Alternating N-continued fraction expansions

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joint work with Niels Langeveld

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Aim

Inspired by the recent study of the dynamical properties of alternate β-expansions, we introduce the notion of alternate N-continued fraction expansions.

N-continued fraction expansions

In 2008, Edward Burger and his co-authors introduced a generalization of the *regular continued fraction* (RCF) *expansion* of the form

$$x = d_0 + \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_k + \ddots}}} = [d_0; d_1, d_2, \ldots, d_k, \cdots]_N$$

where $N, d_i \in \mathbb{N}$, $N, d_i \neq 0 \, (n \geq 1)$. They called such expansions N-expansions.

Burger et al.

Their main object of study was the N-expansions of quadratic irrationals.

They showed that if x is a quadratic irrational, then there are infinitely many integers N for which x can be expressed as an eventually periodic N-expansion with period length one.

In this case they showed that the sequence of truncations of such an N-expansion forms a subsequence of the sequence of convergents of the regular continued fraction expansion of x.

Anselm and Weintraub

In 2011, M. Anselm and S. Weintraub studied further the properties of N-expansions. They showed:

Every positive real number x has always an N-expansion, and for $N \geq 2$ even infinitely many.

Rationals always have finite and infinite expansions.

In case $N \geq 2$ every quadratic irrational has both eventually periodic and non-periodic expansions.

Dynamical approach

Together with C. Kraaikamp and N. van der Wekken (2013), we took a dynamical approach.

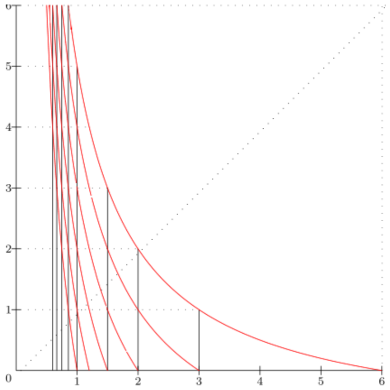


Fig. 1. All possible maps in case $N = 6$.

Dynamical approach

We studied ergodic properties of certain deterministic algorithms.

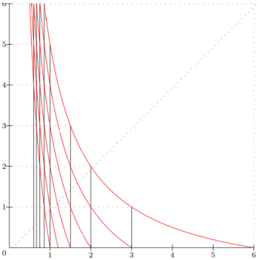


Fig. 1. All possible maps in case $N = 6$.

Choosing the lowest curve, leads to the greedy algorithm. Choosing the highest curve leads to the lazy algorithm. Restrict to a window, leads to expansions with finitely many digits.

Dynamical approach

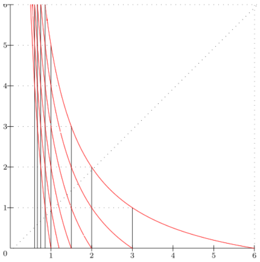


Fig. 1. All possible maps in case $N = 6$.

In 2017, C. Kraaikamp and N. Langeveld concentrated on expansions of points in windows of length 1 to obtain a parametrised family of N-expansions with finitely many digits. For certain values of the parameter they were able to obtain the density of the invariant measure.

In 2018, together with M. Oomen, we studied random algorithms (skew products).

Object of study: Alternating N-expansions

Given non-negative integers $(N_1, \ldots, N_m)$, we would like to study expansions of the form

$$x = \cfrac{N_1}{d_1 + \cfrac{N_2}{d_2 + \cfrac{N_3}{d_3 + \ddots + \cfrac{N_m}{d_m + \cfrac{N_1}{d_{m+1} + \ddots}}}}}$$

Dynamical setup

For $i \in \{1, \ldots, m\}$ let $I_i := [a_i, a_i + 1)$ with $a_i \in \mathbb{N} \cup \{0\}$ be a set of distinct intervals.

We will study continued fraction maps $T : \cup I_i \to \cup I_i$ such that $T(I_i) = I_{(i \mod m)+1}$ for $i \in \{1, \ldots, m\}$.

