Ergodic behavior of transformations associated with alternate base expansions

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May 26, 2020

One World Numeration Seminar

Introduction

Let $\beta > 1$, a β-**representation** of a real $x \in [0,1]$ is an infinite word $a \in (\mathbb{N})^{\mathbb{N}}$ such that

$$x = \frac{a_0}{\beta} + \frac{a_1}{\beta^2} + \frac{a_2}{\beta^3} + \cdots$$
$$= \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}}$$

The **greedy** β-representation of $x \in [0,1]$ is called β-**expansion** of x and denoted $d_\beta(x) = a_0 a_1 \cdots \in [\![0, \lfloor \beta \rfloor]\!]^{\mathbb{N}}$

- $a_0 = \lfloor x\beta \rfloor$ and $r_0 = x\beta - a_0$
- $a_n = \lfloor r_{n-1}\beta \rfloor$ and $r_n = r_{n-1}\beta - a_n, \quad \forall n \geq 1$

On $[0,1)$, the β-**expansion** is generated by iterating the transformation

$$T_\beta\colon\ [0,1) \to [0,1)\,,\ x \mapsto \beta x \bmod 1.$$

We have $a_n = \left\lfloor \beta T_\beta^n(x) \right\rfloor \in [\![0, \lceil \beta \rceil - 1]\!]$.

Consider a probability space $(X, \mathcal{F}, \mu)$ and a map $T\colon X \to X$.

- μ is T-invariant : for all $B \in \mathcal{F}$, $\mu(T^{-1}B) = \mu(B)$.
- μ is equivalent to ν : for all $B \in \mathcal{F}$, $\mu(B) = 0 \Leftrightarrow \nu(B) = 0$.
- T is ergodic : for all $B \in \mathcal{F}$, $T^{-1}B = B \Rightarrow \mu(B) = 0$ or 1.
- T is non-singular : for all $B \in \mathcal{F}$, $\mu(B) = 0 \Leftrightarrow \mu(T^{-1}B) = 0$.

Theorem (Rényi, 1957)

There exists a unique T_β-invariant probability measure μ_β equivalent to Lebesgue. Moreover, the transformation T_β is ergodic.

Cantor real bases

A **Cantor real base** is a sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ over $\mathbb{R}_{>1}$ such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. A **$\beta$-representation** of $x \in [0,1]$ is a sequence $(a_i)_{i\in\mathbb{N}} \in (\mathbb{N})^{\mathbb{N}}$ such that

$$\begin{aligned} x &= \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots \\ &= \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \end{aligned}$$

The **greedy** β-representation of $x \in [0,1]$, denoted $d_\beta(x) = a_0 a_1 \cdots \in \mathbb{N}^{\mathbb{N}}$, is called the **$\beta$-expansion** of x.
We have

- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_n = \lfloor r_{n-1}\beta_n \rfloor$ and $r_n = r_{n-1}\beta_n - a_n, \quad \forall n \geq 1$

For all $n \in \mathbb{N}$, $a_n \in [\![0, \lfloor \beta_n \rfloor]\!]$.

Example

Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$. Consider $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ defined by

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \dots)$ and $d_{\boldsymbol{\beta}}(1) =$

Example

$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
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$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) = 2$

- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$

Example

$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
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$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) = 20$

- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$
- $a_1 = \lfloor (\alpha - 2)\beta \rfloor = 0, \quad r_1 = (\alpha - 2)\beta \simeq 0.43$

Example

$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
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$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) = 200$

- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$
- $a_1 = \lfloor (\alpha - 2)\,\beta \rfloor = 0, \quad r_1 = (\alpha - 2)\,\beta \simeq 0.43$
- $a_2 = \lfloor ((\alpha - 2)\,\beta)\beta \rfloor = 0, \quad r_2 = ((\alpha - 2)\,\beta)\beta \simeq 0.62$

Example

$d_{\boldsymbol\beta}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_n = \lfloor r_{n-1}\beta_n \rfloor$ and $r_n = r_{n-1}\beta_n - a_n$

Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$. Consider $\boldsymbol\beta = (\beta_n)_{n\in\mathbb{N}}$ defined by

