A Spectral Theory of Regular Sequences

Michael Coons

Abstract. A few years ago, Michael Baake and I introduced a probability measure associated to Stern's diatomic sequence, an example of a regular sequence—sequences which generalise constant length substitutions to infinite alphabets. In this talk, I will discuss extensions of these results to more general regular sequences as well as further properties of these measures.

This is joint work with several people, including Michael Baake, James Evans, Zachary Groth and Neil Manibo.

A spectral theory of regular sequences

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Definition (Allouche and Shallit, 1992)
Let $k \geqslant 2$ be an integer. A real sequence f is called k-regular provided its k-kernel, $\ker_k(f) := \left\{(f(k^\ell n + r))_{n \geqslant 0} : \ell \geqslant 0, 0 \leqslant r < k^\ell\right\}$, generates a $\mathbb{R}$-vector space $V_k(f)$ of finite dimension over $\mathbb{R}$.

Equivalently, a sequence f is k-regular provided there exist a positive integer d, a finite set of matrices $\mathcal{A}_f = \{\mathsf{A}_0, \ldots, \mathsf{A}_{k-1}\} \subseteq \mathbb{R}^{d \times d}$, and vectors $\mathsf{v}, \mathsf{w} \in \mathbb{R}^d$ such that

$$f(n) = \mathsf{w}^T \mathsf{A}_{i_0} \cdots \mathsf{A}_{i_s} \mathsf{v},$$

where $(n)_k = i_s \cdots i_0$ is the base-k expansion of n.

Stern's sequence

Stern's sequence, $(s(n))_{n\geqslant 0}$, is defined by the relations $s(0)=0$, $s(1)=1$, and for $n\geqslant 0$, by

$$s(2n)=s(n), \quad \text{and} \quad s(2n+1)=s(n)+s(n+1).$$

Stern's sequence is 2-regular and has linear representation

$$\mathrm{v}^T=\mathrm{w}^T=(1\ 0), \quad \mathrm{A}_0=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad \mathrm{A}_1=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

$$0$$
$$1, 1$$
$$1,2, 1$$
$$1, 3,2, 3, 1$$
$$1, 4, 3, 5,2, 5, 3, 4, 1$$
$$1, 5, 4, 7, 3, 8, 5, 7,2, 7, 5, 8, 3, 7, 4, 5, 1$$
$$\dots$$

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It has some interesting properties:

- Wilf's favourite sequence!
- the ratios $(s(n)/s(n+1))_{n\geqslant 0}$ enumerate the nonnegative rationals, in reduced form, and without repeats!
- The maximum values between powers of 2 are Fibonacci numbers.

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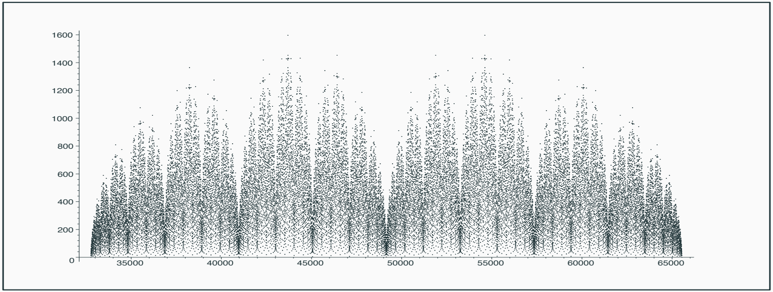
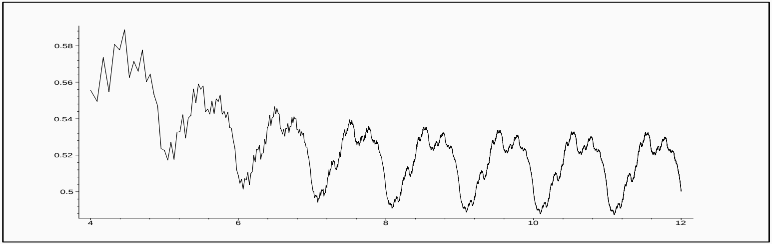
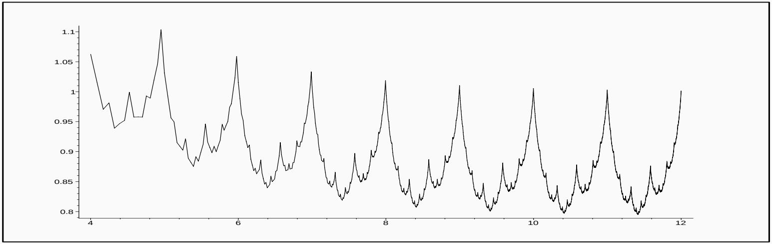