On every interval I_i we choose an $N_i \in \mathbb{N}$. Let $j(x) = i$ when $x \in I_i$ and $k(x) = (j(x) \mod m) + 1$. Furthermore, let $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$.

Define $T : \cup I_i \to \cup I_i$ as

$$T(x) = \frac{N_{j(x)}}{x} - \left\lfloor \frac{N_{j(x)}}{x} \right\rfloor + a_{k(x)},$$

for $x \neq 0$ and $T(0) = 0$ whenever $0 \in \cup_{i=1}^m I_i$. We refer to T as the **N**-continued fraction map. Note that with this definition some digits might be negative!

Avoiding negative digits: the allowable case

To avoid negative digits, we impose an extra restriction.

Definition (allowable)

Let $I_i := [a_i, a_i + 1)$ with $a_i \in \mathbb{N} \cup \{0\}$ be a set of distinct intervals and $\Omega = \cup_{i=1}^{m} I_i$. We call $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ **allowable** for Ω if for all $x \in \Omega$ we have $\left\lfloor \frac{N_{j(x)}}{x} \right\rfloor - a_{k(x)} > 0$.

Since we can pick the values for $\mathbf{N}$ arbitrarily high, we can find a continued fraction algorithm for every choice of distinct intervals with all possible ways of indexing them.

Many of these choices will give rise to dynamical systems that are difficult to study since not all branches will be full (a branch of the map T will not be mapped entirely onto the next interval). We will introduce two special sub-classes of allowable expansions.

Desirable and simple

Definition (desirable and simple)

Let $I_i := [a_i, a_i + 1)$ with $a_i \in \mathbb{N} \cup \{0\}$ be a set of distinct intervals and $\Omega = \cup_{i=1}^{m} I_i$. We call $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ **desirable** for Ω if it is allowable and for all $i \in \{1, \ldots, m\}$ we have that $a_i \mid N_i$ and $a_i + 1 \mid N_i$ whenever $a_i \neq 0$. Furthermore, we call $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ **simple** if it is desirable and $N_i = N_j$ for all $i, j \in \{1, \ldots, m\}$.

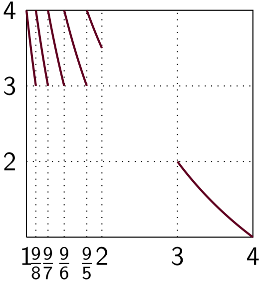
Note that for any set of distinct intervals we can find a simple $\mathbf{N}$ by taking a large enough multiple of the product of the endpoints of the intervals I_i, omitting 0.

Our simple case generalizes the classical case, in the sense that the numerators will not alternate but the sets from which the digits are taken are alternating.

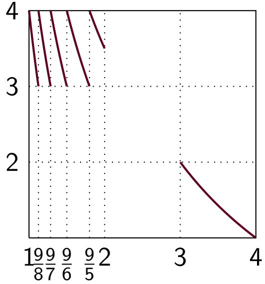
Example: allowable

Let $a_1 = 1$, $a_2 = 3$ and $\mathbf{N} = (9, 12)$. We find that $T : [1,2) \cup [3,4) \to [1,2) \cup [3,4)$ is given by

$$T(x) = \begin{cases} \frac{9}{x} - \lfloor \frac{9}{x} \rfloor + 3 & \text{for } x \in [1,2), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 1 & \text{for } x \in [3,4), \end{cases}$$



Example: allowable



We see that, since $3 \mid 12$ and $4 \mid 12$ we have full branches on $[3,4)$ but since $2 \nmid 9$ we do not have full branches on $[1,2)$. Therefore, this choice of $\mathbf{N}$ is allowable but not desirable.

Example: allowable

At the left end point of any of the intervals, T is not continuous giving us an extra digit in case the left point is not equal to 0.

Setting $d_1(x) = \lfloor \frac{N_{j(x)}}{x} \rfloor - a_{k(x)}$ and $d_n(x) = d_1(T^{n-1}(x))$, iterations of T give the following expressions.