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol\beta = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol\beta}(1) = 2001$

- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$
- $a_1 = \lfloor (\alpha-2)\,\beta \rfloor = 0, \quad r_1 = (\alpha-2)\,\beta \simeq 0.43$
- $a_2 = \lfloor ((\alpha-2)\,\beta)\beta \rfloor = 0, \quad r_2 = ((\alpha-2)\,\beta)\beta \simeq 0.62$
- $a_3 = \left\lfloor ((\alpha-2)\,\beta^2)\alpha \right\rfloor = 1, \quad r_3 = ((\alpha-2)\,\beta^2)\alpha - 1 \simeq 0.43$

Example

$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
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$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) = 20010110^\omega$.

- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$
- $a_1 = \lfloor (\alpha - 2)\,\beta \rfloor = 0, \quad r_1 = (\alpha - 2)\,\beta \simeq 0.43$
- $a_2 = \lfloor ((\alpha - 2)\,\beta)\beta \rfloor = 0, \quad r_2 = ((\alpha - 2)\,\beta)\beta \simeq 0.62$
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$$\vdots$$

Generalization of the results about the β-representations to the $\boldsymbol{\beta}$-representations:

- The $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1]$ is the greatest of all the $\boldsymbol{\beta}$-representations of x with respect to the lexicographic order.
- The function $d_{\boldsymbol{\beta}}$ is increasing.
- Definition of a quasi-greedy representation $d^*_{\boldsymbol{\beta}}(1)$.
- Generalization of Parry's theorem and its corollary.
- Characterization of the $\boldsymbol{\beta}$-shift.

$$\vdots$$

Alternate bases

An **alternate base** is a periodic Cantor base:

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$$

where $p \geq 1$ is called the *length* of $\boldsymbol{\beta}$.
We denote

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$$

and we have

$$\beta_n = \beta_{n \bmod p}, \quad \forall n \in \mathbb{Z}\,.$$

Example

Let $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_{\boldsymbol{\beta}}(1) = 2010^{\omega}.$$

Example

$d_\beta(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_n = \lfloor r_{n-1}\beta_n \rfloor$ and $r_n = r_{n-1}\beta_n - a_n$

Let $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(1) = 2010^\omega.$$

In fact,

- $a_0 = \left\lfloor 1 \left(\frac{1+\sqrt{13}}{2} \right) \right\rfloor = 2, \quad r_0 = \frac{-3+\sqrt{13}}{2} \simeq 0.30$

Example

$d_\beta(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_n = \lfloor r_{n-1}\beta_n \rfloor$ and $r_n = r_{n-1}\beta_n - a_n$

Let $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(1) = 2010^\omega.$$

In fact,

- $a_0 = \left\lfloor 1\left(\frac{1+\sqrt{13}}{2}\right)\right\rfloor = 2, \quad r_0 = \frac{-3+\sqrt{13}}{2} \simeq 0.30$
- $a_1 = \left\lfloor \left(\frac{-3+\sqrt{13}}{2}\right)\left(\frac{5+\sqrt{13}}{6}\right)\right\rfloor = 0, \quad r_1 = \frac{-1+\sqrt{13}}{6} \simeq 0.43$

Example

$d_\beta(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
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In fact,

- $a_0 = \left\lfloor 1\left(\frac{1+\sqrt{13}}{2}\right)\right\rfloor = 2, \quad r_0 = \frac{-3+\sqrt{13}}{2} \simeq 0.30$
- $a_1 = \left\lfloor \left(\frac{-3+\sqrt{13}}{2}\right)\left(\frac{5+\sqrt{13}}{6}\right)\right\rfloor = 0, \quad r_1 = \frac{-1+\sqrt{13}}{6} \simeq 0.43$
- $a_2 = \left\lfloor \left(\frac{-1+\sqrt{13}}{6}\right)\left(\frac{1+\sqrt{13}}{2}\right)\right\rfloor = 1, \quad r_2 = 0$

Consider $\boldsymbol{\beta}$ an alternate base:

- The $\boldsymbol{\beta}$-expansion of 1 can't be purely periodic.
- Better generalization of Parry's corollary.
- Better characterization of the $\boldsymbol{\beta}$-shift.
- Generalization of *Bertrand-Mathis'* theorem : The $\boldsymbol{\beta}$-shift is sofic if and only if all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic, where $\boldsymbol{\beta}^{(i)} = (\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{i-1})$.