The value $\frac{1}{n^{\log_2 3}} \sum_{m \leqslant n} s(m)$ in the $\log_2$-scale.



The value $\frac{1}{n} \left(\sum_{m \leqslant n} s_2(m) - \frac{1}{2} n \log_2 n \right)$ in the $\log_2$-scale.

How complicated are integer sequences?

Traditionally, combinatorial sequences $(f(n))_{n\geqslant 0}$ are classified depending on the diffeo-algebraic properties of their generating functions

$$F(z) = \sum_{n\geqslant 0} f(n) z^n.$$

Some major classes are

- rational functions,
- irrational algebraic functions,
- *D*-finite functions (solutions to homogeneous linear differential equations),
- transcendental functions, and
- *D*-algebraic functions (solutions to algebraic differential equations).

*Adapted from Stanley to apply to complexity problems in enumerative combinatorics.

Three questions...

Question (Maximal values)

Characterise the maximal values of regular sequences.

In particular, where do they occur?

Question (Spectrum of values)

Characterise the image of a regular sequence.

What values can it take? Do they satisfy some sort of density property?

Question (Distribution of values)

Characterise the distribution of values of regular sequences.

How are the values distributed?

Maximal values: the finiteness conjecture

Recall, the *joint spectral radius* of a finite set of matrices $\mathcal{M} := \{\mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_k\}$, by the real number

$$\rho^* = \rho^*(\mathcal{M}) := \lim_{n\to\infty} \max_{1\leqslant i_1, i_2, \ldots, i_n \leqslant k} \left\| \mathsf{M}_{i_1} \mathsf{M}_{i_2} \cdots \mathsf{M}_{i_n} \right\|^{1/n}.$$

Definition (Finiteness property)

The finite set of matrices $\mathcal{M}$ is said to satisfy the *finiteness property* provided there is a finite product $\mathsf{M}_{i_0} \cdots \mathsf{M}_{i_{m-1}}$ of matrices from $\mathcal{M}$ such that

$$\rho(\mathsf{M}_{i_0} \cdots \mathsf{M}_{i_{m-1}})^{1/m} = \rho^*(\mathcal{M}).$$

Conjecture (Lagarias and Wang, 1995)

The finiteness property holds for all finite sets of integer matrices.

Spectrum of values: Zaremba's conjecture

For $x \in (0,1)$ we write the ordinary continued fraction expansion of x as

$$x = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}.$$

The convergents of the number x are the rationals $p_n/q_n := [a_1, \ldots, a_n]$ and can be computed using the well-known relationship

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}.$$

Let $\mathfrak{D}_k$ the set of all denominators of convergents of real numbers $x \in (0,1)$ whose partial quotients are bounded above by k.