Example: allowable

For $x \in [1, 2)$,

$$x = \cfrac{9}{d_1(x) + \cfrac{12}{d_2(x) + \cfrac{9}{\ddots}}},$$

with $d_i \in \{1, \ldots, 6\}$ for i odd (only 6 when $T^{i-1}(x) = 1$) and $d_i \in \{2, 3\}$ for i even (only 3 when $T^{i-1}(x) = 3$). For $x \in [3, 4)$ we find

$$x = \cfrac{12}{d_1(x) + \cfrac{9}{d_2(x) + \cfrac{12}{\ddots}}},$$

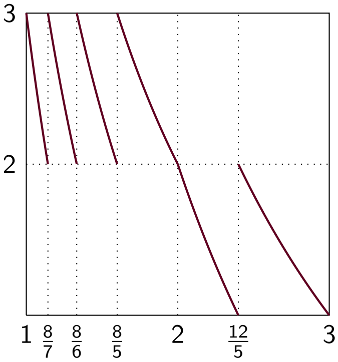
with $d_i \in \{2, 3\}$ for i odd and $d_i \in \{1, \ldots, 6\}$ for i even.

Example: desirable

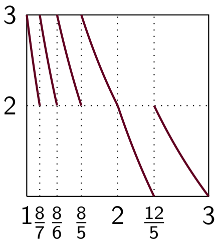
Let $a_1 = 1, a_2 = 2$ and $\mathbf{N} = (8, 12)$. Then we can define $T : [1,3) \to [1,3)$ as

$$T(x) = \begin{cases} \frac{8}{x} - \lfloor \frac{8}{x} \rfloor + 2 & \text{for } x \in [1,2), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 1 & \text{for } x \in [2,3), \end{cases}$$

That this choice of $\mathbf{N}$ is desirable.



Example: desirable



For $x \in [1,2)$ we find the continued fraction

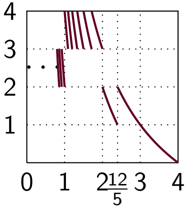
$$x = \cfrac{8}{d_1 + \cfrac{12}{d_2 + \cfrac{8}{\ddots}}},$$

where $d_i \in \{2,3,4,5,6\}$ for i odd (only 6 when $T^{i-1}(x) = 1$) and $d_i \in \{3,4,5\}$ for i even (only 5 when $T^{i-1}(x) = 2$).

Example: simple

Let $a_1 = 0$, $a_2 = 2$, $a_3 = 1$, $a_4 = 3$. This time we choose $\mathbf{N} = (12, 12, 12, 12)$ so that we are in the simple case. Note that since there is an i such that $a_i = 0$ we have infinitely many branches. We find that $T : [0,4) \to [0,4)$ is now defined as

$$T(x) = \begin{cases} \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 2 & \text{for } x \in [0,1), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 3 & \text{for } x \in [1,2), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor + 1 & \text{for } x \in [2,3), \\ \frac{12}{x} - \lfloor \frac{12}{x} \rfloor & \text{for } x \in [3,4), \end{cases}$$



Example: simple

For $x \in (0,1)$ we find

$$x = \cfrac{12}{d_1 + \cfrac{12}{d_2 + \cfrac{12}{\ddots}}},$$

where

$$d_i(x) \in \begin{cases} \mathbb{N}_{\geq 10} & \text{for } i = 1 \mod 4, \\ \{3,4,5\} & \text{for } i = 2 \mod 4, \\ \{3,\dots,9\} & \text{for } i = 3 \mod 4, \\ \{3,4\} & \text{for } i = 0 \mod 4. \end{cases}$$

Here $d_i(x) = 5$ for $i = 2 \mod 4$ only when $T^{i-1}(x) = 2$, $d_i(x) = 9$ for $i = 3 \mod 4$ only when $T^{i-1}(x) = 1$ and $d_i(x) = 4$ for $i = 0 \mod 4$ only when $T^{i-1}(x) = 3$.

