Alternating $N$-continued fraction expansions

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Inspired by the recent study of the dynamical properties of alternate $\beta$-expansions, we introduce the notion of alternate $N$-continued fraction expansions.
In 2008, Edward Burger and his co-authors introduced a generalization of the *regular continued fraction* (RCF) expansion of the form

\[ x = d_0 + \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cdots + \cfrac{N}{d_k + \cdots} \cdot \cdots} = [d_0; d_1, d_2, \ldots, d_k, \cdots]_N \]

where \( N, d_i \in \mathbb{N}, N, d_i \neq 0 \ (n \geq 1) \). They called such expansions \( N \)-expansions.
Their main object of study was the $N$-expansions of quadratic irrationals. They showed that if $x$ is a quadratic irrational, then there are infinitely many integers $N$ for which $x$ can be expressed as an eventually periodic $N$-expansion with period length one. In this case they showed that the sequence of truncations of such an $N$-expansion forms a subsequence of the sequence of convergents of the regular continued fraction expansion of $x$. 
In 2011, M. Anselm and S. Weintraub studied further the properties of \( N \)-expansions. They showed:

Every positive real number \( x \) has always an \( N \)-expansion, and for \( N \geq 2 \) even infinitely many.

Rationals always have finite and infinite expansions.

In case \( N \geq 2 \) every quadratic irrational has both eventually periodic and non-periodic expansions.
Together with C. Kraaikamp and N. van der Wekken (2013), we took a dynamical approach.

Fig. 1. All possible maps in case $N = 6$. 
Dynamical approach

We studied ergodic properties of certain deterministic algorithms.

Choosing the lowest curve, leads to the greedy algorithm. Choosing the highest curve leads to the lazy algorithm. Restrict to a window, leads to expansions with finitely many digits.
In 2017, C. Kraaikamp and N. Langeveld concentrated on expansions of points in windows of length 1 to obtain a parametrised family of $N$-expansions with finitely many digits. For certain values of the parameter they were able to obtain the density of the invariant measure.

In 2018, together with M. Oomen, we studied random algorithms (skew products).
Given non-negative integers \((N_1, \ldots, N_m)\), we would like to study expansions of the form

\[
x = \frac{N_1}{d_1 + \frac{N_2}{d_2 + \frac{N_3}{d_3 + \cdots + \frac{N_m}{d_m + \frac{N_1}{d_{m+1} + \cdots}}}}}
\]
Dynamical setup

For \(i \in \{1, \ldots, m\}\) let \(I_i := [a_i, a_i + 1)\) with \(a_i \in \mathbb{N} \cup \{0\}\) be a set of distinct intervals.

We will study continued fraction maps \(T : \bigcup l_i \rightarrow \bigcup l_i\) such that \(T(I_i) = I_{(i \mod m) + 1}\) for \(i \in \{1, \ldots, m\}\).

On every interval \(l_i\) we choose an \(N_i \in \mathbb{N}\). Let \(j(x) = i\) when \(x \in l_i\) and \(k(x) = (j(x) \mod m) + 1\). Furthermore, let \(\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m\).

Define \(T : \bigcup l_i \rightarrow \bigcup l_i\) as

\[
T(x) = \frac{N_j(x)}{x} - \left\lfloor \frac{N_j(x)}{x} \right\rfloor + a_k(x),
\]

for \(x \neq 0\) and \(T(0) = 0\) whenever \(0 \in \bigcup_{i=1}^m l_i\). We refer to \(T\) as the \(\mathbf{N}\)-continued fraction map. Note that with this definition some digits might be negative!
Avoiding negative digits: the allowable case

To avoid negative digits, we impose an extra restriction.

**Definition (allowable)**

Let $I_i := [a_i, a_i + 1)$ with $a_i \in \mathbb{N} \cup \{0\}$ be a set of distinct intervals and $\Omega = \bigcup_{i=1}^{m} I_i$. We call $N = (N_1, \ldots, N_m) \in \mathbb{N}^m$ **allowable** for $\Omega$ if for all $x \in \Omega$ we have

$$\left\lfloor \frac{N_j(x)}{x} \right\rfloor - a_k(x) > 0.$$

Since we can pick the values for $N$ arbitrarily high, we can find a continued fraction algorithm for every choice of distinct intervals with all possible ways of indexing them.

