(Logarithmic) Densities for Automatic Sequences along Primes and Squares

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joint work with Boris Adamczewski and Clemens Müllner

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Summary

- ★ Thue-Morse and Rudin-Shapiro sequence
- ★ Thue-Morse sequence along subsequences
- ★ Automatic sequences
- ★ Logarithmic densities along subsequences
- ★ Automatic sequences along primes
- ★ Automatic sequences along squares

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

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Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

0

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

01

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n \geqslant 0}$:

0110

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n \geqslant 0}$:

01101001

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

0110100110010110

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

0110100110010110100101100110 1001

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

$$011010011001011010010110011010011001011001101\cdots$$

$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n)$$

$$\boxed{t(n) = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

★ Thue-Morse sequence

Thue-Morse sequence $(t(n))_{n \geqslant 0}$:

$$01101001100101101001011001101001100101100110 1 \cdots$$

$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n) \qquad \text{or} \qquad t_{2k} = t_k,\ t_{2k+1} = 1 - t_k$$

$$\boxed{t(n) = s_2(n) \bmod 2}$$

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$$\#\{0 \leqslant n < N : t(n) = 0\} \sim \frac{N}{2}$$

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Thue-Morse sequence $(t(n))_{n\geqslant 0}$:

$$01101001100101101001011001101001100101100110 1\cdots$$

$$\#\{0\leqslant n<N : t(n)=0\}\sim\frac{N}{2}$$

The letters 0 and 1 appear with asymptotic frequency $\frac{1}{2}$:

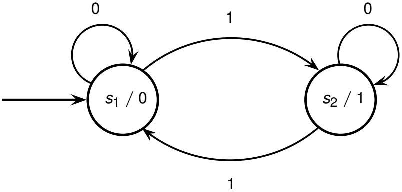
$$\mathrm{dens}(t(n),0)=\mathrm{dens}(t(n),1)=\frac{1}{2}.$$

★ Thue-Morse sequence

- TM sequence is **not periodic** and **cubeless**.
- TM sequence is **almost periodic**:
  Every appearing consecutive block appears infinitely many times with bounded gaps.
- **Subword complexity is linear**: $p_k \leqslant \frac{10}{3} k$
  p_k ... subword complexity (*number of different consecutive blocks of length k that appear in the TM sequence*).
- **Zero topological entropy** of the corresponding dynamical system:
  $$h = \lim_{k\to\infty} \frac{1}{k} \log p_k = 0$$
- **Linear subsequences** $(t_{an+b})_{n\geqslant 0}$ have the same properties.
- The TM sequence and its linear subsequences are **automatic sequences**.

★ Thue-Morse sequence

Automaton that generates the Thue-Morse sequence:
$t(n) = \sum_{j \geqslant 0} \varepsilon_j(n) \bmod 2$



★ Rudin-Shapiro sequence

Rudin-Shapiro sequence $(r(n))_{n\geqslant 0}$:

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$$r_0 = 0, \qquad r_{2k} = r_k, \qquad r_{2k+1} = \begin{cases} r_k & \text{if } k \text{ is even,} \\ 1 - r_k & \text{if } k \text{ is odd.} \end{cases}$$

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$$\boxed{r(n) = \sum_{i\geqslant 0} \varepsilon_i(n)\varepsilon_{i+1}(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}$$

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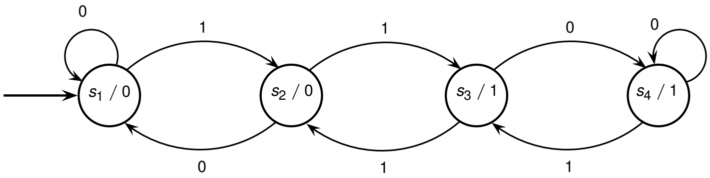

★ Thue-Morse sequence along primes

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★ Thue-Morse sequence along primes

Thue-Morse sequence $(t(n))_{n \geqslant 0}$:

10 0 1 1 1 0 1 0 0 1 1 1 0 $\cdots$

Mauduit and Rivat (2010):

$$\# \{0 \leqslant n < N : t(p_n) = 0\} \sim \frac{N}{2} \quad \text{or} \quad \mathrm{dens}(t(p_n), 0) = \frac{1}{2}.$$

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Solution of a **Conjecture of Gelfond** (1968)

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The same property holds for the Rudin-Shapiro sequence $r(n)$ (Mauduit and Rivat (2015)).

