Automorphisms and Factors of Finite Topological Rank Systems

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Definitions

1. A **system** is a pair (X, T), where X is a compact metrizable space and $T\colon X \to X$ an homeomorphism.
2. We will consider only **minimal** systems (X, T), *i.e.* s.t. every orbit $\mathrm{orb}_T(x) \coloneqq \{T^k x : k \in \mathbb{Z}\}$ is dense in X.
3. For $n \geq 0$, let $\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+$ be a **substitution** *i.e.* a map s.t. $\sigma_n(uv) = \sigma_n(u)\sigma_n(v)$ for all words $u, v \in \mathcal{A}_{n+1}^+$. We write $\sigma_{[n,N)} = \sigma_n \cdots \sigma_{N-1}$.
4. If $\boldsymbol{\sigma} \coloneqq (\sigma_n)_{n\geq 0}$ satisfy $\lim_{n\to\infty} \min_{a\in\mathcal{A}_{n+1}} |\sigma_{[0,n)}(a)| = \infty$, then we call it a **directive sequence**.
5. The substitution σ_n is **proper** if there exist $a, b \in \mathcal{A}_n$ s.t. $\sigma_n(c)$ starts with a and ends with b for all $c \in \mathcal{A}_{n+1}$.

Definitions

1. The n-th **$\mathcal{S}$-adic subshift** $X_{\boldsymbol{\sigma}}^{(n)}$ is the set of all sequences $x \in \mathcal{A}_n^{\mathbb{Z}}$ s.t. for any $\ell \geq 0$, $x_{[-\ell,\ell]}$ occurs in $\sigma_n \cdots \sigma_N(a)$ for some $N \geq n$ and $a \in \mathcal{A}_{N+1}$.
   Alternatively, $x \in X_{\boldsymbol{\sigma}}^{(n)}$ iff $x \in \bigcup_{k \in \mathbb{Z}} T^k \sigma_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}})$ for all $N \geq n$.
   We set $X_{\boldsymbol{\sigma}} := X_{\boldsymbol{\sigma}}^{(0)}$.

Example: Let $\mathcal{A} = \{0, 1\}$, σ be the Fibonacci substitutions (*i.e.* $0 \mapsto 01$, $1 \mapsto 0$) and $\boldsymbol{\sigma} = (\sigma)_{n \geq 0}$. Then $\sigma^2(0) = 010$, $\sigma^2(1) = 01$ and $\sigma^4(0) = 01001010$, $\sigma^4(1) = 01001$.

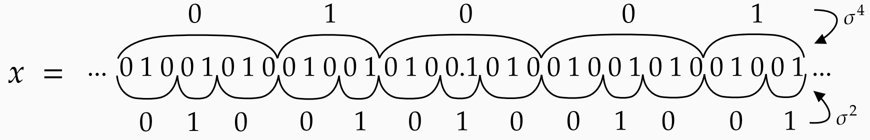

Figure: The point x belongs to $\sigma^2(\mathcal{A}^{\mathbb{Z}})$ and $\sigma^4(\mathcal{A}^{\mathbb{Z}})$.

Definitions

1. The pair $(X_{\boldsymbol{\sigma}}^{(n)}, T)$ is a system, where $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ is the **shift**.
2. When $\boldsymbol{\sigma}$ is constant, $X_{\boldsymbol{\sigma}}$ is the usual substitutive subshift associated to σ.
3. A **contraction** of $\boldsymbol{\sigma}$ is a directive sequence of the form $\boldsymbol{\sigma}' = (\sigma_{[0,n_1)}, \sigma_{[n_1,n_2)}, \dots)$. Since $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ generate the same $\mathcal{S}$-adic subshift $X_{\boldsymbol{\sigma}}$, we usually identify $\boldsymbol{\sigma}$ with its contractions.

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$\mathcal{S}$-adic representations and their importance

One of the things that make the $\mathcal{S}$-adic representations interesting is that it is possible to characterize many classes of zero entropy subshifts as particular $\mathcal{S}$-adic subshifts and study them using this representation.