Many of these choices will give rise to dynamical systems that are difficult to study since not all branches will be full (a branch of the map $T$ will not be mapped entirely onto the next interval). We will introduce two special sub-classes of allowable expansions.
Definition (desirable and simple)

Let \( I_i := [a_i, a_i + 1) \) with \( a_i \in \mathbb{N} \cup \{0\} \) be a set of distinct intervals and \( \Omega = \bigcup_{i=1}^{m} I_i \). We call \( N = (N_1, \ldots, N_m) \in \mathbb{N}^m \) desirable for \( \Omega \) if it is allowable and for all \( i \in \{1, \ldots, m\} \) we have that \( a_i \mid N_i \) and \( a_i + 1 \mid N_i \) whenever \( a_i \neq 0 \). Furthermore, we call \( N = (N_1, \ldots, N_m) \in \mathbb{N}^m \) simple if it is desirable and \( N_i = N_j \) for all \( i, j \in \{1, \ldots, m\} \).

Note that for any set of distinct intervals we can find a simple \( N \) by taking a large enough multiple of the product of the endpoints of the intervals \( I_i \), omitting 0.

Our simple case generalizes the classical case, in the sense that the numerators will not alternate but the sets from which the digits are taken are alternating.
Let $a_1 = 1$, $a_2 = 3$ and $\mathbf{N} = (9, 12)$. We find that

$T : [1, 2) \cup [3, 4) \rightarrow [1, 2) \cup [3, 4)$ is given by

$$T(x) = \begin{cases} \frac{9}{x} - \lfloor \frac{9}{x} \rfloor + 3 & \text{for } x \in [1, 2), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 1 & \text{for } x \in [3, 4), \end{cases}$$
We see that, since 3 \mid 12 and 4 \mid 12 we have full branches on [3, 4) but since 2 \nmid 9 we do not have full branches on [1, 2). Therefore, this choice of \( N \) is allowable but not desirable.
At the left end point of any of the intervals, $T$ is not continuous giving us an extra digit in case the left point is not equal to 0.

Setting $d_1(x) = \left\lfloor \frac{N_j(x)}{x} \right\rfloor - a_k(x)$ and $d_n(x) = d_1(T^{n-1}(x))$, iterations of $T$ give the following expressions.
For \( x \in [1, 2) \),

\[
x = \frac{9}{d_1(x) + \frac{12}{d_2(x) + \frac{9}{\ldots}}}
\]

with \( d_i \in \{1, \ldots, 6\} \) for \( i \) odd (only 6 when \( T^{i-1}(x) = 1 \)) and \( d_i \in \{2, 3\} \) for \( i \) even (only 3 when \( T^{i-1}(x) = 3 \)). For \( x \in [3, 4) \) we find

\[
x = \frac{12}{d_1(x) + \frac{9}{d_2(x) + \frac{12}{\ldots}}}
\]

with \( d_i \in \{2, 3\} \) for \( i \) odd and \( d_i \in \{1, \ldots, 6\} \) for \( i \) even.
Example: desirable

Let $a_1 = 1$, $a_2 = 2$ and $\mathbf{N} = (8, 12)$. Then we can define $T : [1, 3) \rightarrow [1, 3)$ as

$$T(x) = \begin{cases} \frac{8}{x} - \left\lfloor \frac{8}{x} \right\rfloor + 2 & \text{for } x \in [1, 2), \\ \frac{12}{x} - \left\lfloor \frac{12}{x} \right\rfloor + 1 & \text{for } x \in [2, 3), \end{cases}$$

That this choice of $\mathbf{N}$ is desirable.
For $x \in [1, 2)$ we find the continued fraction

$$x = \frac{8}{d_1 + \frac{12}{d_2 + \frac{8}{\ddots}}},$$

where $d_i \in \{2, 3, 4, 5, 6\}$ for $i$ odd (only 6 when $T^{i-1}(x) = 1$) and $d_i \in \{3, 4, 5\}$ for $i$ even (only 5 when $T^{i-1}(x) = 2$).
Let $a_1 = 0, a_2 = 2, a_3 = 1, a_4 = 3$. This time we choose $N = (12, 12, 12, 12)$ so that we are in the simple case. Note that since there is an $i$ such that $a_i = 0$ we have infinitely many branches. We find that $T : [0, 4) \rightarrow [0, 4)$ is now defined as

$$
T(x) = \begin{cases} 
\frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 2 & \text{for } x \in [0, 1), \\
\frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 3 & \text{for } x \in [1, 2), \\
\frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 1 & \text{for } x \in [2, 3), \\
\frac{12}{x} - \lfloor \frac{12}{x} \rfloor & \text{for } x \in [3, 4), 
\end{cases}
$$
For $x \in (0, 1)$ we find