$\vdots$

Transformation

Let $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}} \in (\mathbb{R}_{\geq 1})^{\mathbb{N}}$. For all $n \in \mathbb{N}$, consider

$$T_{\beta_n} \colon\ [0,1) \to [0,1)\,,\ x \mapsto \beta_n x \bmod 1.$$

For all $x \in [0,1)$, we have

$$a_n = \big\lfloor \beta_n\big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)\big)\big\rfloor \in [\![0, \lceil \beta_n \rceil - 1]\!].$$

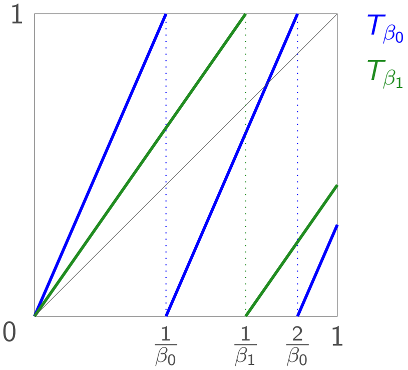
If $\boldsymbol{\beta}$ is an alternate base $(\beta_0, \ldots, \beta_{p-1})$, we have p transformations

$$T_{\beta_0}, \ldots, T_{\beta_{p-1}}.$$

Example

Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

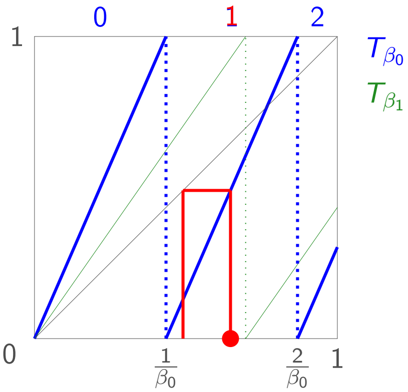
$$d_\beta(\tfrac{1+\sqrt{5}}{5}) =$$



Example

Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

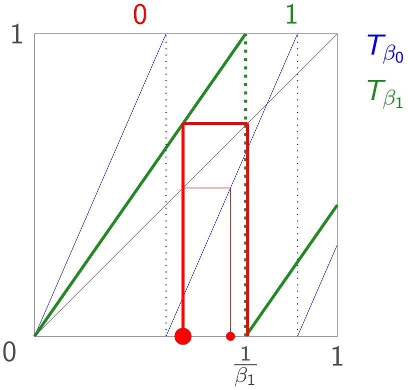
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 1$$



Example

Consider $\boldsymbol{\beta} = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

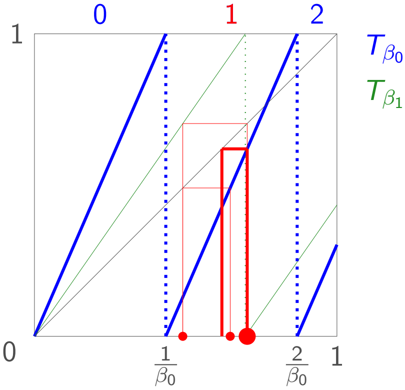
$$d_{\boldsymbol{\beta}}(\tfrac{1+\sqrt{5}}{5}) = 10$$



Example

Consider $\boldsymbol{\beta} = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

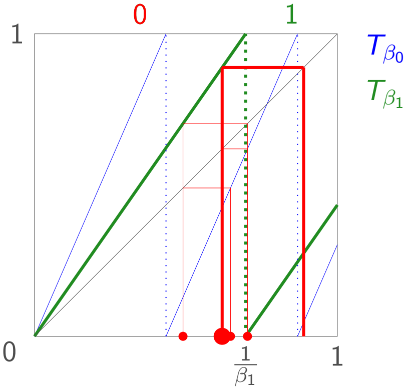
$$d_{\boldsymbol{\beta}}(\tfrac{1+\sqrt{5}}{5}) = 101$$