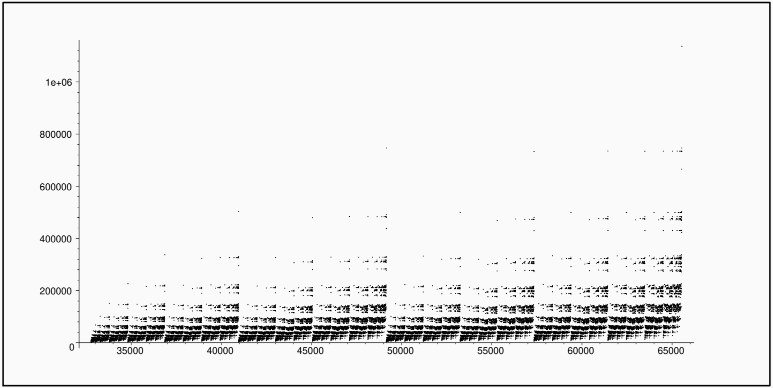
Conjecture (Zaremba, 1972)

There is a positive integer k such that $\mathfrak{D}_k = \mathbb{N}$.

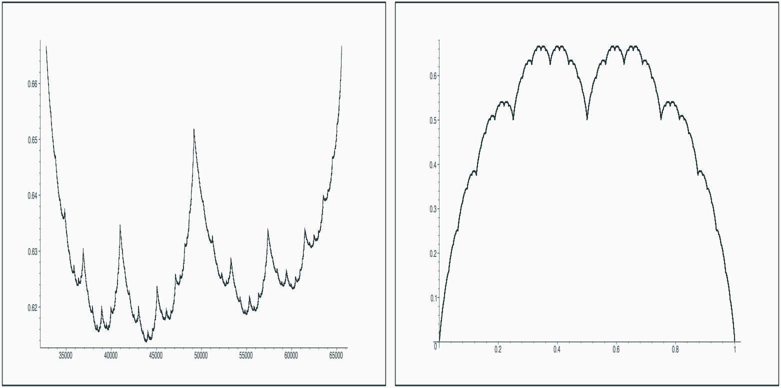
The Zaremba sequence

We define the 2-regular *Zaremba sequence*, $(q(n))_{n\geqslant 1}$, by its linear representation given by $\mathrm{w}^T = \mathrm{v} = (1\ 0)$ and

$$\mathrm{B}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{B}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$



M. Coons,
Mahler takes a regular view of Zaremba,
Integers 18A (2018), #A6, 1–15.



(left) The ratio $x^{-2}\sum_{n\leqslant x} q(n)$ in the interval $[2^{15}, 2^{16}]$ and (right) the Takagi curve.

A geometric approach to regular sequences

...via measures

One can easily show that $\sum_{m=0}^{2^n-1} s(2^n+m) = 3^n$, holds for all $n \in \mathbb{N}_0$. Therefore, we define

$$\mu_n := \frac{1}{3^n} \sum_{m=0}^{2^n-1} s(2^n+m)\, \delta_{m/2^n},$$

and consider $\mu := \lim_{n\to\infty} \mu_n$.

Theorem (Lebesgue decomposition)

Any regular Borel measure μ on $\mathbb{R}^d$ has a unique decomposition

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}$$

where $\mu_{\mathrm{pp}} \perp \mu_{\mathrm{ac}} \perp \mu_{\mathrm{sc}}$ and also $|\mu| = |\mu_{\mathrm{pp}}| + |\mu_{\mathrm{ac}}| + |\mu_{\mathrm{sc}}|$.

Theorem (Baake and C. 2018)

The sequence $(\mu_n)_{n\in\mathbb{N}_0}$ *of probability measures on* $\mathbb{T}$ *converges weakly to a probability measure* μ*. In particular, one has* $\mu_0 = \delta_0$ *and* $\mu_n = \mathop{\ast}_{m=1}^{n} \frac{1}{3}\bigl(\delta_0 + \delta_{2^{-m}} + \delta_{-2^{-m}}\bigr)$ *for* $n \geqslant 1$*. The weak limit as* $n \to \infty$ *is given by the convergent infinite convolution product*

$$\mu = \mathop{\ast}_{m\geqslant 1} \tfrac{1}{3}\bigl(\delta_0 + \delta_{2^{-m}} + \delta_{-2^{-m}}\bigr).$$