Convergence

Suppose $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ is allowable for $\Omega = \cup_{i=1}^{m} [a_i, a_{i+1})$. We write $N_n = N_{(n \mod m} + 1)$. If $x \in \Omega$, after n iterations of the map T we have

$$x = \cfrac{N_1}{d_1 + \cfrac{N_2}{d_2 + \ddots + \cfrac{N_n}{d_n + T^n x}}}$$

Convergence

Given $x \in \Omega$, we denote by $\frac{p_n(x)}{q_n(x)} = \frac{p_n}{q_n}$ the rational number obtained by considering the first n terms of the expansion,

$$\frac{p_n}{q_n} = \cfrac{N_1}{d_1 + \cfrac{N_2}{d_2 + \ddots + \cfrac{N_n}{d_n}}}$$

To prove that

$$\lim_{n\to\infty} |x - \frac{p_n}{q_n}| = 0$$

we use a classical approach.

Convergence

With the help of Möbius transformations, we derive results similar to those of the regular continued fractions.
We define

$$B_{d,N} = \begin{bmatrix} 0 & N \\ 1 & d \end{bmatrix}.$$

$M_n(x) = M_n = B_{d_1,N_1} B_{d_2,N_2} \dots B_{d_n,N_n}$

$\det(M_n) = (-1)^n \prod_{i=1}^{n} N_i$

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}.$$

Convergence

With the help of the matrices M_n, we have

$$p_{n-1}q_n - q_{n-1}p_n = (-1)^n \prod_{i=1}^{n} N_i.$$

$$\begin{aligned} p_{-1} &= 1 \quad & p_0 &= 0 \quad & p_n &= d_n p_{n-1} + N_n p_{n-2}, \\ q_{-1} &= 0 \quad & q_0 &= 1 \quad & q_n &= d_n q_{n-1} + N_n q_{n-2}. \end{aligned}$$

$$x = \frac{p_n + p_{n-1}T^n(x)}{q_n + q_{n-1}T^n(x)}.$$

Convergence

$$x - \frac{p_n}{q_n} = \frac{(-1)^n N_1 \cdots N_n T^n(x)}{q_n(q_n + q_{n-1} T^n(x))}.$$

$$|x - \frac{p_n}{q_n}| < \frac{N_1 \cdots N_n \max \Omega}{q_n^2}.$$

Using the recurrsion relations, we show that

$$\lim_{n\to\infty} \frac{N_1 \cdots N_n}{q_n^2} = 0.$$

Ergodic properties

To prove the existence of an ergodic absolutely continuous T-invariant measure, we use a result of R. Zweimüller.

Ergodic properties

Theorem

(Zweimüller) Let $T : [0,1] \to [0,1]$*, and let* B *be a collection (not necessarily finite) of nonempty pairwise disjoint open subintervals with* $\lambda(\bigcup B) = 1$ *such that* T *restricted to each element* Z *of* B *is continuous and strictly monotone. Suppose* T *satisfies the following three conditions:*

(A) Adler's condition: $T''/(T')^2$ *is bounded on* $\bigcup B$*,*

(B) Finite image condition: $TB = \{TZ : Z \in B\}$ *is finite,*

(C) Uniformly eventually expanding: there is a k *such that* $|(T^k)'| > \gamma > 1$ *on* $\bigcup B$*.*

Then there are a finite number of pairwise disjoint open sets $X_1, \cdots, X_n$ *such that* $TX_i = X_i$ *(modulo sets of* λ*-measure zero) and* $T|_{X_i}$ *is conservative and ergodic with respect to* λ*. Almost all points of* $[0,1] \setminus \bigcup_i X_i$ *are eventually mapped into one of these ergodic components. Furthermore, each* X_i *supports an absolutely continuous invariant measure* μ_i *which is unique up to a constant factor.*

Ergodic properties

Theorem

Let $\Omega = \bigcup_{i=1}^{m} I_i$ where $I_i := [a_i, a_i + 1)$, with $a_i \in \mathbb{N} \cup \{0\}$, are distinct intervals. Assume $\mathbf{N} = (N_1, \ldots, N_m) \in \mathbb{N}^m$ is allowable for Ω and $T : \Omega \to \Omega$ the corresponding $\mathbf{N}$-continued fraction map. Then, T admits a unique absolutely continuous invariant ergodic measure.

