two dynamical systems as in our setting. Furthermore let* $D_1 = \Omega_\alpha \backslash \Omega_\beta$ *and* $A_1 = \Omega_\beta \backslash \Omega_\alpha$*. If there is a* $k \in \mathbb{N}$ *such that* $\mathcal{T}_\alpha^k(D_1) = \mathcal{T}_\beta^k(A_1)$ *then the dynamical systems are isomorphic.*

Since isomorphic systems have the same entropy, by using Rohlin's formula to calculate the entropy for $\alpha = \sqrt{2} - 1$, it will give the following corollary.

Corollary

For $N = 2$*, the entropy function is constant on* $\left(\frac{\sqrt{33}-5}{2}, \sqrt{2} - 1\right)$ *and the value is approximately* 1.14.



F
E
D
C
B
A
$T(\alpha+1)$
$T(\beta+1)$
$T^2(\alpha)$ $T^2(\beta)$
$\beta+1$
$\alpha+1$
α β
$T^2(\alpha+1)$
$T^2(\beta+1)$
$T(\beta)$ $T(\alpha)$

Questions :

The following questions about entropy arise:

i. For every integer $N \geq 2$ such plateaux exist, is there an interval in $(0, \sqrt{N}-1)$ for which the entropy function is constant?

ii. For every integer $N \geq 2$ for which $\alpha \in (0, \sqrt{N}-1)$ do we have matching?

N-expansions with a finite digit set

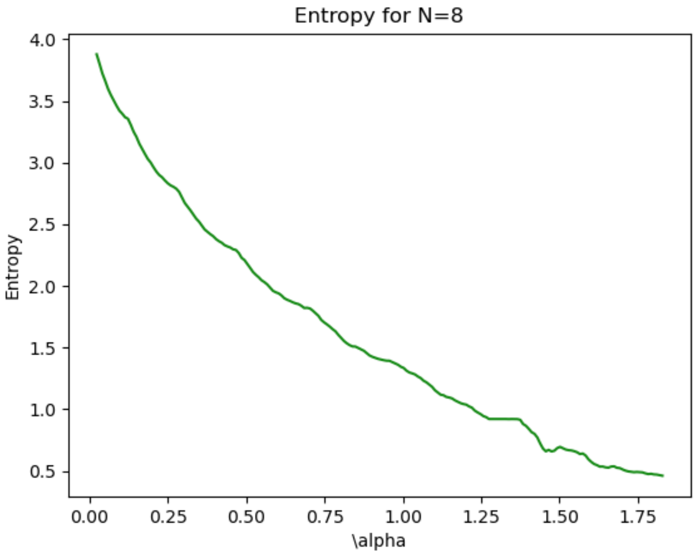
In this paper we proved, that for every integer $N \geq 2$ such matching and plateaux exist, and give them explicitly.

i.The number of such plateaux will be a function of N.

ii. The matching

$$T_\alpha^3(\alpha) = T_\alpha^3(\alpha + 1),$$

always exists in $\alpha \in (0, \sqrt{N} - 1)$.


Entropy for N=8
Entropy
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
\alpha

N-expansions with a finite digit set

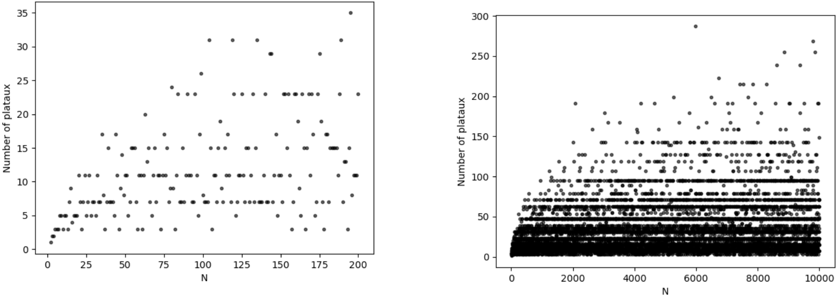


Figure: Number of plataux for $N = 2, \ldots, 200$ (left) and $N = 2, \ldots, 10.000$ (right)

N-expansions with a finite digit set

Map: Let $T_\alpha : [\alpha, \alpha+1] \to [\alpha, \alpha+1)$ be the Gauss map defined as defined as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d : I_\alpha \to$ is defined by $d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor$.