Example

Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

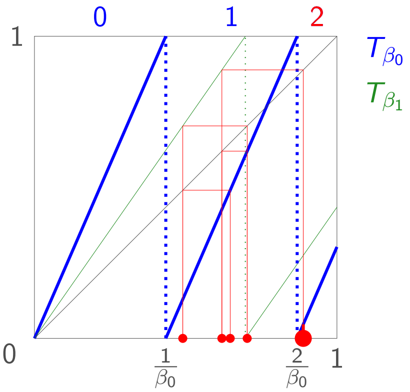
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 1010$$



Example

Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(\tfrac{1+\sqrt{5}}{5}) = 10102\cdots$$



β-expansions and $(\beta_0\beta_1\cdots\beta_{p-1})$-expansions

Let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}) \in (\mathbb{R}_{\geq 1})^p$. By definition of the $d_{\boldsymbol{\beta}}(x)$, we have

$$\begin{aligned}
x = {} & \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \cdots + \frac{a_{p-1}}{\beta_0\cdots\beta_{p-1}} \\
+ {} & \frac{a_p}{\beta_0(\beta_0\cdots\beta_{p-1})} + \frac{a_{p+1}}{\beta_0\beta_1(\beta_0\cdots\beta_{p-1})} + \cdots + \frac{a_{2p-1}}{(\beta_0\cdots\beta_{p-1})^2} \\
+ {} & \cdots .
\end{aligned}$$

That can be rewritten as follows

$$\begin{aligned}
x = {} & \frac{a_0\beta_1\cdots\beta_{p-1}+a_1\beta_2\cdots\beta_{p-1}+\cdots+a_{p-1}}{\beta_0\beta_1\cdots\beta_{p-1}} \\
& + \frac{a_p\beta_1\cdots\beta_{p-1}+a_{p+1}\beta_2\cdots\beta_{p-1}+\cdots+a_{2p-1}}{(\beta_0\beta_1\cdots\beta_{p-1})^2} \\
& + \cdots .
\end{aligned}$$

Lemma

For all $x \in [0,1)$

$$T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}(x) = x\beta_0 \cdots \beta_{p-1} - \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1}$$

where $(c_0, \ldots, c_{p-1})$ is the lexicographically greatest p-tuple in $[\![0, \lceil\beta_0\rceil - 1]\!] \times \cdots \times [\![0, \lceil\beta_{p-1}\rceil - 1]\!]$ such that

$$\frac{\sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1}}{\beta_0 \cdots \beta_{p-1}} \leq x.$$

Question

Can we compare the maps $T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}$ and $T_{\beta_0 \cdots \beta_{p-1}, D_{\boldsymbol{\beta}}}$ defined over the digit set

$$\begin{aligned} D_{\boldsymbol{\beta}} = & \{a_0\beta_1 \cdots \beta_{p-1} + a_1\beta_2 \cdots \beta_{p-1} + \cdots + a_{p-2}\beta_{p-1} + a_{p-1} \colon \\ & \qquad a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!]\} \\ = & \left\{ \sum_{j=0}^{p-1} a_j \beta_{j+1} \cdots \beta_{p-1} \colon a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!] \right\} \end{aligned}$$

on $[0, 1)$?

Let

$$D = \{0 = d_0 < d_1 < \cdots < d_m\} \subset \mathbb{R}$$

be a digit set. The word $a = a_0 a_1 \cdots$ over D is a (β, D)-representation of $x \in \left[0, \frac{d_m}{\beta - 1}\right)$ if

$$x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}}.$$

The **greedy** β-representation is built thanks to the greedy algorithm:

- $a_0 = \max\{d_i : d_i \leq x\beta\}$ and $r_0 = x\beta - a_0$
- $a_n = \max\{d_i : d_i \leq r_{n-1}\beta\}$ and $r_n = r_{n-1}\beta - a_n, \quad \forall n \geq 1$