Its Fourier transform $\widehat{\mu}$ *is given by* $\widehat{\mu}(k) = \prod_{m\geqslant 1} \frac{1}{3}\bigl(1 + 2\cos(2\pi k/2^m)\bigr)$ *for* $k \in \mathbb{Z}$*. Moreover, this infinite product is also well-defined on* $\mathbb{R}$*, where it converges compactly. Furthermore,* μ *is purely singular continuous.*

The generating function of the Stern sequence satisfies

$$S(z) = \sum_{n\geqslant 0} s(n{+}1) z^n = \prod_{j\geqslant 0} \Bigl(1 + z^{2^j} + z^{2^{j+1}}\Bigr) = \prod_{j\geqslant 0} z^{2^j} \Bigl(z^{-2^j} + z^0 + z^{2^j}\Bigr).$$

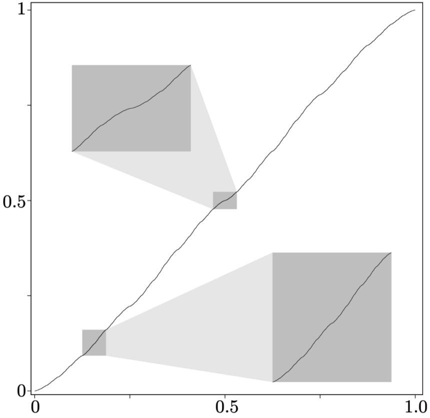


Fig. 2. The distribution function F of the purely singular continuous probability measure μ derived from Stern's diatomic sequence.

General existence of ghost measures, I

Suppose f is a k-regular sequences with digit matrices, $\mathrm{B}_0, \dots, \mathrm{B}_{k-1}$ and set $\mathrm{B} := \mathrm{B}_0 + \cdots + \mathrm{B}_{k-1}$. Let

$$\Sigma_f(n) := \sum_{m=k^n}^{k^{n+1}-1} f(m)$$

$$\mu_{f,n} := \frac{1}{\Sigma_f(n)} \sum_{m=0}^{k^{n+1}-k^n-1} f(k^n + m)\, \delta_{m/k^n(k-1)}.$$

Theorem (C., Evans and Mañibo)

Let f be a real-valued k-regular sequence. Suppose that $\rho(\mathrm{B})$ is the unique dominant eigenvalue of B, $\rho(\mathrm{B}) > \rho^*(\{\mathrm{B}_0, \dots, \mathrm{B}_{k-1}\})$ and that the asymptotical behaviour of $\Sigma_f(n)$ is determined by $\rho(\mathrm{B})$. If the limit $F_f(x)$ of the sequence $\mu_{f,n}([0,x])$ is a function of bounded variation, then $F_f(x) = \mu_f([0,x])$ is the distribution function of a measure μ_f, which is continuous with respect to Lebesgue measure.

We call μ_f a *Ghost Measure* and $F_f(x)$ a *Ghost Distribution*.

Why a *Ghost Measure*?

Before moving on, we explain the name ghost measure. Neither Baake and Coons nor Coons, Evans and Mañibo give a name to their construction. The inspiration for ours comes from Berkeley's critique of infinitesimals in The Analyst, when he says that they are

> *... neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them* **the ghosts of departed quantities?**

The values $f(n)$ are (usually) much smaller than the sum of all terms, so the individual pure points of the μ_N disappear in the averaging as N tends to infinity. The measure μ_N is the ethereal imprint that is left behind, the ghost of the departed pure points of the μ_N.

*From James Evans, *The ghost measures of affine regular sequences*, Houston J. Math., to appear.

Proof via relation to dilation equations

Daubechies and Lagarias,
Two-scale difference equations I, II, SIAM J. Math. Anal., 1991 and 1992.