Digit set: Here the (finite) set of partial quotients (i.e. digits) for T_α is denoted by

$$\{d, d+1, \ldots, d+i\}$$

.

Partition: $\mathcal{P} = \bigcup I_k$ of $[\alpha, \alpha+1]$, where $I_k = \{x \,|\, d_1(x) = k\}$, $k = d, \cdots ,$.

N-expansions with a finite digit set

Lemma

Let $N \in \mathbb{N}$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, we have that $d \in \{1, 2, \ldots, N-1\}$ and $\lim_{\alpha \downarrow 0} d = N - 1$.

$Proof$: From $\alpha < N/(\alpha + 1) - d$, it follows that $\alpha^2 + (d+1)\alpha + d - N < 0$. Since $\alpha, d > 0$ it follows that $d < N$. Furthermore, if α tends to 0 it follows that d tends to $N - 1$. Note that if $\alpha = 0$, we have that $d = N$.

The following result gives bounds on the number $i + 1$ of possible digits.

Lemma

For all $N \in$, $N \geq 2$, and $0 < \alpha \leq \sqrt{N} - 1$, $d \geq 1$, one has $\frac{d}{\alpha} \leq i < \frac{d+1}{\alpha} + 2$, where $i + 1$ is the number of possible digits. Furthermore, $\lim_{\alpha \downarrow 0} i = +\infty$.

N-expansions with a finite digit set

Now define $\mathcal{A}_{N,d,i}$ be the set of all $\alpha \in (0, \sqrt{N}-1]$ with digit set $\{d, d+1, \ldots, d+i\}$. Furthermore, we define the sets $X_{N,d,i}$ and $X_{N,d,i,k}$ as follows, for $k = d, \ldots, d+i-1$:

$$X_{N,d,i} = \{\alpha \in \mathcal{A}_{N,d,i} \,\Big|\, T_\alpha(\alpha) \in I_d^o, T_\alpha(\alpha+1) \in I_{d+i}^o\}; \quad (3)$$

$$X_{N,d,i,k} = \{\alpha \in X_{N,d,i} \,\Big|\, T_\alpha^2(\alpha) \in I_k, T_\alpha^2(\alpha+1) \in I_{k+1}\}. \quad (4)$$

where

$$X_{N,d,i} = \left\{ \frac{N}{d+1+\alpha} < \frac{N}{\alpha} - (d+i) < \alpha+1,\ \ \alpha < \frac{N}{\alpha+1} - d < \frac{N}{d+i+\alpha} \right\}.$$

Actually, it means that we let the first second digit of α and $\alpha+1$ are, respectively, $\alpha : d+i, d$, and $\alpha+1 : d, d+i$

N-expansions with a finite digit set

We obtained the following result (Matching):

Theorem

Let $N \geq 2$ be an integer, and let $d, i \in$, $i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in X_{N,d,i}$, one has that $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha+1) \in I_{k+1}$ for some $k \in \{d, \ldots, d+i-1\}$. Moreover, $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$.

N-expansions with a finite digit set

$Proof$: By definition of $X_{N,d,i}$ and T_α, one has for $\alpha \in X_{N,d,i}$ that $T_\alpha^2(\alpha) = \frac{N}{\frac{N}{\alpha}-(d+i)} - d$, and that $T_\alpha^2(\alpha+1) = \frac{N}{\frac{N}{\alpha+1}-d} - (d+i)$. Then,

$$\begin{aligned}
\frac{N}{T_\alpha^2(\alpha)} &= \frac{N}{\frac{N}{\frac{N}{\alpha}-(d+i)} - d} = -\frac{N(N-(d+i)\alpha)}{Nd-(d^2+di+N)\alpha}, \\
\frac{N}{T_\alpha^2(\alpha+1)} &= \frac{N}{\frac{N}{\frac{N}{\alpha+1}-d} - (d+i)} = -\frac{N(N-d(\alpha+1))}{(d+i-\alpha-1)N-d(d+i)(\alpha+1))},
\end{aligned}$$

Then one easily finds that:

N-expansions with a finite digit set

$$\frac{N}{T^2(\alpha)} - \left(\frac{N}{T^2(\alpha+1)} - 1\right) = \left(d^2 + di - N(i-1)\right) \cdot R_{N,d,i,\alpha},$$

where $R_{N,d,i,\alpha}$ satisfies:

$$R_{N,d,i,\alpha} = \frac{\left((\alpha^2+\alpha)d^2 + ((-2\alpha-1)N + di\alpha(\alpha+1)) + (N - \alpha(i-\alpha-1))N\right)}{(-d^2\alpha + (-i\alpha+N)d - N\alpha)((-\alpha-1)d^2 + (-i\alpha+N-i)d + N(i-\alpha-1))}.$$

Note that if $d^2 + di - N(i-1) = 0$, so if $N = \frac{d(d+i)}{i-1}$, we have that:

$$\frac{N}{T_\alpha^2(\alpha)} = \frac{N}{T_\alpha^2(\alpha+1)} - 1.$$

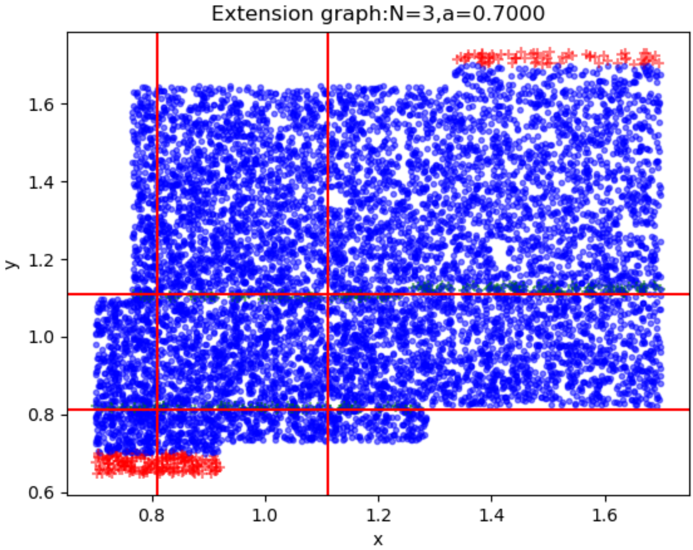
N-expansions with a finite digit set

Since the length of the interval $[\alpha, \alpha+1)$ is 1, we see that for $N = \frac{d(d+i)}{i-1}$ we have *matching* in 3 steps: $T_\alpha^3(\alpha) = T_\alpha^3(\alpha+1)$. Furthermore, $T_\alpha^2(\alpha) \in I_k$ and $T_\alpha^2(\alpha+1) \in I_{k+1}$ for some $k \in \{d, \dots, d+i-1\}$.
Theorem is proved.

Note that an immediate consequence of the proof of Theorem is that for $N = \frac{d(d+i)}{i-1}$,

$$X_{N,d,i} = \bigcup_{k=d}^{d+i-1} X_{N,d,i,k}.$$

By simulation (Figure on the following page) to extension map $\mathcal{T}(x,y) = \left(T(x), \frac{N}{d_1(x)+y}\right)$, we can also figure the shape of the domain of natural extension.


Extension graph:N=3,a=0.7000
1.6
1.4
1.2
1.0
0.8
0.6
y
0.8
1.0
1.2
1.4
1.6
x

N-expansions with a finite digit set

Then, we proved and obtained:

Theorem

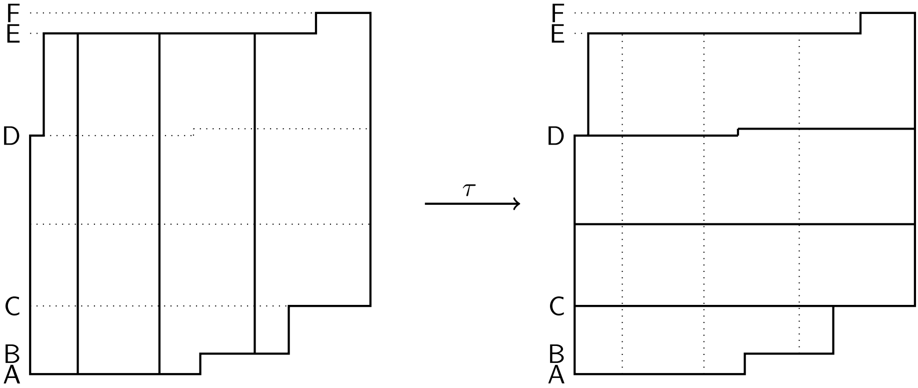
Let $N \geq 2$ *be an integer, and let* $d \geq 1$ *and* $i \geq 2$ *be integers, such that* $N = \frac{d(d+i)}{i-1}$. *Let* $\alpha \in X_{N,d,i}$ *arbitrary, we get the natural extension graph for the planar domain* Ω_α, *which is a polygon bounded by the straight line segments between the vertices (in clockwise order)* (α, A), $(T_\alpha^2(\alpha+1), A)$, $(T_\alpha^2(\alpha+1), B)$, $(T_\alpha(\alpha), B)$, $(T_\alpha(\alpha), C)$, $(\alpha+1, C)$, $(\alpha+1, F)$, $(T_\alpha^2(\alpha), F)$, $(T_\alpha^2(\alpha), E)$, $(T_\alpha(\alpha+1), E)$, $(T_\alpha(\alpha+1), D)$, (α, D), *and finally 'back' to* (α, A), *where* Ω_α *is illustrated for various* α*), where* $0 < A < B < C < D < E < F$.

Theorem

$X_{N,d,i} = (A, B)$, *where*

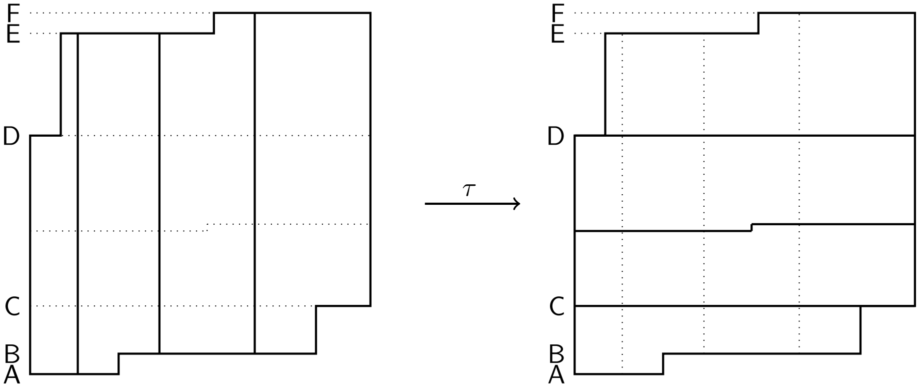
$$X_{N,d,i} = \left\{ \frac{N}{d+1+\alpha} < \frac{N}{\alpha} - (d+i) < \alpha + 1, \ \ \alpha < \frac{N}{\alpha+1} - d < \frac{N}{d+i+\alpha} \right\}.$$

N-expansions with a finite digit set



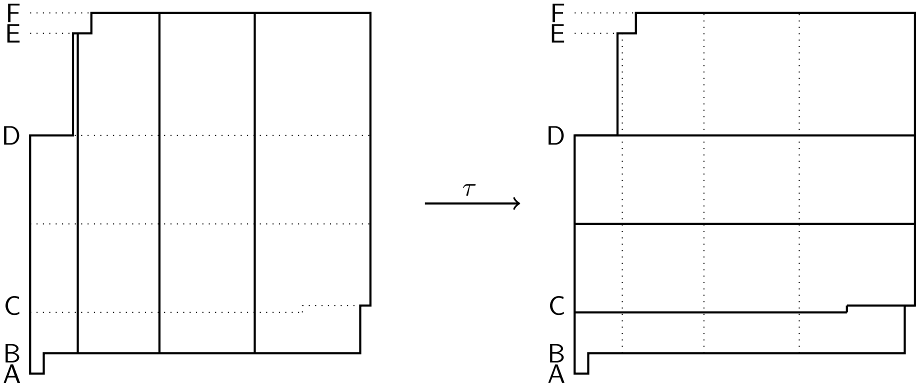
Ω_α and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$; (b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$.

