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Best Diophantine approximations in the complex plane with Gaussian integers

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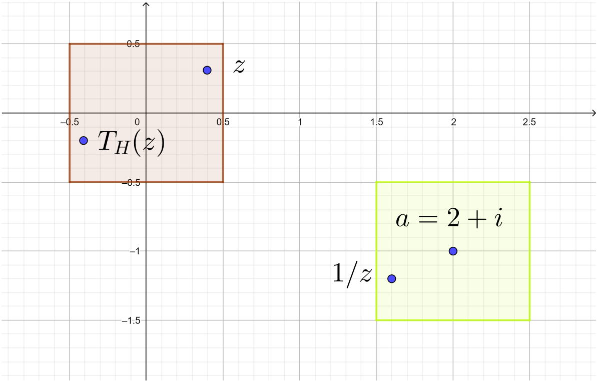
Adolf Hurwitz algorithm

$z \in \mathbb{C}$, $\lfloor z \rceil$ is the nearest Gaussian integer ($\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$)
Adolf Hurwitz map $T_H(z) = \frac{1}{z} - \lfloor \frac{1}{z} \rceil$



$a_0 = \lfloor z \rceil, z_0 = z - a_0, z_{n+1} = T_H(z_n), \; a_{n+1} = \lfloor \frac{1}{z_n} \rceil, \; z_n = \frac{1}{a_{n+1} + z_{n+1}}$

$$z = a_0 + z_0 = a_0 + \frac{1}{a_1 + z_1} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + z_2}} \dots$$

Definition

$(p, q) \in \mathbb{Z}[i]^2$, $q \neq 0$, is a best approximation vector of $z \in \mathbb{C}$ if for all $(p', q') \in \mathbb{Z}[i]^2$,

$$0 < |q'| < |q| \implies |q'z - p'| > |qz - p|$$
$$0 < |q'| \leq |q| \implies |q'z - p'| \geq |qz - p|$$

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \, a_{n-1} + \cfrac{1}{a_n}}}$$

Theorem (R. Lakein, 1973)

If z is not in countable union of lines and circles, all the vectors (p_n, q_n) are best approximation vectors of z.

- For almost all complex numbers z, the sequence of fractions found by Hurwitz algorithm is a subsequence of the sequence of all best approximations vectors.

- **Problem:** Find all the best approximation vectors to a given complex number.

Let E be a discrete subset in $\mathbb{R}^d_{\geq 0}$ that doesn't contain 0.
$\mathbb{R}^d_{\geq 0}$ is equipped with the partial order

$$(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d) \text{ iff for all } i,\ x_i \leq y_i.$$

- A point x in E is *minimal* if there is no $y \in E$ such that $y < x$.
- Voronoï used sequence of 'successive' minimal points.

- Voronoï : find the fundamental units of the ring R of integers in a totally real extension K of $\mathbb{Q}$ of degree 3.

- He used the discrete subsets $E = \{(|\sigma_1(x)|, |\sigma_2(x)|, |\sigma_3(x)|) : x \in R\} \subset \mathbb{R}^3_{\geq 0}$.

- Buchmann extended Voronoï's work to some quartic, quintic and sextic fields.

- For $\theta \in \mathbb{R}$ and $E = \{(|p - q\theta|, |q|) : p, q \in \mathbb{Z}\} \setminus \{0\}$ Voronoï's algorithm leads to the usual continued fraction algorithm.

- You can also use Voronoï's algorithm with $E = \{(|x_1|, |x_2|) : x \in \Lambda \setminus \{0\}\}$, Λ unimodular lattice in $\mathbb{R}^2$. This leads to the natural extension of the Gauss map $x \to \{\frac{1}{x}\}$.

Definition

A Gauss lattice in a finite dimensional $\mathbb{C}$-vector space V is a subset Λ such that

- Λ is submodule over the Gaussian integers,
- Λ is a discrete subset of V,
- Λ generates the vector space V.

$\Lambda = \mathbb{Z}[i]e_1 + \cdots + \mathbb{Z}[i]e_n$ where $(e_1, \ldots, e_n)$ is a basis of the $\mathbb{C}$-vector space V.

Let Λ be Gauss lattice in $\mathbb{C}^2$. Minimal vectors in Λ are defined with

$$E = E(\Lambda) = \{(|x_1|, |x_2|) : (x_1, x_2) \in \Lambda \setminus \{0\}\} \subset \mathbb{R}^2_{\geq 0}.$$

For $u = (u_1, u_2) \in \mathbb{C}^2$,

$$C(u) = \{(x_1, x_2) \in \mathbb{C}^2 : |x_1| \le |u_1| \text{ and } |x_2| \le |u_2|\}$$

Definition

Let Λ be a Gauss lattice in $\mathbb{C}^2$.