We define the matrix-valued function $\mathsf{F}_\rho : \mathbb{R} \to \mathbb{R}^d$ by

$$\mathsf{F}_\rho(x) \cdot \rho = \sum_{a=0}^{k-1} \mathsf{B}_a^T \cdot \mathsf{F}_\rho(kx - a), \quad \text{where} \quad \mathsf{F}_\rho(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ \mathsf{v}_\rho & \text{for } x \geqslant 1. \end{cases}$$

The function F_ρ exists and is unique since $\rho > \rho^*$. The function F_ρ is Hölder continuous with exponent α for any $\alpha < \log_k(\rho/\rho^*)$.

We have the relationship,

$$F_f(x) = \mu_f([0,x]) = \frac{\mathsf{w}^T \left(\mathsf{F}_\rho\left(\frac{1+(k-1)x}{k}\right) - \mathsf{F}_\rho\left(\frac{1}{k}\right) \right)}{\mathsf{w}^T \left(\mathsf{v}_\rho - \mathsf{F}_\rho\left(\frac{1}{k}\right)\right)}.$$

Zaremba, Salem, and the fractal nature of ghost distributions

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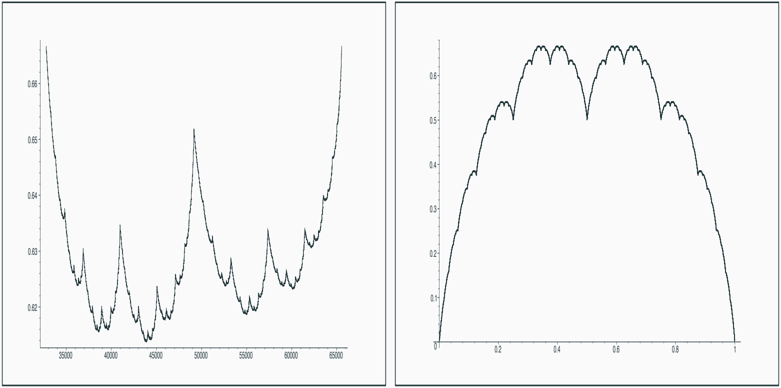
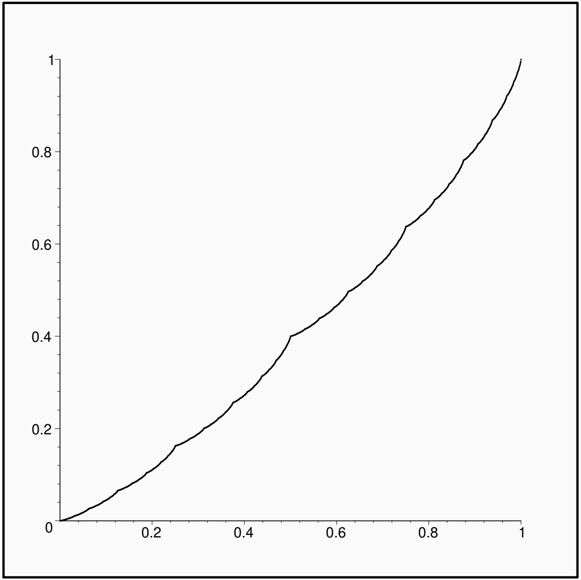


(left) The ratio $x^{-2}\sum_{n\leqslant x} q(n)$ in the interval $[2^{15}, 2^{16}]$ and (right) the Takagi curve.

The points $\left(\frac{k+1}{2^{11}}, \frac{1}{2\cdot 4^{11}} \sum_{j=0}^{k} q(2^{11}+j)\right)$, for $k = 0, \ldots, 2^{11}-1$.