★ Thue-Morse sequence along squares

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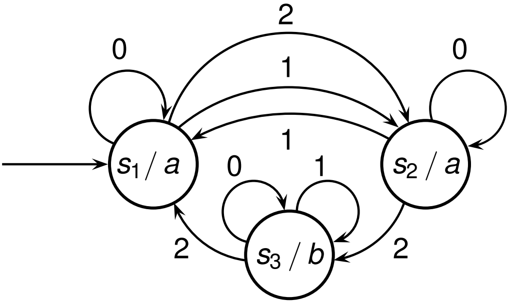
The same property holds for the Rudin-Shapiro sequence $r(n)$ (Mauduit and Rivat (2018)).

★ Automatic sequences

Definition

A sequence $(a(n))_{n\geqslant 0}$ is called a *q-automatic sequence*, if $a(n)$ is the output of an automaton when the input is the q-ary expansion of n.



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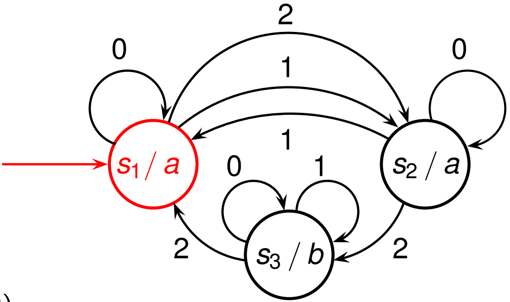

$32 = (1012)_3$

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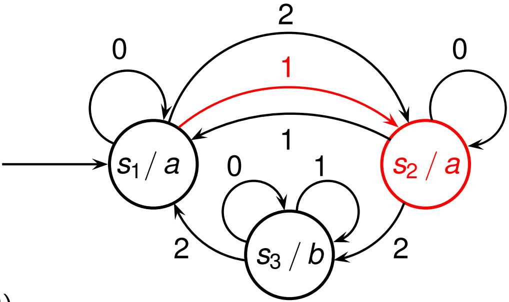

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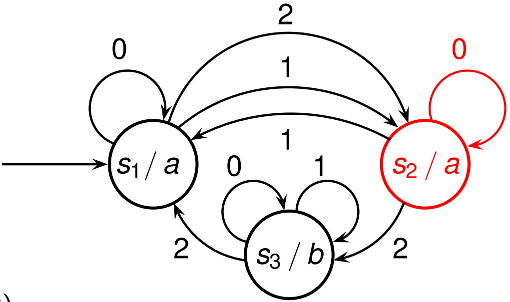

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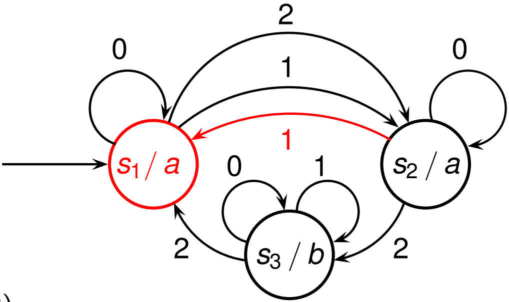

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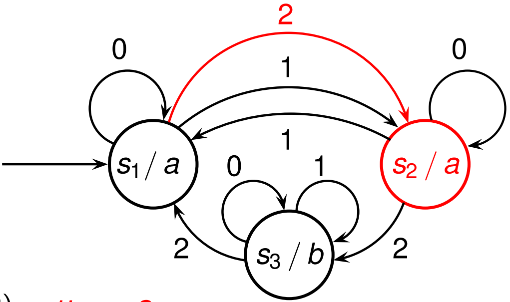

$32 = (1012)_3 \quad u_{32} = a,$

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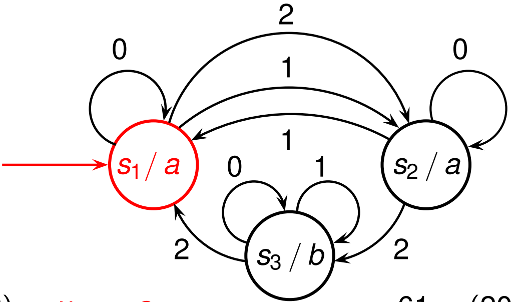