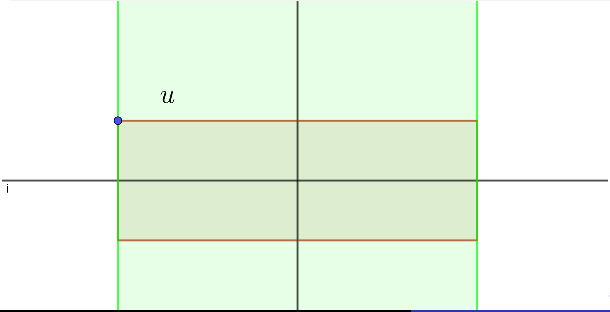
- A non zero vector $u = (u_1, u_2) \in \Lambda$ is a *minimal* vector in Λ if for every non zero $v \in \Lambda$, $v \in C(u) \Rightarrow |v_1| = |u_1|$ and $|v_2| = |u_2|$.
- Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are *consecutive* if $|u_2| < |v_2|$ and if there is no minimal vector $w = (w_1, w_2)$ in Λ with $|u_2| < |w_2| < |v_2|$.

If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two minimal vectors in a lattice then either

$$\begin{cases} |u_1| = |v_1| \\ |u_2| = |v_2| \end{cases} \quad \textit{or} \quad \begin{cases} |u_1| < |v_1| \\ |u_2| > |v_2| \end{cases} \quad \textit{or} \quad \begin{cases} |u_1| > |v_1| \\ |u_2| < |v_2| \end{cases} .$$

Lemma

If $u = (u_1, u_2)$ is a minimal vector with $|u_1| > 0$ there exists a minimal vector $v = (v_1, v_2)$ such that u and v are consecutive.

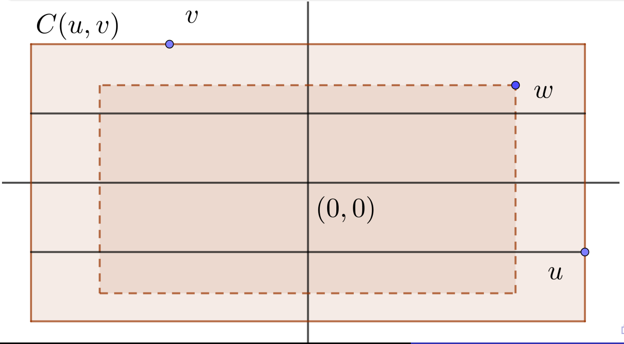


For $u = (u_1, u_2),\ v = (v_1, v_2) \in \mathbb{C}^2$,

$$C(u, v) = \{(x_1, x_2) \in \mathbb{C}^2 : |x_1| \leq |u_1| \text{ and } |x_2| \leq |v_2|\}$$

Lemma

Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are consecutive iff $|u_2| < |v_2|$ and zero is the only vector of Λ in the interior of the cylinder $C(u, v)$.



Minimal vectors in a lattice $\Lambda \subset \mathbb{C}^2$

- Sequence of "all" minimal vectors,

$$(u_n(\Lambda))_{n\in D} = (u_{n1}, u_{n2})_{n\in D}$$

  D interval $\subset \mathbb{Z}$.

  $u_n(\Lambda)$ and $u_{n+1}(\Lambda)$ are consecutive.

  For all $x = (x_1, x_2)$ minimal in Λ, there exists $n \in D$ such that $|u_{n1}| = |x_1|, |u_{n2}| = |x_2|$.

$$r_n(\Lambda) = |u_{n1}| \downarrow \text{ and } q_n(\Lambda) = |u_{n2}| \uparrow$$

- It can happen that there exists linearly independent minimal vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ such that $|x_1| = |y_1|$ and $|x_2| = |y_2|$.

Minimal vector and best approximation vector

- $(p, q) \in \mathbb{Z}[i]^2$ with $q \neq 0$ is a best approximation vector of $z \in \mathbb{C}$ if for all $(a, b) \in \mathbb{Z}[i]^2$,

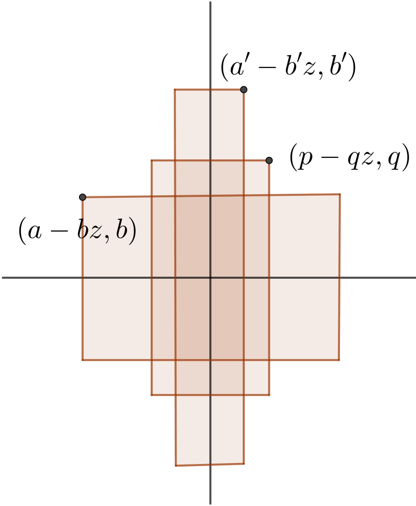
$$\begin{cases} 0 < |b| < |q| \Rightarrow |a - bz| > |p - qz| \\ |b| = |q| \Rightarrow |a - bz| \geq |p - qz|. \end{cases}$$