In the early 1940s, Salem gave a geometric construction of (the graphs of) a family of strictly increasing singular continuous functions from $[0,1]$ to $[0,1]$. Consider Salem's example, which is the unique attractor of the iterated function system $\mathcal{S}_s = \{S_0, S_1\} : [0,1]^2 \to [0,1]^2$, where

$$S_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad S_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 2/5 \end{pmatrix}.$$

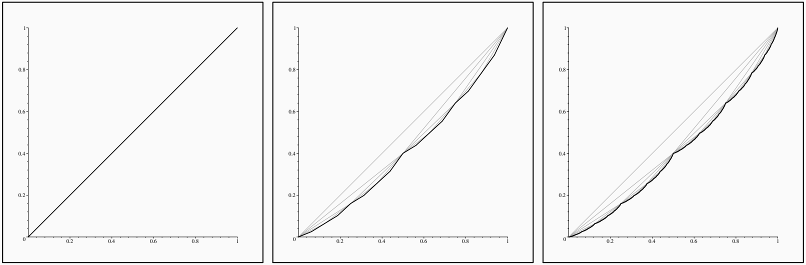


Figure 1: The line segment from $(0,0)$ to $(1,1)$ after applying $\mathcal{S}_s$, Salem's iterated function system, 0, 4 and 10 times.

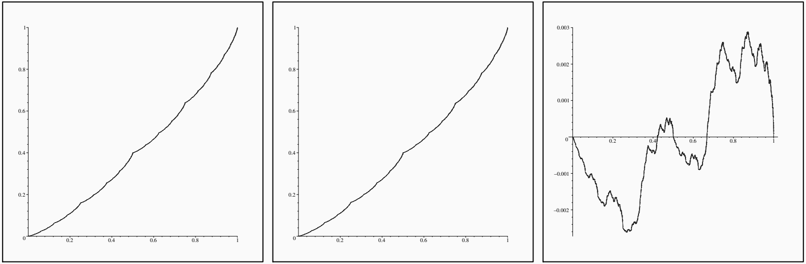


Figure 2: Salem's attractor (left) and the Zaremba ghost distribution (centre), and their point-wise difference, Salem minus Zaremba (right).

From dilation equations to iterated function systems

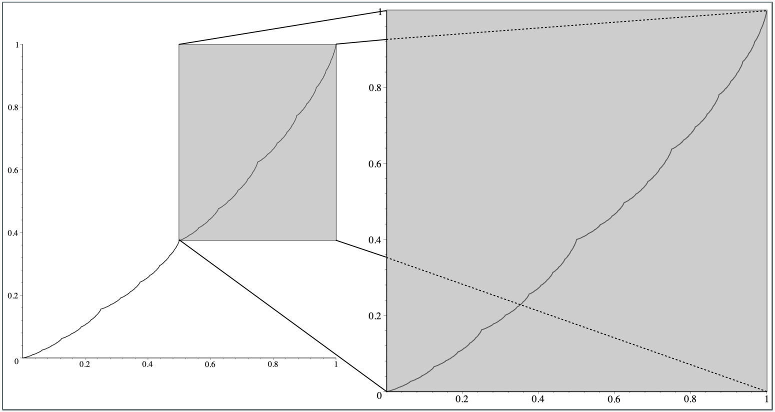
Let $k \geqslant 2$ be an integer and f be a k-regular sequence where the spectral radius $\rho(\mathrm{B})$ is the unique dominant eigenvalue of B and $\rho(\mathrm{B}) > \rho^*(\mathcal{B})$, and suppose that $\mathrm{F}(x)$ is the solution of the associated dilation equation. Then the graph

$$\mathcal{F}_f := \{(x, \mathrm{F}(x)) : x \in [0,1]\} \subset [0,1]^{d+1}$$

of $\mathrm{F}(x)$ is a section of a self-affine set. In particular, $\mathcal{F}_f$ is the attractor of the iterated function system $\mathcal{S}_f = \{S_0, \ldots, S_{k-1}\} : [0,1]^{d+1} \to [0,1]^{d+1}$ where, for $j \in \{0, 1, \ldots, k-1\}$, we have

$$S_j \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix} = \begin{pmatrix} 1/k & 0^{1\times(k-1)} \\ 0^{(k-1)\times 1} & \rho^{-1}\,\mathrm{B}_j \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ y_d \end{pmatrix} + \begin{pmatrix} j/k \\ \sum_{a=0}^{j-1} \rho^{-1}\,\mathrm{B}_a \mathrm{v}_\rho \end{pmatrix}.$$