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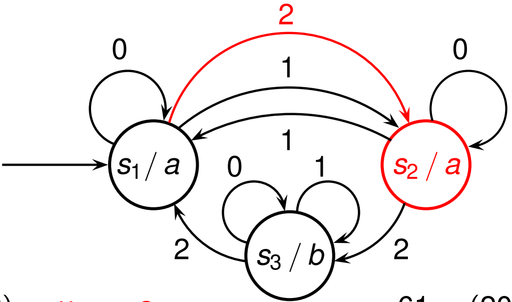

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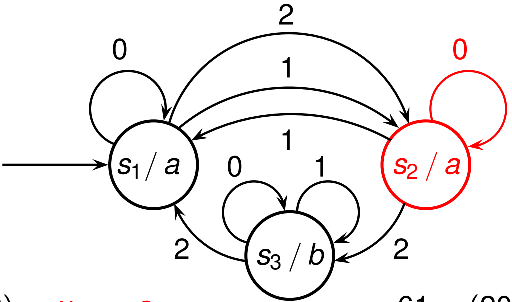

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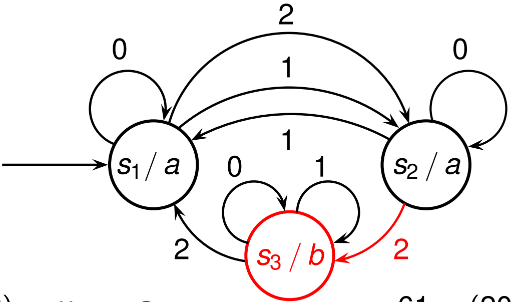

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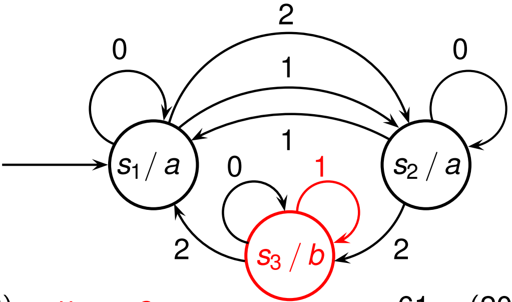

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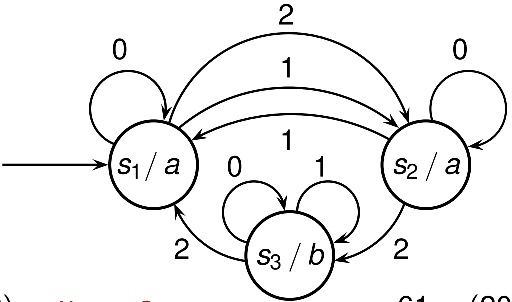
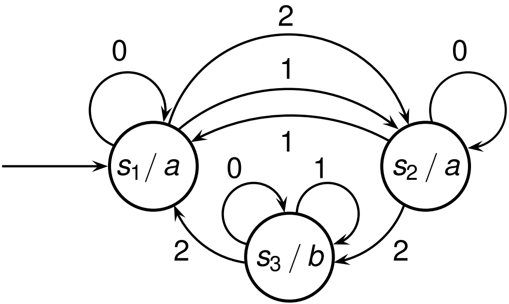
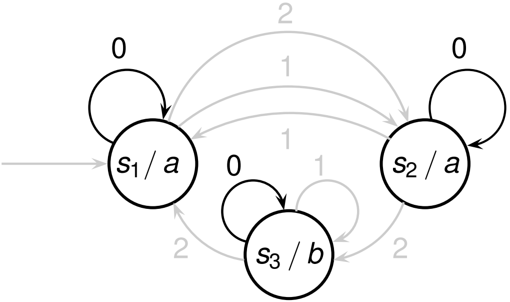


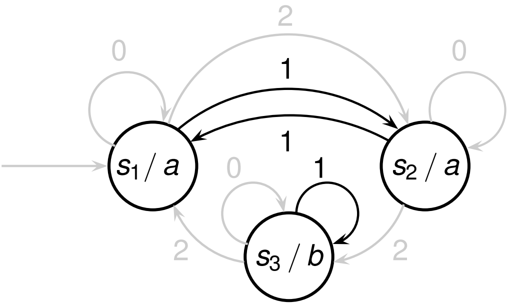
$32 = (1012)_3 \quad u_{32} = a$, $\qquad$ $61 = (2021)_3 \quad u_{61} = b$

$(a(n))_{n\geqslant 0}$: $aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba\ldots$