Class of system		Type of directive sequence
Sturmian		Belonging to $\{\sigma_0, \sigma_1\}^{\mathbb{N}}$ [1]
Substitutional		Constant
Linearly Recurrent		Proper, primitive and bounded
Interval Exchange	$\Longleftrightarrow$	Infinite path in Rauzy graph
Dendric		Infinite path in a known graph
Toeplitz		Proper and constant length
Sublinear complexity		?

[1] $\sigma_i \colon \{0,1\}^+ \to \{0,1\}^+$ is defined by $\sigma_i(1-i) = i \cdot (1-i) \in \{0,1\}^2$ and $\sigma_i(i) = 1-i$.

Finite topological rank systems

1. We say that $\boldsymbol{\sigma}$ is **recognizable** if for all $x \in X_{\boldsymbol{\sigma}}$ and $n \geq 1$ there is a unique pair $(y, k) \in X_{\boldsymbol{\sigma}}^{(n)} \times \mathbb{Z}$ s.t. $x = T^k \sigma_{[0,n)}(y)$ and $k \in \{0, 1, \ldots, |\sigma_{[0,n)}(y_0)| - 1\}$.

2. The recognizable property is equivalent to the fact that $\{T^k \sigma_{[0,n)}([a]) : a \in \mathcal{A}_n, k \in \{0, 1, \ldots, |\sigma_{[0,n)}(a)| - 1\}\}$ is a partition of X for all $n \geq 1$.

3. We say that a minimal subshift (X, T) has **topological rank at most $\boldsymbol{K}$** if it is generated by a *recognizable* and *proper* directive sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n \geq 0}$ s.t. $\sup_{n \geq 0} \# \mathcal{A}_n \leq K$.

4. The examples of the previous slide have finite topological rank (with the exception of some Toeplitz subshifts).

Known general properties of finite top. rank systems

Although the finite top. rank class contains a broad spectrum of minimal subshifts, it is possible to prove general theorems for it.

For example, in a finite top. rank system (X, T),

1. there is zero entropy (Handelman and Boyle, 1994);
2. there are finitely many ergodic measures (Bezuglyi, Kwiatkowski, Medynets, Solomyak, 2013);
3. the rational dimension of its dimension group is finite (Herman, Putnam, Skau, 1999);
4. there is criteria for deciding if a given complex number is a measurable or topological eigenvalue (Durand, Frank, Maass, 2019).

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Automorphisms and asymptotic pairs

1. Recall that $\mathrm{Aut}(X, T) = \{g \colon X \to X : \text{g bicontinuous}, \; g \circ T = T \circ g\}$ and $\langle T \rangle := \{ T^n : n \in \mathbb{Z}\}$.
2. An **asymptotic pair** in (X, T) is a tuple $(x, y) \in X \times X$ satisfying $\lim_{n\to\infty} \mathrm{dist}(T^n x, T^n y) = 0$. We define the equivalence relation $x \sim y$ iff $(x, T^k y)$ is an asymptotic pair for some $k \in \mathbb{Z}$, and $\mathrm{AC}(X, T)$ as the $\sim$-equivalence classes which are not the orbit of a single point.
3. $\mathrm{Aut}(X, T)/\langle T \rangle$ acts faithfully on $\mathrm{AC}(X, T)$ when (X, T) is an infinite minimal subshift [DDMP16].
4. (X, T) has **non-superlinear complexity** if $\liminf_{n\to\infty} p_X(n)/n < \infty$, where $p_X(n)$ is the number of words of length n occurring in X.

Theorem (Donoso, Durand, Maass, Petite (2016))

An infinite minimal subshift with non-superlinear complexity has finitely many asymptotic pairs up to shifts. Consequently, $\mathrm{Aut}(X, T)/\langle T \rangle$ is finite.

Automorphisms and asymptotic pairs

It is possible to generalize the previous result:

Theorem (E., Maass (2019))

A minimal subshift of finite top. rank has finitely many asymptotic pairs. Consequently, $\mathrm{Aut}(X,T)/\langle T\rangle$ *is finite.*

The main tool of the proof is the following purely combinatorial theorem.