$$x = \frac{12}{d_1 + \frac{12}{d_2 + \frac{12}{\ldots}}}$$

where

$$d_i(x) \in \begin{cases} 
\mathbb{N}_{\geq 10} & \text{for } i = 1 \mod 4, \\
\{3, 4, 5\} & \text{for } i = 2 \mod 4, \\
\{3, \ldots, 9\} & \text{for } i = 3 \mod 4, \\
\{3, 4\} & \text{for } i = 0 \mod 4.
\end{cases}$$

Here $d_i(x) = 5$ for $i = 2 \mod 4$ only when $T^{i-1}(x) = 2$, $d_i(x) = 9$ for $i = 3 \mod 4$ only when $T^{i-1}(x) = 1$ and $d_i(x) = 4$ for $i = 0 \mod 4$ only when $T^{i-1}(x) = 3$. 
Suppose $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ is allowable for $\Omega = \bigcup_{i=1}^m [a_i, a_{i+1})$. We write $N_n = N_{(n \mod m + 1)}$. If $x \in \Omega$, after $n$ iterations of the map $T$ we have

$$
x = \frac{N_1}{d_1 + \frac{N_2}{d_2 + \cdots + \frac{N_n}{d_n + T^n x}}}$$
Given $x \in \Omega$, we denote by $\frac{p_n(x)}{q_n(x)} = \frac{p_n}{q_n}$ the rational number obtained by considering the first $n$ terms of the expansion,

$$
\frac{p_n}{q_n} = \frac{N_1}{d_1 + \frac{N_2}{d_2 + \cdots + \frac{N_n}{d_n}}}
$$

To prove that

$$
\lim_{n \to \infty} \left| x - \frac{p_n}{q_n} \right| = 0
$$

we use a classical approach.
With the help of Möbius transformations, we derive results similar to those of the regular continued fractions.

We define

$$B_{d,N} = \begin{bmatrix} 0 & N \\ 1 & d \end{bmatrix}.\quad (1)$$

$$M_n(x) = M_n = B_{d_1,N_1}B_{d_2,N_2}\cdots B_{d_n,N_n}.$$

$$\det(M_n) = (-1)^n \prod_{i=1}^{n} N_i.$$  

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}.$$
With the help of the matrices $M_n$, we have

$$p_{n-1}q_n - q_{n-1}p_n = (-1)^n \prod_{i=1}^{n} N_i.$$ 

$$p_{-1} = 1 \quad p_0 = 0 \quad p_n = d_n p_{n-1} + N_n p_{n-2},$$

$$q_{-1} = 0 \quad q_0 = 1 \quad q_n = d_n q_{n-1} + N_n q_{n-2}.$$ 

$$x = \frac{p_n + p_{n-1} T^n(x)}{q_n + q_{n-1} T^n(x)}.$$
\[
\frac{x - \frac{p_n}{q_n}}{q_n} = \frac{(-1)^n N_1 \cdots N_n T^n(x)}{q_n(q_n + q_{n-1} T^n(x))}.
\]

\[
|x - \frac{p_n}{q_n}| < \frac{N_1 \cdots N_n \max \Omega}{q_n^2}.
\]

Using the recursion relations, we show that

\[
\lim_{n \to \infty} \frac{N_1 \cdots N_n}{q_n^2} = 0.
\]
To prove the existence of an ergodic absolutely continuous $T$-invariant measure, we use a result of R. Zweimüller.
Theorem

(Zweimüller) Let $T : [0, 1] \to [0, 1]$, and let $B$ be a collection (not necessarily finite) of nonempty pairwise disjoint open subintervals with $\lambda(\bigcup B) = 1$ such that $T$ restricted to each element $Z$ of $B$ is continuous and strictly monotone. Suppose $T$ satisfies the following three conditions:

(A) Adler’s condition: $T''/(T')^2$ is bounded on $\bigcup B$,
(B) Finite image condition: $TB = \{TZ : Z \in B\}$ is finite,
(C) Uniformly eventually expanding: there is a $k$ such that $|(T^k)'| > \gamma > 1$ on $\bigcup B$.