2
0
1
0
1
s_1 / a
0
1
s_2 / a
2
s_3 / b
2



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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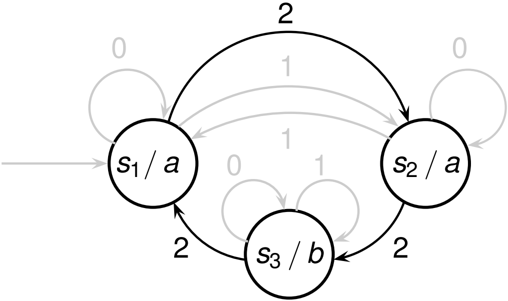

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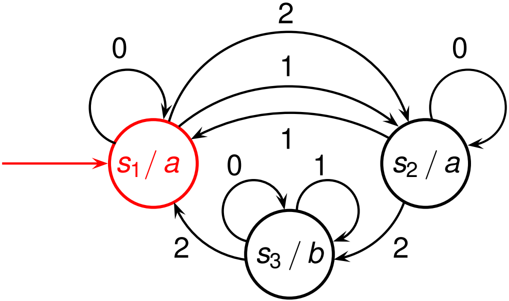

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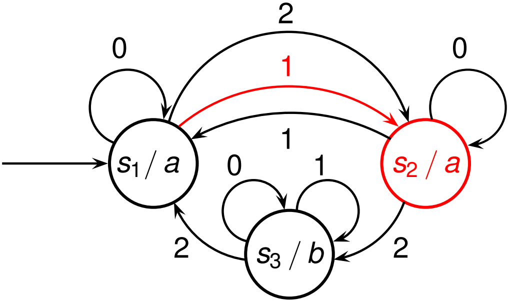

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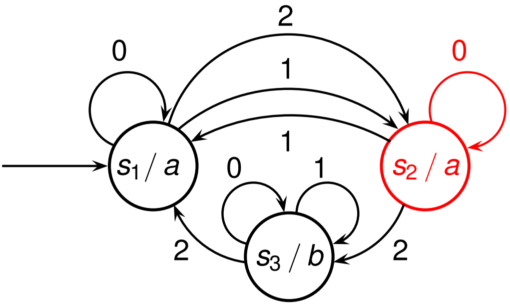
$$M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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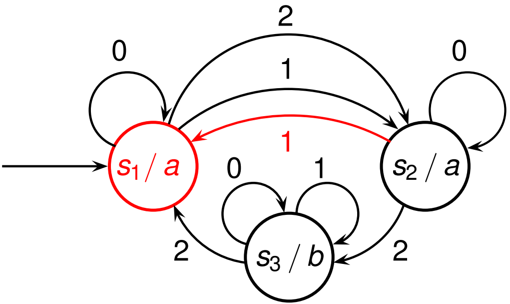
$$32 = (1012)_3 : \qquad M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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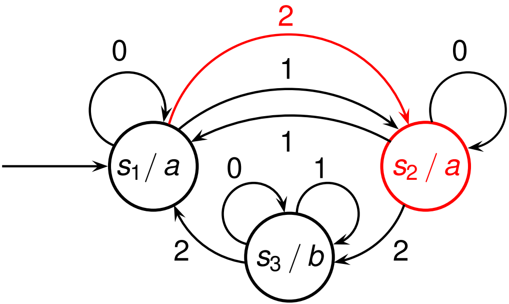

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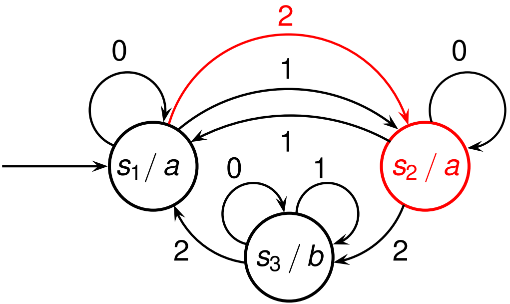
$$32 = (1012)_3 : \qquad M_2 \circ M_1 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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$$\boxed{S(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}}$$

$$\boxed{a(n) = f(S(n)\mathbf{e}_1)} \qquad \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$$