The IFS and ghost distribution of Zaremba's sequence



$$S_0\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1/2&0&0\\0&1/4&1/4\\0&1/4&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix},\ S_1\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1/2&0&0\\0&1/2&1/4\\0&1/4&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+\begin{pmatrix}1/2\\3/8\\1/4\end{pmatrix}.$$

Lebesgue decomposition of regular ghost measures

General existence of ghost measures, II

Theorem (C., Evans and Mañibo)

Let f be a nonnegative real-valued k-regular sequence with reduced representation $g = (\mathsf{w}, \mathsf{B}_0, \ldots, \mathsf{B}_{k-1}, \mathsf{x})$. If the spectral radius $\rho(\mathsf{B})$ is the unique simple maximal eigenvalue of B and there is a linear cone K fixed by each B_i, then $\mu_f = \mu_g$ exists.

Let f be a nonnegative real-valued k-regular sequence of degree d with $d \times d$ linear representation $(\mathsf{u}, \mathsf{A}_0, \ldots, \mathsf{A}_{k-1}, \mathsf{v})$ and suppose that $\mathsf{A} := \sum_{i=0}^{k-1} \mathsf{A}_i$ has a unique (not necessarily simple) maximal eigenvalue ρ. Then there is a nonnegative k-regular sequence g of degree $d_g \leqslant d$ with $d_g \times d_g$ linear representation $(\mathsf{w}, \mathsf{B}_0, \ldots, \mathsf{B}_{k-1}, \mathsf{x})$ such that

- $\Sigma_g(N) = \rho^N$ for all $N \geqslant 0$,
- x is a ρ-eigenvector of $\mathsf{B} := \sum_{i=0}^{k-1} \mathsf{B}_i$, and
- ρ has equal algebraic and geometric multiplicities as a B-eigenvalue.

Moreover, if the ghost measure μ_f exists, so does μ_g, and they are equal.

Spectral purity of ghost measures

Theorem (Lebesgue decomposition)

Any regular Borel measure μ on $\mathbb{R}^d$ has a unique decomposition

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}$$

where $\mu_{\mathrm{pp}} \perp \mu_{\mathrm{ac}} \perp \mu_{\mathrm{sc}}$ and also $|\mu| = |\mu_{\mathrm{pp}}| + |\mu_{\mathrm{ac}}| + |\mu_{\mathrm{sc}}|$.

Theorem (C., Evans and Mañibo)

The measure μ_f provided by the above theorem is spectrally pure. That is, μ_f is either discrete, or singular continuous, or absolutely continuous, with respect to Lebesgue measure.

The proof follows by showing D-ergodicity. Recall that a set Z is D-invariant if $Z + d = Z$ for all $d \in D$ and a regular Borel measure μ is D-ergodic if $\mu(Z) \in \{0, 1\}$ for every D-invariant set Z. The countable subgroup of $\mathbb{T}$ we use is $D := \frac{1}{k-1} \cdot \mathbb{Z}\left[\frac{1}{k}\right] \pmod 1$.