The transformation that generates the expansions is

$$T_{\beta,D}(x) = \begin{cases} \beta x - d_i & \text{if } x \in \left[\frac{d_i}{\beta}, \frac{d_{i+1}}{\beta}\right), \text{ for } i \in [\![0, m-1]\!] \\ \beta x - d_m & \text{if } x \in \left[\frac{d_m}{\beta}, \frac{d_m}{\beta-1}\right). \end{cases}$$

Example

Consider $\beta = \frac{\varphi^3}{2}$ and

$$D = \{0 < 1 < \varphi < \varphi^2\}.$$

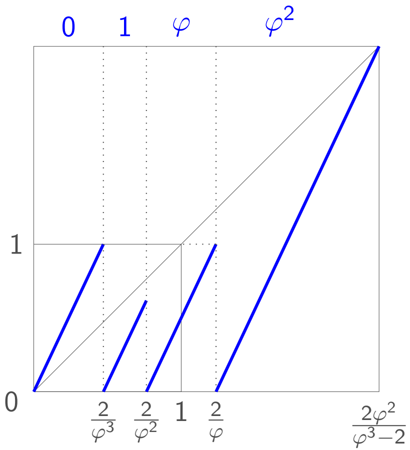
We have

$$T_{\beta,D}(x) = \begin{cases} \beta x & \text{if } x \in \left[0, \frac{2}{\varphi^3}\right), \\ \beta x - 1 & \text{if } x \in \left[\frac{2}{\varphi^3}, \frac{2}{\varphi^2}\right), \\ \beta x - \varphi & \text{if } x \in \left[\frac{2}{\varphi^2}, \frac{2}{\varphi}\right), \\ \beta x - \varphi^2 & \text{if } x \in \left[\frac{2}{\varphi}, \frac{2\varphi^2}{\varphi^3 - 2}\right). \end{cases}$$

Example

Consider $\beta = \frac{\varphi^3}{2}$ and

$$D = \{0 < 1 < \varphi < \varphi^2\}.$$



Define

$$f_{\boldsymbol{\beta}}\colon [\![0, \lceil \beta_0 \rceil - 1]\!] \times \cdots \times [\![0, \lceil \beta_{p-1} \rceil - 1]\!] \to D_{\boldsymbol{\beta}},$$
$$(a_0, \ldots, a_{p-1}) \mapsto \sum_{j=0}^{p-1} a_j \beta_{j+1} \cdots \beta_{p-1}.$$

We get

$$\begin{aligned} D_{\boldsymbol{\beta}} &= \left\{ \sum_{j=0}^{p-1} a_j \beta_{j+1} \cdots \beta_{p-1} \colon a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!] \right\} \\ &= \{ f_{\boldsymbol{\beta}}(a_0, \ldots, a_{p-1}) \colon a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!] \} \\ &= \{ 0 = d_0 < d_1 < \cdots < d_m \} \end{aligned}$$

with $m \leq \prod_{k=0}^{p-1} \lceil \beta_k \rceil$.

Lemma

The function $f_{\boldsymbol{\beta}}$ is non decreasing if and only if for all $j \in [\![1, p-2]\!]$,

$$\sum_{i=j}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \leq \beta_j \cdots \beta_{p-1}.$$

Theorem

The function $f_{\boldsymbol{\beta}}$ is non decreasing if and only if

$$T_{\beta_0\beta_1\cdots\beta_{p-1}, D_{\boldsymbol{\beta}}} = T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}$$

on $[0, 1)$.

In particular, for all length-2 alternate bases $\boldsymbol{\beta} = (\beta_0, \beta_1)$, we get $T_{\beta_0\beta_1, D_{\boldsymbol{\beta}}} = T_{\beta_1} \circ T_{\beta_0}$ on $[0, 1)$.