★ Automatic sequences

- For every q-automatic sequence $a(n)$ (on an alphabet $\mathcal{A}$) there exists the **logarithmic density** (for every letter $\alpha \in \mathcal{A}$)

$$\mathrm{logdens}(a(n),\alpha) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{1\leqslant n\leqslant N} \frac{1}{n}\cdot \mathbf{1}_{[a(n)=\alpha]}$$

  which is also computable.
- If the **densities**

$$\mathrm{dens}(a(n),\alpha) = \lim_{N\to\infty} \frac{1}{N}\#\{n\leqslant N : a(n)=\alpha\}$$

  exist then they coincide with the logarithmic densities.
- It can be effectively decided whether densities exist (as for the Thue-Morse sequence $t(n)$ and the Rudin-Shapiro sequence $r(n)$).
- The matrix $M = M_0 + \cdots + M_{q-1}$ plays an important role:

$$\sum_{n<q^\lambda} S(n) = \sum_{n<q^\lambda} M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)} = \left(M_0 + \cdots + M_{q-1}\right)^\lambda$$

★ Automatic sequences

We will say that an automatic sequence is **primitive and prolongable** if the directed graph of the corresponding minimal automaton is strongly connected and the initial state has a 0-labeled loop.

Lemma

For every primitive and prolongable automatic sequence $a(n)$ the densities

$$\mathrm{dens}(a(n),\alpha) = \lim_{N\to\infty} \frac{1}{N} \#\{n \leqslant N : a(n) = \alpha\}$$

exist

Proof with matrix product representation and Perron-Frobenius theory.

★ Automatic sequences

Proof

$\mathbf{b}(n) = S(n)\mathbf{e}_1$. The j-component of $\mathbf{b}(n)$ equals 1 if we stop in state j.

$M = M_0 + \cdots + M_{q-1}$ is primitive matrix;
q is dominant eigenvalue;
P projection to dominant eigen-direction.

$$\sum_{n<q^\lambda} \mathbf{b}(n) = M^\lambda \mathbf{e}_1 = q^\lambda P\mathbf{e}_1 + O(q^{\lambda(1-\varepsilon)})$$

$\Longrightarrow \quad P\mathbf{e}_1 \quad$ is vector of (rational) frequencies.

★ Automatic sequences

Example. Leading digit, $q \geqslant 3$

$a(n)$ ... leading digits of n in the q-ary expansion.

$$a(n) = \ell \quad \Longleftrightarrow \quad \ell q^k \leqslant n < (\ell+1)q^k \quad \text{for some } k \geqslant 0$$

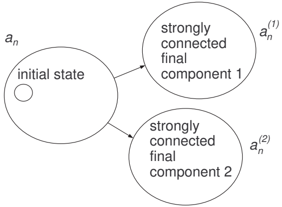
$$\frac{1}{\ell q^k} \#\{n < \ell q^k : a(n) = \ell\} = \frac{1}{\ell q^k} \frac{q^k - 1}{q-1} \sim \frac{1}{\ell(q-1)}$$

$$\frac{1}{(\ell+1)q^k} \#\{n < (\ell+1)q^k : a(n) = \ell\} = \frac{1}{(\ell+1)q^k} \frac{q^{k+1} - 1}{q-1} \sim \frac{q}{(\ell+1)(q-$$

Hence, no densities exist!!

$$\sum_{\ell q^k \leqslant n < (\ell+1)q^k} \frac{1}{n} \sim \log\left(1 + \frac{1}{\ell}\right) \quad \Longrightarrow \quad \mathrm{logdens}(a(n), \ell) = \log_q\left(1 + \frac{1}{\ell}\right)$$

★ Automatic sequences



Suppose that $a_n^{(1)}$ and $a_n^{(2)}$ are produced by the two final components.

- If the densities for $a_n^{(1)}$ and $a_n^{(2)}$ exist and are equal then a_n has the same densities.
- If the densities for $a_n^{(1)}$ and $a_n^{(2)}$ exist but are not equal then a_n has logarithmic densities.

★ Automatic sequences

Theorem

For every automatic sequence $a(n)$ the logarithmic densities $\mathrm{logdens}(a(n),\alpha)$ exist and are computable.
Furthermore, if the densities of those automatic sequences that correspond to the final strongly connected components coincide then the densities $\mathrm{dens}(a(n),\alpha)$ exist and are computable rational numbers.