Determining *the* spectral type

Given a finite set of matrices $\{\mathsf{B}_1, \dots, \mathsf{B}_k\}$, in our case associated to a nonnegative k-regular sequence f, we have a fundamental inequality,

$$\frac{\rho}{k} \leqslant \rho^* \leqslant \rho := \rho(\mathsf{B}_1 + \mathsf{B}_2 + \cdots + \mathsf{B}_k).$$

- The ghost measure μ_f is continuous if and only if $\rho^* < \rho$. That is, μ_f is pure point if and only if $\rho^* = \rho$.
- If μ_f is continuous and $\rho/k < \rho^*$, then μ_f is singular continuous.
- Suppose that $\rho^* = \rho/k$ and additionally that there is a $d > 0$ such that

  $$\max_{1 \leqslant i_1, i_2, \dots, i_n \leqslant \ell} \left\| \mathsf{B}_{i_1} \mathsf{B}_{i_2} \cdots \mathsf{B}_{i_n} \right\| \leqslant d(\rho^*)^n,$$

  for each $n \geqslant 1$. Then μ_f is absolutely continuous.

Continuous ghost measures have a level-set construction

Define the set of intervals

$$\mathcal{I}_k := \left\{ I_{\ell,m} = \left[\frac{m - k^\ell}{k^\ell(k-1)}, \frac{m+1-k^\ell}{k^\ell(k-1)} \right) : \ell \geqslant 0, k^\ell \leqslant m < k^{\ell+1} \right\}.$$

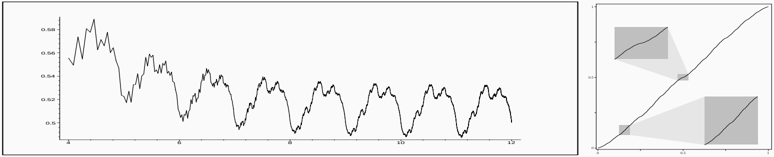
Proposition (Level-set construction)

Let f be a nonnegative real-valued k-regular sequence with reduced linear representation $(\mathrm{w}, \mathrm{B}_0, \dots, \mathrm{B}_{k-1}, \mathrm{x})$ and suppose that μ_f exists and is continuous. If ν is the measure on $[0,1)$ defined by

$$\nu(I_{\ell,m}) = \frac{\mathrm{w}^T \mathrm{B}_{(m)_k} \mathrm{x}}{\rho^\ell}$$

for all $I_{\ell,m} \in \mathcal{I}_k$, then $\nu = \mu_f$.

Example: Stern sequence



The value $\frac{1}{n^{\log_2 3}} \sum_{m \leqslant n} s(m)$ in the $\log_2$-scale and distribution of μ_s.

The matrices associated to Stern's sequence are

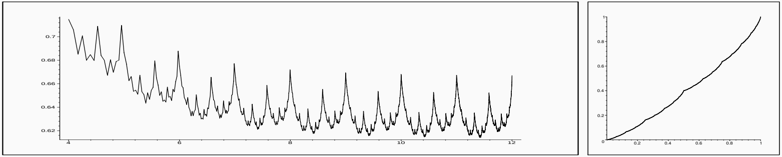
$$\mathsf{A}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The fundamental inequality for Stern's sequence is

$$\frac{3}{2} < \frac{1+\sqrt{5}}{2} < 3,$$

so μ_s is *singular continuous*.

Examples: 2-Zaremba sequence



The value $\frac{1}{n^2}\sum_{m\leqslant n} q(m)$ in the $\log_2$-scale and dist. of μ_q.

The matrices associated to the 2-Zaremba sequence are

$$\mathrm{B}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{B}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

The fundamental inequality for the 2-Zaremba sequence is

$$2 < 1 + \sqrt{2} < 4,$$

so μ_q is *singular continuous*.

Some questions and further work

- Work out the characterisation of the absolutely continuous case.
- To what extent does specialising to the (nonnegative) integer case of regular sequences help? (possibility of Diophantine-type gaps?)

$$\bar{\rho} \leqslant \frac{\rho}{k} \leqslant \rho^* \leqslant \rho$$

- Determine multifractal spectrum of singular continuous ghost measures.
- Explore the possibility of "effective" versions of certain F_α sets.

Thank you, Merci, Danke.