We verify the conditions of Zweimüller's Theorem:

Adler's condition: $T''/(T')^2 < 2$

Finite image condition: at most $2m$ possible images

Uniformly eventually expanding: $|(T^k)'| > 4$ for $k \geq 2$

Ω is the smallest forward invariant set.

Invariant density

The above theorem gives the existence of a T-invariant measure μ of the form

$$\mu(A) = \int f \, d\lambda,$$

where f is in $L^1(\Omega)$ and λ is Lebesgue.

Can we find the explicit form of f?

A useful tool in finding the explicit form is to use the notion of natural extensions.

Natural extension

A natural extension of a dynamical system $(\Omega, \mathcal{B}, T, \mu)$ is the *"smallest"* invertible dynamical system containing $(\Omega, \mathcal{B}, T, \mu)$ as a factor.

A natural extension is unique up to an isomorphism

Rohlin (1961) gave a canonical construction of a natural extension that resembles the idea of converting a one-sided shift to a two-sided shift.

Natural extension: which version?

Given an allowable $\mathbf{N}$-continued fraction map T defined on $(\Omega, \mathcal{B}, \mu)$, we want to find a version $\mathcal{T}$ of the natural extension of T with the following properties:
$\mathcal{T}$ is defined on a domain $X \subset \mathbb{R}^2$ of positive Lebesgue measure

$$\mathcal{T}(x,y) = \left(T(x), \frac{N_{j(x)}}{d_1(x) + y} \right).$$

There exists an absolutely continuous $\mathcal{T}$-invariant measure $\overline{\mu}$ with an explicit two-dimensional density.

Projecting $\overline{\mu}$ in the first coordinate gives us an explicit form of the density of μ.

Natural extension: the simple two interval case

We are able to build the desired version of the natural extension for the simple case of two intervals.

Theorem

For a simple system $(\Omega, \mathcal{B}_\Omega, \mu, T)$ *with* $\Omega = [a_1, a_1+1) \cup [a_2, a_2+1)$ *we define* $X = [a_1, a_1+1) \times [a_2, a_2+1] \cup [a_2, a_2+1) \times [a_1, a_1+1]$*. The natural extension is given by* $(X, \mathcal{B}_X, \overline{\mu}, \overline{\mathcal{T}})$ *where* $\overline{\mathcal{T}} : X \to X$ *is defined as*

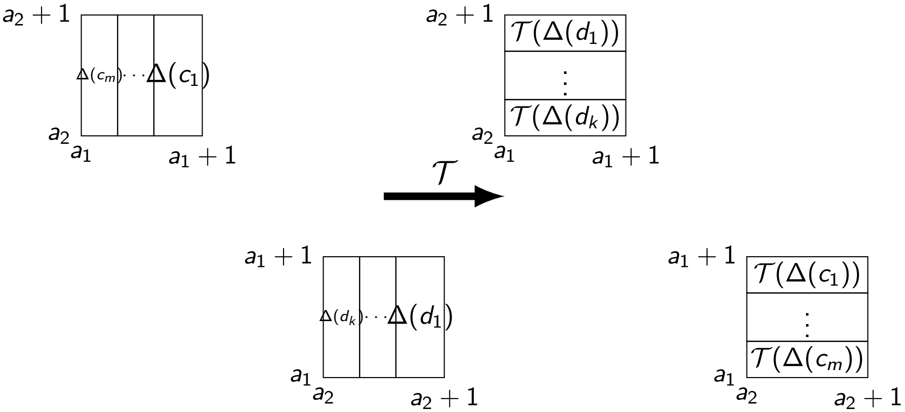
$$\overline{\mathcal{T}}(x,y) = \left(T(x), \frac{N}{d_1(x)+y} \right).$$

Furthermore, the invariant measure $\overline{\mu}$ *is given by*

$$\overline{\mu}(A) = C \iint\limits_A \frac{N}{(N+xy)^2} dydx,$$

where $C^{-1} = \iint\limits_X \frac{N}{(N+xy)^2} dydx = 2 \ln \left[1 + \frac{N}{(N+a_1(a_2+1))(N+a_2(a_1+1))} \right]$.