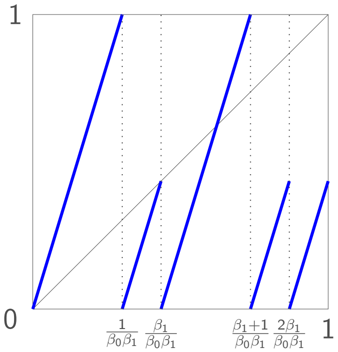
Example

Let $\boldsymbol{\beta} = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We obtain

$$T_{\beta_0\beta_1, D_{\boldsymbol{\beta}}} = T_{\beta_1} \circ T_{\beta_0}$$

on $[0,1)$ where

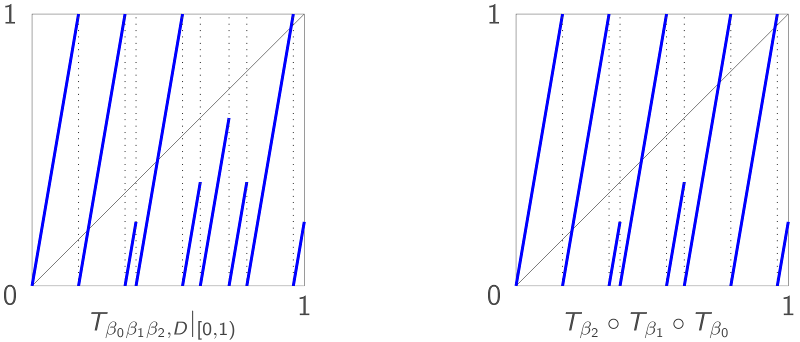
$$\begin{aligned} D_{\boldsymbol{\beta}} &= \{a_0\beta_1 + a_1 \colon a_0 \in \{0,1,2\}, a_1 \in \{0,1\}\} \\ &= \{0 < 1 < \beta_1 < \beta_1 + 1 < 2\beta_1 < 2\beta_1 + 1\}. \end{aligned}$$



Example

Let $\beta = (\beta_0, \beta_1, \beta_2) = (\varphi, \varphi, \sqrt{5})$. We get

$$D_\beta = \{a_0\beta_1\beta_2 + a_1\beta_2 + a_2 \colon a_0, a_1 \in \{0,1\}, a_2 \in \{0,1,2\}\}$$



We have $f_\beta(0,1,2) = \varphi + 2$ and $f_\beta(1,0,0) = \varphi^2 = \varphi + 1$.

Invariant measure and ergodicity

Let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}) \in (\mathbb{R}_{\geq 1})^p$. We define the transformation

$$T_{\boldsymbol{\beta}} : [\![0, p-1]\!] \times [0,1) \to [\![0, p-1]\!] \times [0,1)\,,$$
$$(i, x) \mapsto (i + 1 \bmod p,\, T_{\beta_i}(x))\,.$$

Iterating the map $T_{\boldsymbol{\beta}}$, we get

$$T_{\boldsymbol{\beta}}^n(0, x) = (n \bmod p,\ \ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)).$$

Then, the $\boldsymbol{\beta}$-expansion of $x \in [0,1)$ is obtained by iterating the transformation $T_{\boldsymbol{\beta}}$ on $(0, x)$.

Question

Can we find a $T_{\boldsymbol{\beta}}$-invariant probability measure $\mu_{\boldsymbol{\beta}}$ equivalent to "Lebesgue" and such that the transformation $T_{\boldsymbol{\beta}}$ is ergodic?

Steps :

- For all $i \in [\![0, p-1]\!]$, we consider the alternate base

$$(\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{i-1})$$

  and we find a $\left(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}\right)$-invariant probability measure μ_i, equivalent to Lebesgue and such that $T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}$ is ergodic.
- With the probability measures μ_i we construct a probability measure $\mu_{\boldsymbol{\beta}}$.
- We show the properties of $\mu_{\boldsymbol{\beta}}$.

Consider $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$, the map $T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}$ is piecewise linear with a slope $\beta_0 \cdots \beta_{p-1} > 1$.

Theorem

The map $T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}$ admits a unique invariant probability measure μ equivalent to Lebesgue. Moreover, $T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_0}$ is ergodic.