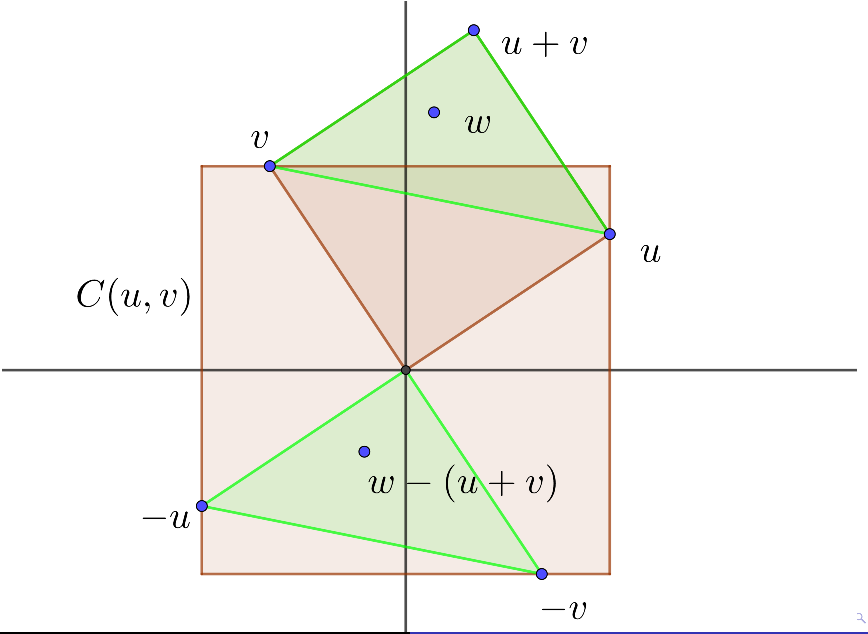
- For $z \in \mathbb{C}$, let

$$\Lambda_z = \{(p - qz, q) : p, q \in \mathbb{Z}[i]\}.$$

The vector $(p - qz, q)$ with $q \neq 0$, is minimal in Λ_z iff $(p, q) \in \mathbb{Z}[i]^2$ is a best approximation vector of z: for all $(a, b) \in \mathbb{Z}[i]^2$,

$$0 < |b| < |q| \Rightarrow |a - bz| > |p - qz|$$




$u+v$
w
v
u
$C(u,v)$
$w-(u+v)$
$-u$
$-v$

Theorem

If u and v are two consecutive minimal vectors in a Gauss lattice Λ in $\mathbb{C}^2$, then the sub lattice $\mathbb{Z}[i]u + \mathbb{Z}[i]v$ is of index 1 or 2 in Λ. Furthermore, when $\mathbb{Z}[i]u + \mathbb{Z}[i]v$ is of index two,

$$\Lambda = \langle u, v \rangle_J \stackrel{def}{=} \{gu + hv : (g, h) \in \mathbb{Z}[i]^2 \cup J^2\}$$

where $J = \frac{1}{1+i}\mathbb{Z}[i] \setminus \mathbb{Z}[i]$.

$\Lambda = \mathbb{Z}[i]e_1 + \mathbb{Z}[i]e_2$ is unimodular if $\det(e_1, e_2)$ is a unit of $\mathbb{Z}[i]$, i.e. $\det(e_1, e_2) \in \mathbb{U}_4 = \{\pm 1, \pm i\}$

Theorem (Continued fraction algorithm part 1)

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two consecutive minimal vectors in a unimodular lattice Λ with $|u_2| < |v_2|$. Let

$$w_1 = \frac{v_1}{u_1} \text{ and } w_2 = \frac{u_2}{v_2}.$$

If $w_1 \neq 0$ then there exists $v' \in \Lambda$ a minimal vectors that follows immediately v.

Lexicographic preorder on $\mathbb{C}^2$: $(x_1, x_2) \prec (y_1, y_2)$ if $|x_2| < |y_2|$ or, $|x_2| = |y_2|$ and $|x_1| \leq |y_1|$.

Theorem (Continued fraction part 2)

If $\det_{\mathbb{C}}(u, v) = 1$*, then* v' *is any vector that is minimal for the lexicographic preorder* $\prec$ *in the set*

$$\left\{ z = -au + bv : a \in \{1, 1+i\},\ b \in \mathbb{Z}[i],\ |\tfrac{a}{w_1} - b| < 1 \right\}.$$

Moreover with $u' = v = (u'_1, v'_2 w'_2)$ *and* $v' = -au + bv = (u'_1 w'_1, v'_2)$*, we have*

$$w'_1 = b - \frac{a}{w_1}, \qquad w'_2 = \frac{1}{b - aw_2}. \tag{1}$$

$u = (u_1, u_2)$ and $v = (v_1, v_2)$ two consecutive minimal vectors of Λ

$$w_1 = v_1/u_1, \; w_2 = u_2/v_2$$

$$\mathcal{C}(v_1) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |v_1| = |u_1 w_1|\}.$$

- If $w_1 \neq 0$, $\mathcal{C}(v_1) \cap \Lambda \neq \emptyset$.
- The successor v' is any minimal element in $\mathcal{C}(v_1) \cap \Lambda$ for the lexicographic preorder $\prec$.
- $v' = -au + bv, \; a, b \in \mathbb{Z}[i]$,
- $\det_{u,v}(v, v') = \det \begin{pmatrix} 0 & -a \\ 1 & b \end{pmatrix} = 1$ or $1 + i$ according to the index of v, v'. Therefore $a = 1$ or $1 + i$.
- $|-au_1 + bv_1| = |-au_1 + bu_1 w_1| < |u_1 w_1| \Leftrightarrow |-\frac{a}{w_1} + b| < 1.$