Natural extension: the simple two interval case



Explicit invariant density: the simple two interval case

Projecting the measure $\overline{\mu}$ in the first coordinate gives us the following result.

Corollary

Suppose $(\Omega, \mathcal{B}_\Omega, \mu, T)$ *is a simple system with* $\Omega = [a_1, a_1+1) \cup [a_2, a_2+1)$ *then the invariant measure* μ *is given by*

$$\mu(A) = C \int_A \left(\frac{a_2+1}{N+(a_2+1)x} - \frac{a_2}{N+a_2x} \right) \mathbf{1}_{[a_1,a_1+1)} + \left(\frac{a_1+1}{N+(a_1+1)x} - \frac{a_1}{N+a_1x} \right) \mathbf{1}_{[a_2,a_2+1)} dx$$

for $A \in \mathcal{B}_\Omega$ *and* C *as in previous theorem.*

Approximation coefficients

For $x \in \Omega$ we define the n^{th} approximation coefficient by

$$\theta_n(x) = \frac{q_n^2}{\prod_{i=1}^n N_i} \left| x - \frac{p_n}{q_n} \right|.$$

By writing $t_n = T^n(x)$, $v_n = \frac{N_n q_{n-1}}{q_n}$ and using

$$x - \frac{p_n}{q_n} = \frac{(-1)^n N_1 \cdots N_n T^n(x)}{q_n(q_n + q_{n-1} T^n(x))},$$

we have $\theta_n(x) = \frac{N_n t_n}{N_n + t_n v_n}$ and $\mathcal{T}^n(x, 0) = (t_n, v_n)$.

Approximation coefficients: the simple two interval case

Recall that $t_n = T^n(x)$, $v_n = \frac{N_n q_{n-1}}{q_n}$. $\theta_n(x) = \frac{N_n t_n}{N_n + t_n v_n}$ and $\mathcal{T}_T^n(x, 0) = (t_n, v_n)$.

In the simple case $\mathbf{N} = (N, N)$, we have $\theta_n(x) = \dfrac{N t_n}{N + t_n v_n}$

For $c \in \mathbb{R}$, let $A_c := \{(x, y) \in X : \frac{Nx}{N + xy} \leq c\}$.

We have $\theta_n(x) \leq c$ if and only if $\mathcal{T}^n(x, 0) \in A_c$.

Using the Ergodic theorem (and that the orbit of $(x, 0)$ is generic for irrational x), we have a Doeblin-Lenstra type theorem.

Theorem

In the simple two-interval case

$$\lim_{n \to \infty} \frac{1}{n} \{1 \leq i \leq n : \theta_i(x) \leq c\} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{I}_{A_c}(\overline{\mathcal{T}}^i(x, 0)) = \overline{\mu}(A_c).$$

Conjecture

Let $(\Omega, \mathcal{B}_\Omega, \mu, T)$ be a simple system that is ergodic with respect to μ an a.c.i.m., and $\mathcal{B}_\Omega$ the Borel σ-algebra on Ω. Let $\mathcal{T} : \Omega \times [0, \infty) \to \Omega \times [0, \infty)$ be defined as

$$\mathcal{T}(x, y) = \left(T(x), \frac{N}{d_1(x) + y} \right).$$

Now let $X = \bigcap_{i=0}^{\infty} \mathcal{T}^i(\Omega \times [0, \infty))$. Then $\overline{\mathcal{T}} := \mathcal{T}|_X$ is bijective Lebesgue almost everywhere and $(X, \mathcal{B}_X, \overline{\mu}, \overline{\mathcal{T}})$ is the natural extension of $(\Omega, \mathcal{B}_\Omega, \mu, T)$ where $\overline{\mu}$ is an invariant measure that is absolutely continuous with respect to the two dimensional Lebesgue measure given by

$$\overline{\mu}(A) = C \iint_A \frac{N}{(N + xy)^2} dydx$$

with $A \in \mathcal{B}_X$ where $\mathcal{B}_X$ is the Borel σ-algebra restricted to X and C a normalising constant.