Then there are a finite number of pairwise disjoint open sets $X_1, \ldots, X_n$ such that $TX_i = X_i$ (modulo sets of $\lambda$-measure zero) and $T|_{X_i}$ is conservative and ergodic with respect to $\lambda$. Almost all points of $[0, 1] \setminus \bigcup_i X_i$ are eventually mapped into one of these ergodic components. Furthermore, each $X_i$ supports an absolutely continuous invariant measure $\mu_i$ which is unique up to a constant factor.
Ergodic properties

**Theorem**

Let $\Omega = \bigcup_{i=1}^{m} I_i$ where $I_i := [a_i, a_i + 1)$, with $a_i \in \mathbb{N} \cup \{0\}$, are distinct intervals. Assume $N = (N_1, \ldots, N_m) \in \mathbb{N}^m$ is allowable for $\Omega$ and $T : \Omega \to \Omega$ the corresponding $N$-continued fraction map. Then, $T$ admits a unique absolutely continuous invariant ergodic measure.

We verify the conditions of Zweimüller’s Theorem:

- **Adler’s condition:** $T''/(T')^2 < 2$
- **Finite image condition:** at most $2m$ possible images
- **Uniformly eventually expanding:** $|(T^k)'| > 4$ for $k \geq 2$
- $\Omega$ is the smallest forward invariant set.
The above theorem gives the existence of a $T$-invariant measure $\mu$ of the form

$$\mu(A) = \int f \, d\lambda,$$

where $f$ is in $L^1(\Omega)$ and $\lambda$ is Lebesgue.

Can we find the explicit form of $f$?

A useful tool in finding the explicit form is to use the notion of natural extensions.
A natural extension of a dynamical system \((\Omega, \mathcal{B}, T, \mu)\) is the "smallest" invertible dynamical system containing \((\Omega, \mathcal{B}, T, \mu)\) as a factor.

A natural extension is unique up to an isomorphism.

Rohlin (1961) gave a canonical construction of a natural extension that resembles the idea of converting a one-sided shift to a two-sided shift.
Given an allowable $\mathbf{N}$-continued fraction map $T$ defined on $(\Omega, \mathcal{B}, \mu)$, we want to find a version $\mathcal{T}$ of the natural extension of $T$ with the following properties:

$\mathcal{T}$ is defined on a domain $X \subset \mathbb{R}^2$ of positive Lebesgue measure

$$\mathcal{T}(x, y) = \left( T(x), \frac{N_j(x)}{d_1(x)} + y \right).$$

There exists an absolutely continuous $\mathcal{T}$-invariant measure $\overline{\mu}$ with an explicit two-dimensional density.

Projecting $\overline{\mu}$ in the first coordinate gives us an explicit form of the density of $\mu$. 
We are able to build the desired version of the natural extension for the simple case of two intervals.

**Theorem**

For a simple system \((\Omega, \mathcal{B}_\Omega, \mu, T)\) with \(\Omega = [a_1, a_1 + 1) \cup [a_2, a_2 + 1)\) we define \(X = [a_1, a_1 + 1) \times [a_2, a_2 + 1) \cup [a_2, a_2 + 1) \times [a_1, a_1 + 1]\). The natural extension is given by \((X, \mathcal{B}_X, \bar{\mu}, \bar{T})\) where \(\bar{T} : X \to X\) is defined as

\[
\bar{T}(x, y) = \left( T(x), \frac{N}{d_1(x) + y} \right).
\]

Furthermore, the invariant measure \(\bar{\mu}\) is given by

\[
\bar{\mu}(A) = C \int \int_A \frac{N}{(N + xy)^2} \, dy \, dx,
\]

where \(C^{-1} = \int \int_X \frac{N}{(N + xy)^2} \, dy \, dx = 2 \ln \left[ 1 + \frac{N}{(N + a_1(a_2 + 1))(N + a_2(a_1 + 1))} \right].\)
Natural extension: the simple two interval case

\[ T(a_1 + 1, a_2 + 1) \]

\[ T(\Delta(c_1)) \]

\[ T(\Delta(c_m)) \]

\[ \Delta(d_1) \]

\[ \Delta(d_k) \]

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Projecting the measure $\overline{\mu}$ in the first coordinate gives us the following result.