Theorem (Continued fraction part 3)

If $\det_{\mathbb{C}}(u,v) = 1+i$, *then* v' *is any vector that is minimal for the lexicographic preoder* $\prec$ *in the set*

$$\left\{ z = -\tfrac{1}{1+i}(u+v) + bv : b \in \mathbb{Z}[i],\ \left|\tfrac{1}{(1+i)w_1} + \tfrac{1}{(1+i)} - b\right| < 1 \right\}.$$

Moreover with $u' = v = (u_1', v_2' w_2')$ *and* $v' = -au + bv = (u_1' w_1', v_2')$, *we have*

$$w_1' = b - \tfrac{1}{(1+i)w_1} - \tfrac{1}{(1+i)}, \qquad w_2' = \frac{1}{b - \frac{1}{(1+i)} w_2 - \frac{1}{(1+i)}}.$$

Döblin, Lenstra, Bosma, Jager, Wiedijk

Theorem

For almost all real numbers θ,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{0 \le n < N : |(p_n - q_n\theta)q_n| \le t\} = \int_0^t \phi(s)ds$$

for all $t \in [0, \infty[$, where

$$\phi(s) = \tfrac{1}{\ln 2} \times \begin{cases} 1, & s \in [0, \frac{1}{2}] \\ \frac{1-s}{s}, & s \in [\frac{1}{2}, 1] \\ 0, & s \ge 1 \end{cases}$$

This result is sometimes referred to as the Lenstra conjecture. Döblin stated the above theorem in 1940, *Compositio Mathematica*, 7. Doeblin only sketched the proof of this result and it is difficult to reconstitute a proof from his paper.
A complete proof was given by Bosma, Jager and Wiedijk, *Nederl. Akad. Wetensch. Indag. Math.* 45 (1983), no. 3, 281–299.

Döblin, Lenstra, Bosma, Jager, Wiedijk for lattices in $\mathbb{C}^2$

- Let $z \in \mathbb{C}$ and let $(p_n, q_n)_n$ be its sequence of best approximations vectors. We want to study the limit distribution of the sequence $((p_n - q_n z)q_n)_n$.

- The sequence of minimal vectors of $\Lambda_z = \{(p - qz, q) : (p, q) \in \mathbb{Z}[i]^2\}$ is $(p_n - q_n z, q_n)_n$.

- Let Λ be lattice in $\mathbb{C}^2$ and let $(u_n(\Lambda) = (u_{n1}(\Lambda), u_{n2}(\Lambda)))_n$ be its sequence of minimal vectors. What is the limit distribution of the sequence $(u_{n1}(\Lambda)u_{n2}(\Lambda))_n$?

Theorem

There exists a density $\Phi : \mathbb{C} \to \mathbb{R}_+$ such that for almost all unimodular lattices Λ in $\mathbb{C}^2$ and all Borel sets $B \subset \mathbb{C}$ with negiligible boundaries,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \in \{1, \dots, N\} : u_{n1}(\Lambda)u_{n2}(\Lambda) \in B\} = \int_B \Phi(z)dz.$$

Φ *is Lipschitz*
$\Phi = Cste > 0$ *on* $D(0, \frac{1}{2})$
$\Phi = 0$ *on* $\mathbb{C} \setminus \mathbb{D}$.

- Ω_1 = the set of all Gauss lattices in $\mathbb{C}^2$ whose bases have determinants in $\mathbb{U}_4 = \{\pm 1, \pm i\}$.
- The space Ω_1 can be identified with $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i])$ using the map

$$M\,\mathrm{SL}(2,\mathbb{Z}[i]) \in \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{Z}[i]) \to M\mathbb{Z}[i]^2 \in \Omega_1 .$$

- μ the Haar measure on Ω_1.
- The flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

Idea of the proof 1

- If $u = (u_1, u_2)$ is a minimal vector of a lattice $\Lambda \in \Omega_1$ then $g_t u$ is a minimal vector of $g_t\Lambda$ and one has

$$e^t u_1 \times e^{-t} u_2 = u_1 u_2.$$

- Therefore the limit distribution of the sequence $(u_{n1}(\Lambda)u_{n2}(\Lambda))$ depends only on the flow trajectory.
- Therefore it is enough to prove the theorem for the lattices in a transversal (a cross section) T of the flow that cuts almost all flow trajectories.
- Main idea of the proof: use Birkhoff's theorem with the first return map R of the flow on a transversal T and a function $f : T \to \mathbb{C}$ such that

$$f \circ R^n(\Lambda) = u_{n1}(\Lambda)u_{n2}(\Lambda)$$

for all $\Lambda \in T$.