For all $i \in [\![0, p-1]\!]$, consider the alternate base $(\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{i-1})$ and let μ_i be the measure given by the theorem for

$$T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}.$$

Note that we use the convention

$$\mu_n = \mu_{n \bmod p}, \quad \forall n \in \mathbb{Z}.$$

We define the σ-algebra

$$\mathcal{F}_p = \Big\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) \colon \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}([0,1)) \Big\}.$$

We define the probability measure μ_β on $\mathcal{F}_p$ as follows:
For all $i \in [\![0, p-1]\!]$ and $B \in \mathcal{B}([0,1))$

$$\mu_\beta(\{i\} \times B) = \frac{1}{p^2} \sum_{j=0}^{p-1} \mu_{i-j}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-j}})^{-1} B\big)$$

and then for all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$,

$$\mu_\beta\Big(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \Big) = \sum_{i=0}^{p-1} \mu_\beta(\{i\} \times B_i).$$

Example

Consider $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ and
$A = (\{0\} \times B_0) \cup (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{F}_3$. We get

$$\mu_{\boldsymbol{\beta}}(A) =$$

Example

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$A = (\{0\} \times B_0) \cup (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{F}_3$. We get

$$\mu_{\boldsymbol{\beta}}(A) = \quad \mu_{\boldsymbol{\beta}}(\{0\} \times B_0) + \mu_{\boldsymbol{\beta}}(\{1\} \times B_1) + \mu_{\boldsymbol{\beta}}(\{2\} \times B_2)$$

Example

$$\mu_\beta(\{i\} \times B) = \frac{1}{p^2} \sum_{j=0}^{p-1} \mu_{i-j}((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-j}})^{-1} B)$$

Consider $\beta = (\beta_0, \beta_1, \beta_2)$ and
$A = (\{0\} \times B_0) \cup (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{F}_3$. We get

$$\begin{aligned} \mu_\beta(A) = & \quad \mu_\beta(\{0\} \times B_0) + \mu_\beta(\{1\} \times B_1) + \mu_\beta(\{2\} \times B_2) \\ = & \quad \frac{1}{9}\Big(\mu_0(B_0) + \mu_2(T_{\beta_2}^{-1} B_0) + \mu_1((T_{\beta_2} \circ T_{\beta_1})^{-1} B_0)\Big) \end{aligned}$$

Example

$$\mu_\beta(\{i\} \times B) = \frac{1}{p^2} \sum_{j=0}^{p-1} \mu_{i-j}((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-j}})^{-1} B)$$

Consider $\beta = (\beta_0, \beta_1, \beta_2)$ and
$A = (\{0\} \times B_0) \cup (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{F}_3$. We get

$$\begin{aligned}
\mu_\beta(A) = \;& \mu_\beta(\{0\} \times B_0) + \mu_\beta(\{1\} \times B_1) + \mu_\beta(\{2\} \times B_2) \\
= \;& \frac{1}{9}\Big(\mu_0(B_0) + \mu_2(T_{\beta_2}^{-1} B_0) + \mu_1((T_{\beta_2} \circ T_{\beta_1})^{-1} B_0)\Big) \\
& + \frac{1}{9}\Big(\mu_1(B_1) + \mu_0(T_{\beta_0}^{-1} B_1) + \mu_2((T_{\beta_0} \circ T_{\beta_2})^{-1} B_1)\Big)
\end{aligned}$$

Example

$$\mu_\beta(\{i\} \times B) = \frac{1}{p^2} \sum_{j=0}^{p-1} \mu_{i-j}((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-j}})^{-1} B)$$

Consider $\beta = (\beta_0, \beta_1, \beta_2)$ and $A = (\{0\} \times B_0) \cup (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{F}_3$. We get

$$\begin{aligned}
\mu_\beta(A) = \;& \mu_\beta(\{0\} \times B_0) + \mu_\beta(\{1\} \times B_1) + \mu_\beta(\{2\} \times B_2) \\
= \;& \frac{1}{9}\Big(\mu_0(B_0) + \mu_2(T_{\beta_2}^{-1} B_0) + \mu_1((T_{\beta_2} \circ T_{\beta_1})^{-1} B_0)\Big) \\
& + \frac{1}{9}\Big(\mu_1(B_1) + \mu_0(T_{\beta_0}^{-1} B_1) + \mu_2((T_{\beta_0} \circ T_{\beta_2})^{-1} B_1)\Big) \\
& + \frac{1}{9}\Big(\mu_2(B_2) + \mu_1(T_{\beta_1}^{-1} B_2) + \mu_0((T_{\beta_1} \circ T_{\beta_0})^{-1} B_2)\Big)
\end{aligned}$$

Proposition

The measure μ_{β} is T_{β}-invariant.