★ Automatic sequences along primes

Theorem (Adamczewski+D.+Müllner 2020+)

For every automatic sequence $a(n)$ the logarithmic densities $\mathrm{logdens}(a(p_n),\alpha)$ of the subsequence along **prime numbers** *exist and are computable.*
Furthermore, if the densities along primes on those automatic sequences that correspond to the final strongly connected components coincide then the densities $\mathrm{dens}(a(p),\alpha)$ exist and are computable rational numbers.

Theorem (Adamczewski+D.+Müllner 2020+)

For any automatic sequence $a(n)$ there exists a computable positive integer m such that, for all α, $\mathrm{logdens}(a(p_n),\alpha)$ is equal to the logarithmic density of $a(n)$ along the integers n satisfying $(n,m)=1$.

★ Automatic sequences along squares

Theorem (Adamczewski+D.+Müllner 2020+)

For every automatic sequence $a(n)$ the logarithmic densities $\mathrm{logdens}(a(n^2), \alpha)$ of the subsequence along **squares** *exist and are computable.*
Furthermore, if the densities along squares on those automatic sequences that correspond to the final strongly connected components coincide then the densities $\mathrm{dens}(a(n^2), \alpha)$ exist and are also computable. If the input base q is prime, then these densities are rational numbers.

★ Automatic sequences along primes squares

Equivalently one can say that for all automatic sequences $a(n)$ the following two limits exist:

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N} \frac{1}{n} a(n)\Lambda(n) \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N} \frac{1}{n} a(n^2).$$

And it can be decided when the logarithmic means can be replaced by the usual means:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} a(n)\Lambda(n) \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} a(n^2).$$

$\Lambda(n)$ denotes the von Mangoldt Λ-function
($\Lambda(n) = \log p$ for prime powers $n = p^k$ and $\Lambda(n) = 0$ else.)

★ Paper folding sequence

$$(a(n)) = 1|1|0|1|100|1|1100100|\cdots$$

- **primes**: $m = 2$ (it suffices to consider odd numbers)

$$\mathrm{dens}(a(p_n), 0) = \mathrm{dens}(a(p_n), 1) = \frac{1}{2}$$

- **squares**:

$$\mathrm{dens}(a(n^2), 1) = 1, \qquad \mathrm{dens}(a(n^2), 0) = 0.$$

★ Automatic sequences along subsequences

We call a strictly increasing subsequences $(n_\ell)_{\ell\geqslant 0}$ of the positive integers **regularly varying sequences**, if

$$\boxed{n_\ell = \ell^\gamma L(\ell)},$$

where $\gamma \geqslant 1$ and $L(n)$ is slowly varying in the sense that

$$\lim_{\ell\to\infty} \frac{L(\lceil \delta\ell \rceil)}{L(\ell)} = 1 \qquad (\text{for all } 0 < \delta < 1)$$

The sequence of **primes**, **polynomial sequences**, and **Piatetski-Shapiro sequences** (*i.e.,* $\lfloor n^c \rfloor$, where $c > 1$) are regularly varying sequences.

★ Automatic sequences along subsequences

Theorem (Adamczewski+D.+Müllner 2020+)

Suppose that $(n_\ell)_{\ell \geqslant 0}$ *is a regularly varying sequence and suppose that for any primitive and prolongable automatic sequence* $\tilde{a}(n)$ *the densities along the subsequence* (n_ℓ) *exit:*

$$\mathrm{dens}(\tilde{a}(n_\ell), \alpha) := \lim_{N\to\infty} \frac{\{\ell \leqslant N : \tilde{a}(n_\ell) = \alpha\}}{N}$$

Then the two following properties hold.

(i) *Then for every automatic sequence* $a(n)$ *the logarithmic densities* $\mathrm{logdens}(a(n_\ell), \alpha)$ *exist and can be explicitly computed.*

(ii) *Furthermore, if the densities along the subsequence* n_ℓ *corresponding to those automatic sequences that are generated by the final strongly connected components of the directed graph are all equal then the densities* $\mathrm{dens}(a(n_\ell), \alpha)$ *exist and are equal to them.*

★ Automatic sequences along subsequences

Proposition (Müllner 2017)

Let $a(n)$ be a prolongable and primitive automatic sequence. Then the density of $a(n) = \alpha$ exist along the subsequence of **primes**.

Proposition

Let $a(n)$ be a prolongable and primitive automatic sequence. Then the density of $a(n) = \alpha$ exist along the subsequence of **squares**.