Idea of the proof 2

If $B \subset \mathbb{C}$ is a Borel set then by Birkhoff's theorem

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \leq N : u_{n1}(\Lambda)u_{n2}(\Lambda) \in B\} &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_B \circ f \circ R^n(\Lambda) \\ &= \int_T 1_B \circ f \, d\nu\end{aligned}$$

where ν is the measure induced by the flow on the transversal T.
So the function Φ is the density of the image of ν by the function f.

Transversal

Let $\mathbb{U}_4 = \{\pm 1, \pm i\}$ be the group of units in $\mathbb{Z}[i]$. Let T be the set of Gauss unimodular lattices Λ in $\mathbb{C}^2$ such that there exists two vectors $u = u_T(\Lambda) = (u_1, u_2)$ and $v = v_T(\Lambda) = (v_1, v_2)$ in Λ

1. $r = |u_1| = |v_2| > |u_2|, |v_1|$,
2. the only nonzero vectors of Λ in the ball $B_\infty(0, r)$ are in $\mathbb{U}_4 u \cup \mathbb{U}_4 v$.

The vectors u and v are two consecutive minimal and unique up to multiplicative factors in $\mathbb{U}_4$.

The lattice $L = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ has index 1 or 2 in Λ. Therefore the transversal T is the union of two disjoint pieces T_1 and T_2 according to the index of L.
Roughly,

$$T = \{\Lambda \in \Omega_1 : \lambda_1(\Lambda, |.|_\infty) = \lambda_2(\Lambda, |.|_\infty)\}$$

$$\dim_{\mathbb{R}} T = 5$$

The function f

Let Λ be in T and let $u_T(\Lambda) = (u_1, u_2)$ and $v_T(\Lambda) = (v_1, v_2)$ be the two vectors in Λ associated with T by the definition.
The function $f : T \to \mathbb{C}$ is defined by

$$f(\Lambda) = u_1 u_2.$$

Since $u_T(\Lambda)$ is define up to a factor in $\mathbb{U}_4$, f is defined only modulo ± 1. But we can always suppose that $\arg u_1 \in [0, \frac{\pi}{2}[$.

Visiting times in the transversal

Let Λ be a unimodular lattice in $\mathbb{C}^2$. Let $u_n(\Lambda) = (u_{n1}, u_{n2})$, $n \in \mathbb{Z}$, be the sequence of all minimal vectors in Λ.

$$r_n = |u_{n1}| \searrow, \; q_n = |u_{n2}| \nearrow .$$

- For every n, there is a time t_n such that $e^{t_n} r_n = e^{-t_n} q_{n+1}$, therefore $g_{t_n} C(u_n(\Lambda), u_{n+1}(\Lambda)) = B_\infty(0, e^{t_n} r_n)$ and $g_{t_n}\Lambda \in T$.
- Conversely if $\Lambda' = g_t \Lambda \in T$ then $g_{-t} u_T(\Lambda')$ and $g_{-t} v_T(\Lambda')$ are two consecutive minimal vectors of Λ.
  Therefore $t = t_n$ for some n.
- If $\Lambda \in T$, then there exists $n_0 = n_0(\Lambda) \in \mathbb{Z}$ such that $u_T(\Lambda) = u_{n_0}(\Lambda)$ and $v_T(\Lambda) = u_{n_0+1}(\Lambda)$.
- $R(\Lambda) = g_{t_{n_0+1}} \Lambda$ and $R^n(\Lambda) = g_{t_{n_0+n}} \Lambda$

$$f \circ R^n(\Lambda) = f(g_{t_{n+n_0}} \Lambda) = u_{(n+n_0)1} u_{(n+n_0)2}$$

Parametrizations of T_1 and T_2

$\Psi_k : \mathbb{R} \times \mathbb{D}^2 \to \Omega_1,\ k = 1, 2$ be the maps defined by

$$\Psi_1(\theta, w_1, w_2) = \mathbb{Z}[i]u + \mathbb{Z}[i]v$$
$$\Psi_2(\theta, w_1, w_2) = \mathbb{Z}[i]u + \tfrac{1}{1+i}\mathbb{Z}[i](u+v)$$

where

$$u = u(\theta, w_1, w_2) = (u_1, v_2 w_2),$$
$$v = v(\theta, w_1, w_2) = (u_1 w_1, v_2),$$
$$u_1 = r \exp i\theta,\ v_2 = r \exp i\theta'$$
$$r = \frac{k^{1/4}}{\sqrt{|1 - w_1 w_2|}},\ \theta' = (k-1)\tfrac{\pi}{4} - \theta - \arg(1 - w_1 w_2).$$

Then for all Λ in T_k there exists exactly one element $(\theta, w_1, w_2) \in [0, \frac{\pi}{2}[\times \mathbb{D}^2$ such that $\Lambda = \Psi_k(\alpha, w_1, w_2)$.

The function $f : \Lambda \in T \to u_1 u_2$

Let $\Lambda = \Psi_1(\theta, w_1, w_2) = \mathbb{Z}[i]u + \mathbb{Z}[i]v \in T_1$.

$$
\begin{aligned}
u &= u(\theta, w_1, w_2) = (u_1, v_2 w_2), \\
v &= v(\theta, w_1, w_2) = (u_1 w_1, v_2).
\end{aligned}
$$

$$1 = \det(u, v) = u_1 v_2 (1 - w_1 w_2)$$

$$f(\Lambda) = u_1 v_2 w_2 = \frac{w_2}{1 - w_1 w_2}.$$

If $\Lambda \in T_2$, $\quad f(\Lambda) = \frac{(1+i)w_2}{1 - w_1 w_2}$.