N-expansions with a finite digit set



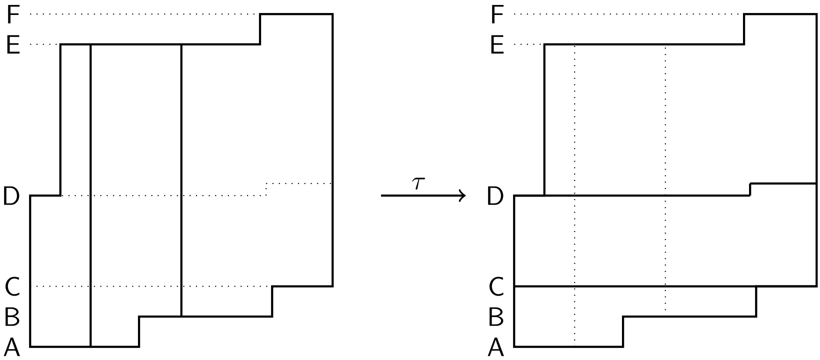
Ω_α and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$;(b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$.

N-expansions with a finite digit set



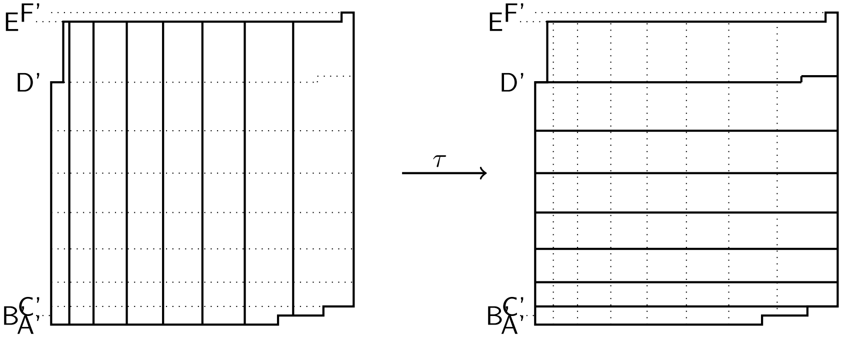
Ω_α and $T(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$;(b): $\alpha \in X_{N,d,i,2}$; (c): $\alpha \in X_{N,d,i,3}$, for $N = 2, d = 1, i = 3$.

N-expansions with a finite digit set



Ω_α and $\mathcal{T}_\alpha(\Omega_\alpha)$ with (a): $\alpha \in X_{N,d,i,1}$, $d = 1, i = 3$; (b): $\alpha \in X_{N,d,i,1}$, $d = 2, i = 7$ for $N = 3$.

N-expansions with a finite digit set



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N-expansions with a finite digit set

By projecting $\mathcal{T}(x,y)$ onto the first coordinate, we obtained the following result:

Theorem

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha \in \overline{X}_{N,d,i}$ arbitrary, the density $f_\alpha(x)$ of the T_α-invariant measure μ_α is given by

$$
\begin{aligned}
& f_\alpha(x) \\
= \quad & H\Big(\frac{D}{N+Dx}\mathbf{1}_{(\alpha,T(\alpha)+1)}(x) + \frac{E}{N+Ex}\mathbf{1}_{(T(\alpha)+1,T^2(\alpha))}(x) + \frac{F}{N+Fx}\mathbf{1}_{(T^2(\alpha),\alpha+1)}(x) \\
& -\frac{A}{N+Ax}\mathbf{1}_{(\alpha,T^2(\alpha)+1)}(x) - \frac{B}{N+Bx}\mathbf{1}_{(T^2(\alpha)+1,T(\alpha))}(x) - \frac{C}{N+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}(x)\Big).
\end{aligned}
$$

where

$$H^{-1} = 2\log A + 2\log(B+1) - \log\left(N-(A+1)d\right) - \log\left(N-(d+i)B\right),$$

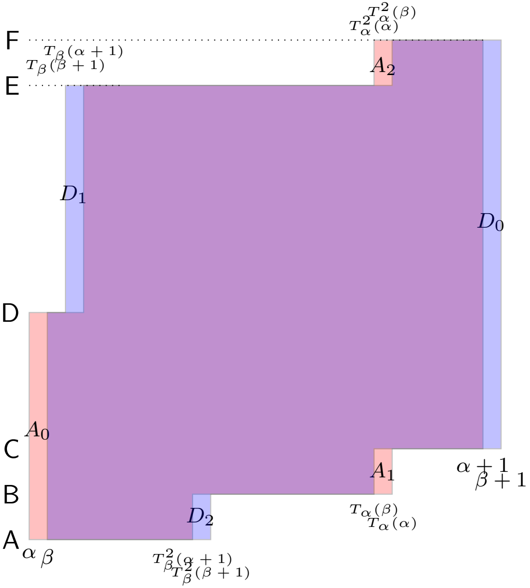
.