Proof (with $p = 3$ and $i = 1$): Let $B \in \mathcal{B}([0,1))$.

$$
\begin{aligned}
&\mu_{\beta}(T_{\beta}^{-1}(\{1\} \times B)) \\
=&\mu_{\beta}(\{0\} \times T_{\beta_0}^{-1}B) \\
=&\frac{1}{9}\left(\mu_0(T_{\beta_0}^{-1}B) + \mu_2(T_{\beta_2}^{-1}(T_{\beta_0}^{-1}B)) + \mu_1((T_{\beta_2} \circ T_{\beta_1})^{-1}(T_{\beta_0}^{-1}B))\right) \\
=&\frac{1}{9}\Bigg(\mu_0(T_{\beta_0}^{-1}B) + \mu_2((T_{\beta_0} \circ T_{\beta_2})^{-1}B) + \underbrace{\mu_1((T_{\beta_0} \circ T_{\beta_2} \circ T_{\beta_1})^{-1}B)}_{\substack{=\mu_1(B) \text{ by invariance of } \mu_1 \\ \text{w.r.t } T_{\beta_0} \circ T_{\beta_2} \circ T_{\beta_1}}}\Bigg) \\
=&\mu_{\beta}(\{1\} \times B).
\end{aligned}
$$

Theorem

The β-transformation T_β is ergodic.

Proof: Consider $A = (\{0\} \times B_0) \cup \cdots \cup (\{p-1\} \times B_{p-1}) \in \mathcal{F}_p$ s.t. $T_\beta^{-1} A = A$. We use the convention that $B_n = B_{n \pmod p}$ for all $n \in \mathbb{Z}$. We thus have

$$T_\beta^{-1}(\{i\} \times B_i) = \{i-1\} \times T_{\beta_{i-1}}^{-1} B_i$$

and therefore

$$T_{\beta_{i-1}}^{-1} B_i = B_{i-1} \quad \text{for all } i \in [\![0, p-1]\!].$$

Then, for all $i \in [\![0, p-1]\!]$ and $n \in \mathbb{N}$,

$$(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-n}})^{-1} B_i = B_{i-n}.$$

With $n = p$ it shows that B_i is $(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i})$-invariant, so by ergodicity we have $\mu_i(B_i) = 0$ or 1.

Proof (cont.): We obtain

$$
\begin{aligned}
\mu_{\boldsymbol\beta}(A) &= \sum_{i=0}^{p-1} \mu_{\boldsymbol\beta}(\{i\} \times B_i) \\
&= \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \mu_{i-j}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-j}})^{-1} B_i\big) \\
&= \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \mu_{i-j}(B_{i-j}) \\
&= \frac{1}{p} \sum_{i=0}^{p-1} \mu_i(B_i).
\end{aligned}
$$

We show that $\mu_i(B_i) = 0 \Leftrightarrow \mu_{i+1}(B_{i+1}) = 0$. By equivalence to Lebesgue of the measures μ_i and μ_{i+1}, we need to show that $\lambda(B_i) = 0 \Leftrightarrow \lambda(B_{i+1}) = 0$. We conclude by non-singularity of Lebesgue w.r.t any β-transformation since that $B_i = T_{\beta_i}^{-1}(B_{i+1})$.

We define an *extended Lebesgue measure* λ_p over $\mathcal{F}_p$ as follows. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$,

$$\lambda_p\Big(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\Big) = \frac{1}{p}\sum_{i=0}^{p-1}\lambda(B_i).$$

Proposition

The probability measure μ_β is equivalent to λ_p.

Further work

- Frequencies of letters and block of letters.
- Can we compute an alternate lazy representation? If yes, can we study in the same way the associated transformation?
- For the real bases, $([0,1), T_\beta)$ is isomorphic to (S_β, σ). Do we have a similar result?
- What is the entropy of our system?

$$\vdots$$