★ Reduction of automatic sequences

Lemma (Müllner 2017)

Let $a(n)$ be a primitive and prolongable q-automatic sequences. Then it can be represented in the form

$$a(n) = f(s(n), T(n)),$$

where $s(n)$ is a pure synchronizing q-automatic sequence and $T(n)$ takes values in a finite group G with the following property. For every $j < q$ and every q there exists $g_{j,q} \in G$ such that

$$T(n \cdot q + j) = T(n) \cdot g_{j,s(n)}$$

holds for all $n \in \mathbb{N}$.

The synchronizing part can be handled with the help of **residue class** considerations, however, for the group structure we need a **representation theoretic analysis.**

★ Carry propery

$\mathbb{U}_d$ ... set of unitrary $d \times d$ matrices, $f_\lambda(n) = f(n \bmod q^\lambda)$

Definition

A function $f : \mathbb{N} \to \mathbb{U}_d$ has the *Carry property* if there exists $\eta > 0$ such that uniformly for $(\lambda, \alpha, \rho) \in \mathbb{N}^3$ with $\rho < \lambda$, the number of integers $0 \le \ell < k^\lambda$ such that there exists $(n_1, n_2) \in \{0, \ldots, q^\alpha - 1\}^2$ with

$$f(\ell q^\alpha + n_1 + n_2)^H f(\ell q^\alpha + n_1) \ne f_{\alpha+\rho}(\ell q^\alpha + n_1 + n_2)^H f_{\alpha+\rho}(\ell q^\alpha + n_1)$$

is at most $O(q^{\lambda - \eta\rho})$ where the implied constant may depend only on q and f.

★ Fourier propery

Definition

Given a non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{\lambda\to\infty} \gamma(\lambda) = +\infty$ and $c > 0$ we let $F_{\gamma,c}$ denote the set of functions $f : \mathbb{N} \to \mathbb{U}_d$ such that for $(\alpha, \lambda) \in \mathbb{N}^2$ with $\alpha \le c\lambda$ and $t \in \mathbb{R}$:

$$\left\| q^{-\lambda} \sum_{u<q^\lambda} f(uq^\alpha)\mathrm{e}(-ut) \right\|_F \le q^{-\gamma(\lambda)}.$$

We say in this case that f has the *Fourier property*.

★ Automatic sequences along primes

Theorem (Müllner 2017, generalization of Mauduit+Rivat 2015)

Let $\gamma : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing function satisfying* $\lim_{\lambda\to\infty} \gamma(\lambda) = +\infty$, *and* $f \in \mathcal{F}_{\gamma,c}$ *be a function satisfying the Carry property for some* $\eta \in (0,1]$ *and the Fourier property for some* $c \geq 10$*. Then for any* $\theta \in \mathbb{R}$ *we have*

$$\left\| \sum_{n\le x} \Lambda(n) f(n) \mathrm{e}(\theta n) \right\|_2 \ll c_1(q) (\log x)^{c_2(q)} x q^{-\eta\gamma(2\lfloor (\log x)/(80\log q)\rfloor)/20}.$$

★ Automatic sequences along squares

Theorem (Generalization of Mauduit+Rivat 2018)

Let $\gamma : \mathbb{R} \to \mathbb{R}$ *be a nondecreasing function satisfying* $\lim_{\lambda\to\infty} \gamma(\lambda) = \infty$, *and let* $f \in \mathcal{F}_{\gamma,c}$ *be a function satisfying the Carry property for some* $\eta \in (0,1]$ *and the Fourier property for some* $c \geq 18$. *Then for any* $\theta \in \mathbb{R}$, *we have*

$$\left\| \sum_{0<n\leq x} f(n^2)\mathrm{e}(n\theta) \right\|_2 \ll_{d,f,q} (\log x)^{\omega(q)+2} \left(xq^{-\frac{\eta\gamma(2\lfloor (3\log x)/(100\log q)\rfloor)}{56}} \right),$$

where the absolute implied constant only depends on d, f *and* q.

★ Automatic sequences along primes and squares

Application for

$$f(n) = D(T(n)),$$

where $D : G \to \mathbb{U}_d$ is a unitary and irreducible representation of the group G.

By Fourier analysis on G and by considering residue classes (for $s(n)$) leads to the (limiting) **distributional behavior** of

$$a(n) = f(s(n), T(n))$$

along **primes** and **squares**.

Thank you!

(Logarithmic) Densities for Automatic Sequences along Primes and Squares

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