N-expansions with a finite digit set

Theorem

Let $N \geq 2$ be an integer, and let $d \geq 1$ and $i \geq 2$ be integers, such that $N = \frac{d(d+i)}{i-1}$. Let $\alpha, \beta \in [A, B] = \overline{X}_{N,d,i}$, $\alpha < \beta$ arbitrary. Then the dynamical systems $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ and $(\Omega_\beta, \bar{\mathcal{B}}_\beta, \bar{\mu}_\beta, \mathcal{T}_\beta)$ are metrically isomorphic.

N-expansions with a finite digit set



N-expansions with a finite digit set

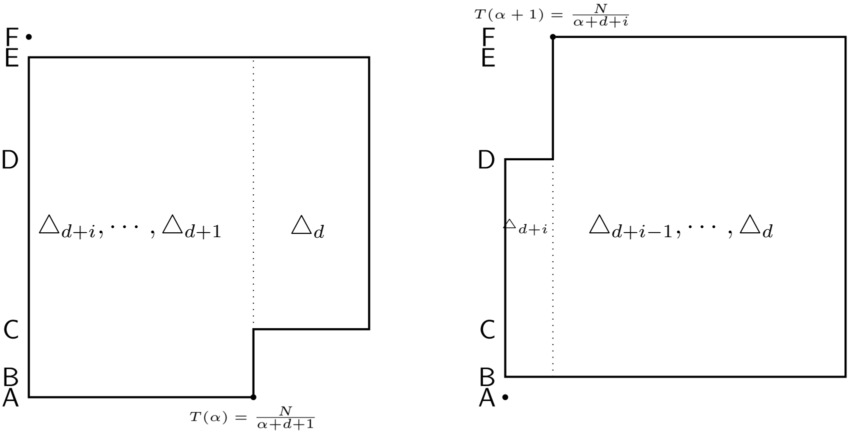


Figure: Ω_B (left) and Ω_A (right).

N-expansions with a finite digit set

By using Rohlin's formula to calculate the entropy for $\alpha = B$, we get the entropy is constant.

Theorem

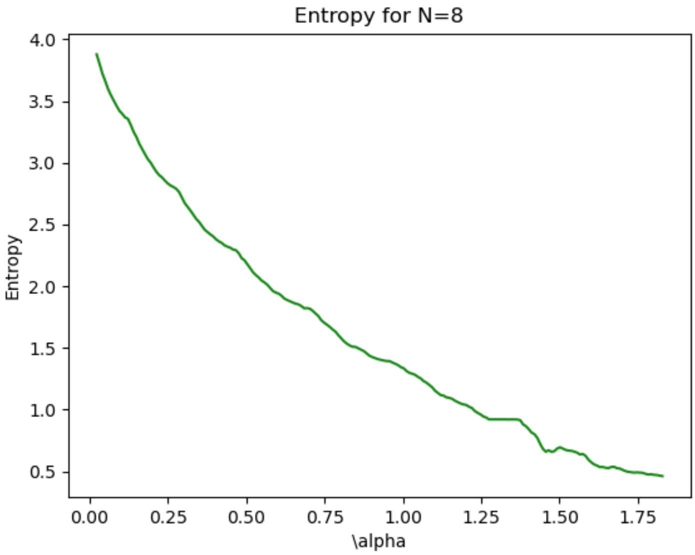
Let $N \geq 2$ be an integer, and let $d, i \in, i \geq 2$, be such, that $N = \frac{d(d+i)}{i-1}$. Then for any $\alpha \in [A, B] = \overline{X}_{N,d,i}$, one has that the entropy function $h(T_\alpha)$ is constant on $[A, B] = \overline{X}_{N,d,i}$, and is given by:

$$
\begin{aligned}
h(T_\alpha) \quad = \quad & \log N - 2H\Bigg(\left(\mathrm{Li}_2(-\tfrac{Ex}{N}) + (\log x)\log(\tfrac{Ex}{N}+1)\right)\Big|_B^{B+1} \\
& -\left(\mathrm{Li}_2(-\tfrac{Ax}{N}) + (\log x)\log(\tfrac{Ax}{N}+1)\right)\Big|_B^{D} \\
& -\left(\mathrm{Li}_2(-\tfrac{Cx}{N}) + (\log x)\log(\tfrac{Cx}{N}+1))\right)\Big|_D^{B+1}\Bigg),
\end{aligned}
$$

where
$H^{-1} = 2\log A + \log(A+1) + \log(B+1) - \log\big(N-(A+1)d\big) - \log\big(N-(d+i)B\big)$ *is the normalising constant for the T_α-invariant measure μ_α for $\alpha \in X_{N,d,i}$.*

N-expansions with a finite digit set

In case $N = 2$, our method yields only one plateau with equal entropy which follows from our method. This is the interval $[A, B] = [\frac{\sqrt{33}-5}{2}, \sqrt{2} - 1] = [0.3722813\cdots, 0.4142136\cdots]$, which was already found by Cor and Niels, where it was also determined that for $\alpha \in [A, B]$ we have that $h(T_\alpha) = 1.137779584292255\cdots$ and $H = 3.965116120651161\cdots$.

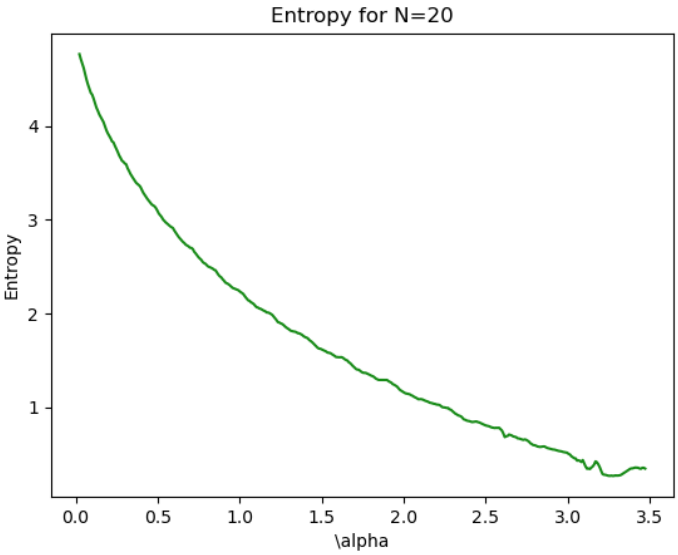

Entropy for N=8
Entropy
\alpha
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75

Example

In case $N = 8$ it follows from our method that there are five plateaux of equal entropy; see Table 1.

(d, i)	Plateau intervals	Approximation of interval	H_α	$h(T_\alpha)$
(2, 2)	$\left[\frac{\sqrt{57}-5}{2}, \frac{\sqrt{33}-3}{2}\right]$	[1.2749,1.3723]	18.377877038370	0.9212748062044
(4, 6)	$\left[\frac{3\sqrt{17}-11}{2}, \frac{\sqrt{41}-5}{2}\right]$	[0.6847,0.7016]	11.239480662654	1.8212263472923
(5, 11)	$\left[\frac{3\sqrt{97}-29}{2}, \frac{\sqrt{57}-7}{2}\right]$	[0.2733,0.2749]	9.9626774452815	2.7933207303296
(6, 22)	$\left[\frac{\sqrt{321}-17}{2}, \frac{2\sqrt{3}-3}{2}\right]$	[0.4582,0.4641]	9.2212359716540	2.2547418855378
(7, 57)	$\left[\frac{3\sqrt{473}-65}{2}, \frac{\sqrt{17}-4}{2}\right]$	[0.1228,0.1231]	8.7715446381451	3.3495778601659

Table: The pairs of integers $d \geq 1, i \geq 2$, the related plateau intervals $[A, B]$ and constant entropy $h(T_\alpha)$ for $\alpha \in [A, B]$. Here $N = 8$.


Entropy for N=20
Entropy
4
3
2
1
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
\alpha

Thank you for your attention!

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