Conditions to be in T

- Let $u = (u_1, u_2)$ and $v = (v_1, v_2) \in \mathbb{C}^2$ and let $\Lambda = \mathbb{Z}u + \mathbb{Z}v$. Suppose that $|u_1| > |v_1|$ and $|v_2| > |u_2|$. Let

$$C(u, v) = \{(z_1, z_2) : |z_1| \le |u_1|, \ |z_2| \le |v_2|\}$$

For $g, h \in \mathbb{C}^*$,

$$\begin{aligned} gu - hv \notin C(u, v) &\iff |gu_1 - hv_1| > |u_1| \text{ or } |gu_2 - hv_2| > |v_2| \\ &\iff \left|\tfrac{g}{h} - \tfrac{v_1}{u_1}\right| > \tfrac{1}{|h|} \text{ or } \left|\tfrac{h}{g} - \tfrac{u_2}{v_2}\right| > \tfrac{1}{|g|} \\ &\iff \mathrm{d}(w_1, \tfrac{g}{h}) > \tfrac{1}{|h|} \text{ or } \mathrm{d}(w_2, \tfrac{h}{g}) > \tfrac{1}{|g|} \end{aligned}$$

with $w_1 = \frac{v_1}{u_1}$ and $w_2 = \frac{u_2}{v_2}$.

- Suppose that $\det(u, v) \in \mathbb{U}_4$ and that $|u_1| = |v_2|$. $\Lambda \in T_1$ iff for all nonzero $g, h \in \mathbb{Z}[i]$,

$$\mathrm{d}(w_1, \tfrac{g}{h}) > \tfrac{1}{|h|} \text{ or } \mathrm{d}(w_2, \tfrac{h}{g}) > \tfrac{1}{|g|}$$

Theorem

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vectors in $\mathbb{C}^2$ such that $|u_1| > |v_1|$ and $|v_2| > |u_2|$.

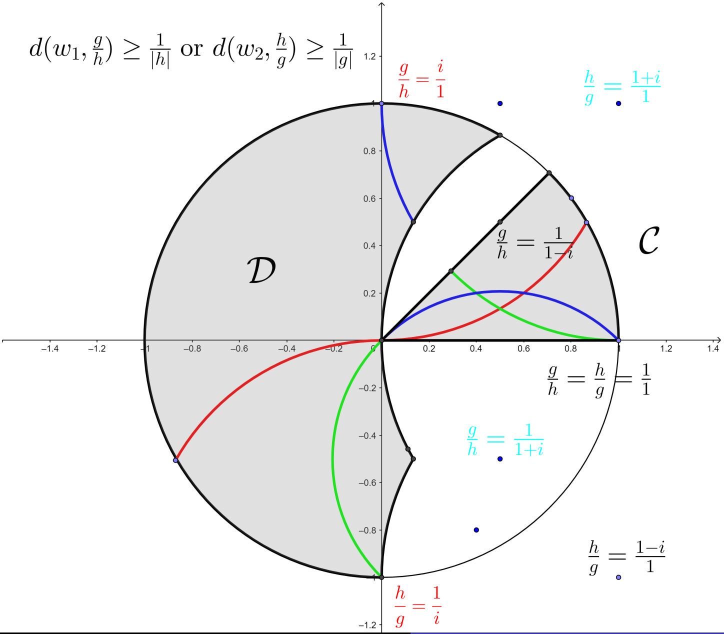
1. *The only elements of $\mathbb{Z}[i]u + \mathbb{Z}[i]v$ in the cylinder*

$$C(u, v) = \{(z_1, z_2) : |z_1| \leq |u_1|, \ |z_2| \leq |v_2|\}$$

   are in $\{0\} \cup \mathbb{U}_4 u \cup \mathbb{U}_4 v$ iff $gu - hv \notin C(u, v)$ for all nonzero $g, \ h \in \mathbb{Z}[i]^2$ with $|g| \times |h| \leq \sqrt{2}$.
2. *The only elements of $\langle u, v \rangle_J$ in the cylinder $C(u, v)$ are in $\{0\} \cup \mathbb{U}_4 u \cup \mathbb{U}_4 v$ iff $gu - hv \notin C(u, v)$ for all $(g, h) \in (\frac{1}{1+i}\mathbb{Z}[i])^2$ with $|g| = |h| = \frac{1}{\sqrt{2}}$.*

Conditions when $w_1 \in \mathcal{C} = \{z \in \mathbb{D} : \arg \in [0, \frac{\pi}{2}]\}$

$d(w_1, \frac{g}{h}) \geq \frac{1}{|h|}$ or $d(w_2, \frac{h}{g}) \geq \frac{1}{|g|}$



Invariant measure

Theorem

The Haar measure and the flow g_t induce on the transversal T a measure ν with density

$$\varphi(\theta, w_1, w_2) = \frac{32}{|1 - w_1 w_2|^4}$$

where (θ, w_1, w_2) are the coordinates associated with the parametrizations Ψ_k, $k = 1, 2$.