Theorem

Let $\mathcal{W} \subseteq \mathcal{A}^+$ *be a finite set of words and* $\langle \mathcal{W} \rangle = \min_{w \in \mathcal{W}} |w|$. *Then, there exists* $\mathcal{B} \subseteq \mathcal{A}^{\langle \mathcal{W} \rangle}$ *with no more than* $122\#\mathcal{W}^7$ *elements s.t.*

> *if* $x, y \in \mathcal{A}^{\mathbb{Z}}$ *are factorizable over* $\mathcal{W}$ *(i.e.* $T^k x, T^\ell y \in \mathcal{W}^{\mathbb{Z}}$ *for some* $k, \ell \in \mathbb{Z}$*),* $x_{(-\infty,0)} = y_{(-\infty,0)}$ *and* $x_0 \neq y_0$*, then* $x_{[-\langle \mathcal{W} \rangle, 0)} \in \mathcal{B}$.

Symbolic factors and their top. rank

Theorem (E. 2020)

Let $\pi\colon (X_{\boldsymbol{\sigma}}, T) \to (Y, T)$ *be a factor between minimal subshifts, where* $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\geq 0}$ *is a recognizable and proper directive sequence. Then, (up to a contraction of* $\boldsymbol{\sigma}$*) there exist sequences of substitutions* $\boldsymbol{\tau} = (\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\geq 0}$, $\phi = (\phi_n\colon \mathcal{A}_n^+ \to \mathcal{B}_n^+)_{n\geq 1}$ *s.t.*

1. $\boldsymbol{\tau}$ *is a recognizable directive sequence generating* (Y, T)*;*
2. $\#\mathcal{B}_n \leq \#\mathcal{A}_n$ *for all* $n \geq 0$*;*
3. *the following diagram commutes for any* $n \geq 1$*:*

$$\begin{array}{ccc} \mathcal{A}_{n+1}^+ & \xrightarrow{\phi_{n+1}} & \mathcal{B}_{n+1}^+ \\ \sigma_n \big\downarrow & & \big\downarrow \tau_n \\ \mathcal{A}_n^+ & \xrightarrow[\phi_n]{} & \mathcal{B}_n^+ \end{array}$$

In particular, the top. rank of (Y, T) *is at most the one of* (X, T)*.*

More on symbolic factors

Corollary

A minimal subshift (X, T) *has finite top. rank iff*

$$\liminf_{n\to\infty} \min\{\#\mathcal{W} : \mathcal{W} \subseteq \mathcal{A}^+,\ \langle \mathcal{W} \rangle \geq n,\ X \subseteq \bigcup_{k\in\mathbb{Z}} T^k \mathcal{W}^{\mathbb{Z}}\} < \infty,$$

where $\langle \mathcal{W} \rangle := \min\{|w| : w \in \mathcal{W}\}$.

In particular, (X, T) *has finite top. rank iff it is generated by a directive sequence* $\boldsymbol{\tau} = (\tau_n \colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\geq 0}$ *s.t.* $\liminf_{n\geq 0} \mathcal{B}_n < \infty$.

Corollary

Let $\pi \colon (X, T) \to (Y, T)$ *be a factor, where* (X, T) *has finite top. rank and* Y *is totally disconnected. Then,* (Y, T) *has finite top. rank (in particular, it is a subshift).*

More on symbolic factors

Corollary

*A finite top. rank system is **coalescent**. Moreover, in any infinite chain of symbolic factors, $(X, T) \overset{\pi_1}{\to} (X_1, T) \overset{\pi_2}{\to} \dots$, there are only finitely many maps π_j which are not conjugacies.*

Theorem

A symbolic factor $\pi\colon (X, T) \to (Y, T)$ between aperiodic minimal finite top. rank subshifts is almost k-to-1 for some finite $k \in \mathbb{N}$, this is, for all y in a residual subset of Y we have $\#\pi^{-1}(y) = k$.

Theorem

Let (X, T) be a finite top. rank subshift. Then, it has finitely many aperiodic symbolic factors up to conjugacy.

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Open problems

Let (X, T) be a finite top. rank system.

1. We have a good description of totally disconnected factors of (X, T). What about the connected factors? We conjecture they are all equicontinuous.
2. What can be said about the maximal equicontinuous factor of (X, T)? *e.g.* size of the fibers, structure theorem for it.
3. Find an $\mathcal{S}$-adic characterization of the class of non-superlinear-complexity minimal subshifts.

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