**Corollary**

Suppose $(\Omega, \mathcal{B}_\Omega, \mu, T)$ is a simple system with
$\Omega = [a_1, a_1 + 1) \cup [a_2, a_2 + 1)$ then the invariant measure $\mu$ is given by

$$\mu(A) = C \int_A \left( \frac{a_2 + 1}{N + (a_2 + 1)x} - \frac{a_2}{N + a_2 x} \right) 1_{[a_1, a_1 + 1)}$$

$$+ \left( \frac{a_1 + 1}{N + (a_1 + 1)x} - \frac{a_1}{N + a_1 x} \right) 1_{[a_2, a_2 + 1)} dx$$

for $A \in \mathcal{B}_\Omega$ and $C$ as in previous theorem.
For \( x \in \Omega \) we define the \( n^{th} \) approximation coefficient by

\[
\theta_n(x) = \frac{q_n^2}{\prod_{i=1}^n N_i} \left| x - \frac{p_n}{q_n} \right|.
\]

By writing \( t_n = T^n(x) \), \( v_n = \frac{N_n}{q_n} q_{n-1} \) and using

\[
x - \frac{p_n}{q_n} = \frac{(-1)^n N_1 \cdots N_n T^n(x)}{q_n (q_n + q_{n-1} T^n(x))},
\]

we have \( \theta_n(x) = \frac{N_n t_n}{N_n + t_n v_n} \) and \( T^n(x, 0) = (t_n, v_n) \).
Approximation coefficients: the simple two interval case

Recall that \( t_n = T^n(x), \ v_n = \frac{N_nq_n-1}{q_n}. \ \theta_n(x) = \frac{N_nt_n}{N_n+t_nv_n} \) and \( T^n(x, 0) = (t_n, v_n). \)

In the simple case \( N = (N, N), \) we have \( \theta_n(x) = \frac{Nt_n}{N + t_nv_n}. \)

For \( c \in \mathbb{R}, \) let \( A_c := \{(x,y) \in X : \frac{Nx}{N+xy} \leq c\}. \)

We have \( \theta_n(x) \leq c \) if and only if \( T^n(x, 0) \in A_c. \)

Using the Ergodic theorem (and that the orbit of \( (x, 0) \) is generic for irrational \( x \)), we have a Doeblin-Lenstra type theorem.

**Theorem**

*In the simple two-interval case*

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ 1 \leq i \leq n : \theta_i(x) \leq c \right\} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{I}_{A_c}(T^i(x, 0)) = \bar{\mu}(A_c).
\]
Let \((\Omega, B_\Omega, \mu, T)\) be a simple system that is ergodic with respect to \(\mu\) an a.c.i.m., and \(B_\Omega\) the Borel \(\sigma\)-algebra on \(\Omega\). Let \(\mathcal{T} : \Omega \times [0, \infty) \to \Omega \times [0, \infty)\) be defined as

\[
\mathcal{T}(x, y) = \left( T(x), \frac{N}{d_1(x) + y} \right).
\]

Now let \(X = \bigcap_{i=0}^{\infty} \mathcal{T}^i(\Omega \times [0, \infty))\). Then \(\overline{\mathcal{T}} := \mathcal{T}|_X\) is bijective Lebesgue almost everywhere and \((X, B_X, \overline{\mu}, \overline{\mathcal{T}})\) is the natural extension of 
\((\Omega, B_\Omega, \mu, T)\) where \(\overline{\mu}\) is an invariant measure that is absolutely continuous with respect to the two dimensional Lebesgue measure given by

\[
\overline{\mu}(A) = C \int \int_A \frac{N}{(N + xy)^2} dydx
\]

with \(A \in B_X\) where \(B_X\) is the Borel \(\sigma\)-algebra restricted to \(X\) and \(C\) a normalising constant.
Why is this a conjecture: unable to prove that \( X = \bigcap_{i=0}^{\infty} T^i(\Omega \times [0, \infty)) \) has positive Lebesgue measure.

We will now numerically support our conjecture. Our algorithm works as follows: we determine the rectangles \( X_n := \bigcap_{i=0}^{n} T^i(\Omega \times [0, \infty)) \). On these rectangles we use the density \( \frac{N}{(N+xy)^2} \), project this to the first coordinate and normalise it to find a smooth approximation for the invariant density.
First, we test our method in the case of two intervals. Let $a_1 = 1$, $a_2 = 2$ and $N = (12, 12)$.

**Figure:** On the left is the new method plotted against the theoretical density (dashed line). On the right is the classical method (following orbits) plotted against the theoretical density (dashed line).
A new example: take $a_1 = 1$, $a_2 = 3$, $a_3 = 2$ and $N = (12, 12, 12)$,
Figure: From left to right, the domains $X_1$, $X_2$ and $X_7$. 
Figure: The simulated density using $X_7$. 
Thank You!