The constant 32 depends on the normalization of the Haar measure.

Summary for T_1

$\theta \in [0, \frac{\pi}{2}[$,
$w_1, w_2 \in \mathbb{D}$ satisfy

$$\max(\mathrm{d}(w_1, \frac{a}{b}) > \frac{1}{|b|}, \mathrm{d}(w_2, \frac{b}{a}) > \frac{1}{|a|})$$

for all nonzero $a, b \in \mathbb{Z}[i]$ such that $|ab| \leq \sqrt{2}$.

$$f(\theta, w_1, w_2) = \frac{w_1}{1 - w_1 w_2}.$$

Find the image by f of the measure ν with density

$$\varphi(\theta, w_1, w_2) = \frac{1}{|1 - w_1 w_2|^4}.$$

$$F_1 = \{(g,h) \in \mathbb{Z}[i]^2 : 0 < |g||h| \le \sqrt{2}\},$$

$$\mathcal{W}_1 = \{(w_1, w_2) \in \mathbb{D}^2 : \forall (g,h) \in F_1,\ \mathrm{d}(w_1, \tfrac{g}{h}) > \frac{1}{|h|} \text{ or } \mathrm{d}(w_2, \tfrac{h}{g}) > \frac{1}{|g|}\}.$$

For any Borel set $\omega \subset \mathbb{C}$,

$$f_*\nu_1(\omega) = \int_{\mathbb{D}^2} 1_{\mathcal{W}_1}(w_1, w_2) 1_\omega(\tfrac{w_2}{1 - w_1 w_2}) \frac{1}{|1 - w_1 w_2|^4} dw_1 dw_2$$

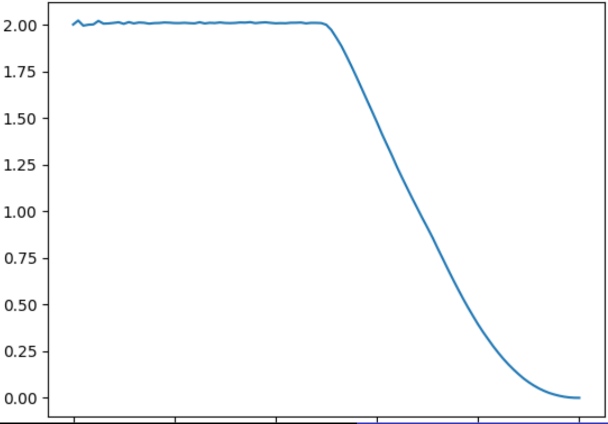
With the change of variable $(w_1, w_2) = \psi_i(z, w) = (w - \frac{1}{z}, \frac{1}{w})$,

$$\begin{aligned} f_*\nu_1(\omega) &= \int_{\mathbb{C}^* \times \mathbb{C}^*} 1_{\mathcal{W}_1}(w - \tfrac{1}{z}, \tfrac{1}{w}) 1_\omega(z) \frac{1}{|1 - (w - \frac{1}{z})\frac{1}{w}|^4} |\operatorname{Jac} \psi_1(z,w)| dz dw \\ &= \int_{\mathbb{C}^* \times \mathbb{C}^*} 1_{\mathcal{W}_1}(w - \tfrac{1}{z}, \tfrac{1}{w}) 1_\omega(z) dz dw \\ &= \int_\omega (\int_{\mathbb{C}^*} 1_{\mathcal{W}_1}(w - \tfrac{1}{z}, \tfrac{1}{w}) dw) dz \\ \implies \Phi_1(z) &= \int_{\mathbb{C}^*} 1_{\mathcal{W}_1}(w - \tfrac{1}{z}, \tfrac{1}{w}) dw \text{ is the density of } f_*\nu_1 \end{aligned}$$

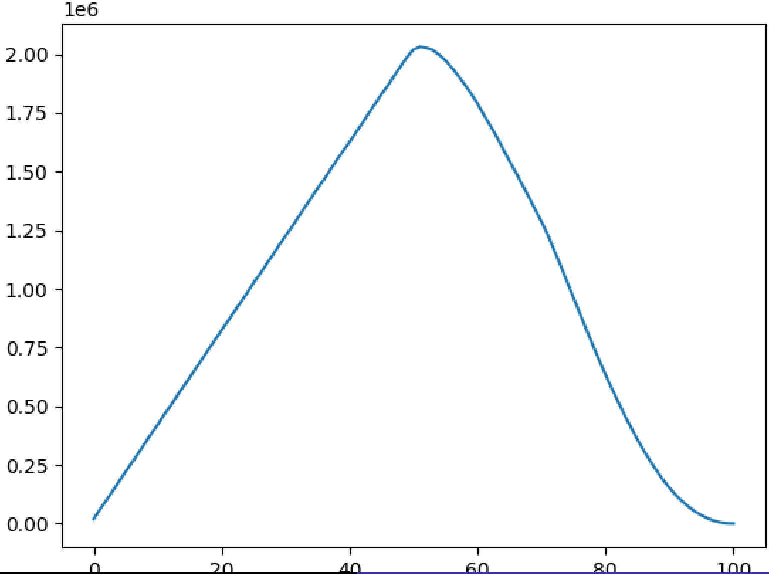
Experimental result, iterating the first return map