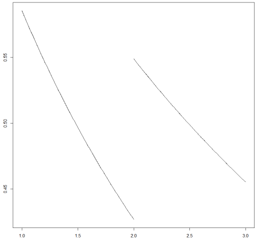
Numerical justification

Why is this a conjecture: unable to prove that $X = \bigcap_{i=0}^{\infty} \mathcal{T}^i(\Omega \times [0, \infty))$ has positive Lebesgue measure.

We will now numerically support our conjecture. Our algorithm works as follows: we determine the rectangles $X_n := \bigcap_{i=0}^{n} \mathcal{T}^i(\Omega \times [0, \infty))$. On these rectangles we use the density $\frac{N}{(N+xy)^2}$, project this to the first coordinate and normalise it to find a smooth approximation for the invariant density.

Numerical justification

First, we test our method in the case of two intervals. Let $a_1 = 1$, $a_2 = 2$ and $\mathbf{N} = (12, 12)$.



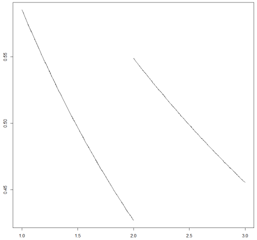
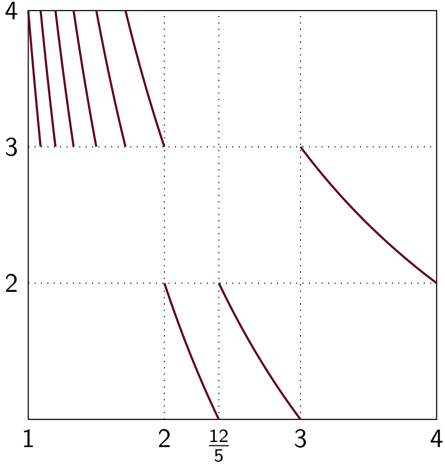


Figure: On the left is the new method plotted against the theoretical density (dashed line). On the right is the classical method (following orbits) plotted against the theoretical density (dashed line).

Numerical justification

A new example: take $a_1 = 1$, $a_2 = 3$, $a_3 = 2$ and $\mathbf{N} = (12, 12, 12)$,



Numerical justification

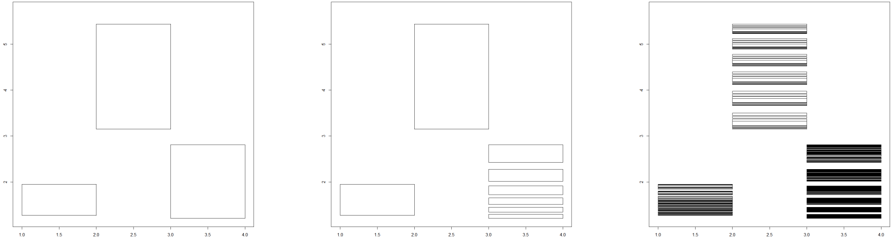


Figure: From left to right, the domains X_1, X_2 and X_7.

Numerical justification

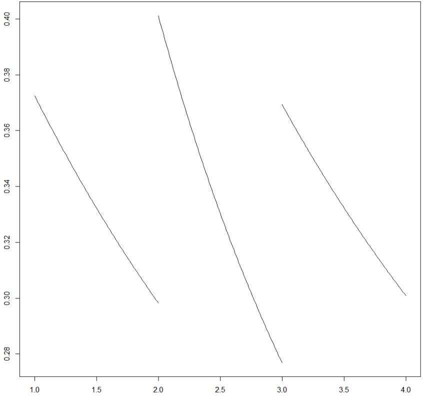


Figure: The simulated density using X_7.

Thank You!