Histogram 100 bins, annulli $D(0, \frac{k+1}{100}) \setminus D(0, \frac{k}{100},)$ $k = 0, \ldots, 99$,

$$N \text{ iterates,} \quad \frac{100^2}{2k+1} \frac{\operatorname{card}\{0 \le n < N : \frac{k}{100} \le |u_{n1} u_{n2}| < \frac{k+1}{100}\}}{N}$$



Density in $[0, 1]$ of the limit distribution of $|u_{n1}u_{n2}|$



Nakada's reasonning,

Assume that

- For almost all $z \in \mathbb{C}$, $\lim_{n\to\infty} \frac{1}{n} \ln |q_n| = K$
- Legendre constant: There exists $L > 0$ such that, if $u = (u_1, u_2)$ is a primitive vector in a unimodular lattice Λ and if $|u_1 u_2| < L$ then u is a minimal vector.
- For almost all $z \in \mathbb{C}$, for any $k > 0$,

$$\lim_{N\to\infty} \frac{\mathrm{card}\{\frac{p}{q} \in \mathbb{Q}[i] : \gcd(p,q) = 1, |q| \leq N, |qz - p| < \frac{k}{|q|}\}}{\ln N} = Mk^2$$

Then for almost all $z \in \mathbb{C}$, for all $0 \leq t < L$

$$\lim_{N\to\infty} \mathrm{card}\{n \leq N : |q_n(q_n z - p_n)| \leq t\} = MKt^2$$

therefore, $\phi(t) = MKt$ is the density on the interval $[0, L]$.

If $t < L$, any coprime (p, q) such that $|qz - p| < \frac{t}{|q|}$ is a best approximation of vector of z. Therefore,

$$\begin{aligned}
&\frac{1}{N}\operatorname{card}\{n \in \mathbb{N} : n \leq N,\ |q_n||q_n z - p_n| < t\} \\
&= \frac{\ln|q_N|}{N}\frac{\operatorname{card}\{\frac{p}{q} \in \mathbb{Q}[i] : \gcd(p, q) = 1, |q| \leq |q_N|, |qz - p| < \frac{k}{|q|}\}}{\ln|q_N|} \\
&\to KMt^2
\end{aligned}$$

for almost all z.

Theorem (Legendre's theorem)

Let $\Lambda \subset \mathbb{C}^2$ be a unimodular Gauss lattice. Let $u = (u_1, u_2) \in \Lambda$ be primitive. If $|u_1 u_2| < \frac{1}{2}$ then u is minimal.

If u were not minimal there would exists a nonzero $v = (v_1, v_2) \in \Lambda$ with say $|v_1| < |u_1|$ and $|v_2| \leq |u_2|$.
If $v = \lambda u$ for some $\lambda \in \mathbb{C}$ then $|\lambda| = \frac{|v_1|}{|u_1|} < 1$. We can assume that $|\lambda| > 0$ is minimal.
Now $\frac{1}{\lambda} = g + \alpha$ where $g \in \mathbb{Z}[i]$ and $|\alpha| < 1$. Now $\alpha \neq 0$ because u is primitive, hence $w = u - gv = \alpha v = \alpha\lambda u \in \Lambda$.
Therefore u and v are linearly independent.
By definition of the determinant we have

$$1 = |\det \Lambda| \leq |\det(u, v)| = |u_1 v_2 - u_2 v_1| \leq 2|u_1 u_2| < 2\frac{1}{2} = 1,$$

a contradiction.

Dirichlet theorem in the space of lattices

Theorem

We have

$$\sup |u_1||v_2| = \frac{\sqrt{2}}{3-\sqrt{3}}.$$

where the supremum is taken over all pairs $u = (u_1, u_2)$ *and* $v = (v_1, v_2)$ *of consecutive minimal vectors in all unimodular lattices* $\Lambda \subset \mathbb{C}^2$.

Theorem (Complex Dirichlet constant)

For every complex number z and for every real number $Q > 1$, there exist Gaussian integers p and q such that

$$\begin{cases} 0 < |q| < Q, \\ |qz - p| \le \frac{\sqrt{2}}{3-\sqrt{3}} \times \frac{1}{Q}, \end{cases}$$

where $\frac{\sqrt{2}}{3-\sqrt{3}} = \frac{1}{\sqrt{6-3\sqrt{3}}} = 1.115355\ldots$. Furthermore the set of complex numbers z for which the constant $\frac{\sqrt{2}}{3-\sqrt{3}}$ can be improved, is of zero Lebesgue measure.

Thank you for your attention

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