On upper and lower fast Khintchine spectra in continued fractions

Lulu Fang

Nanjing University of Science and Technology

fanglulu@njust.edu.cn

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Outline

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I. Khintchine Spectrum

Continued fractions

Every irrational number $x \in (0,1)$ admits a continued fraction expansion of the form:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x) + \ddots}}} =: [a_1(x), a_2(x), \cdots, a_n(x), \cdots],$$

where $a_n(x)$ are positive integers, and are called the partial quotients. See Khintchine (1964) for more information of continued fractions.

Khintchine exponent

For $x \in (0,1)\backslash\mathbb{Q}$, we define the Khintchine exponent of x as the growth rate of the geometric average of partial quotients, namely,

$$\mathrm{k}(x) := \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{n}$$

if the limit exists. For example,

- $\frac{\sqrt{5}-1}{2} = [1,1,\cdots,1,\cdots]$, we have $\mathrm{k}(\frac{\sqrt{5}-1}{2}) = 0$;
- $e-2 = [1,2,1,1,4,1,1,6,1,1,8,1,\cdots]$, we have $\mathrm{k}(e-2) = \infty$.

In fact, for any $\alpha \in [0,\infty]$, there exists $x \in (0,1)\backslash\mathbb{Q}$ such that $\mathrm{k}(x) = \alpha$. Moreover, the level set $\{x \in (0,1)\backslash\mathbb{Q} : \mathrm{k}(x) = \alpha\}$ is uncountable and dense, but is of first Baire category.

Theorem (Khintchine, 1935)

For Lebesgue almost all $x \in (0,1)$,

$$\mathrm{k}(x) = \gamma,$$

where γ is given by

$$\gamma := \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \cdot \log\left(1 + \frac{1}{n(n+2)}\right) \approx 0.987849.$$

The dimensional function

$$[0,\infty] \ni \alpha \mapsto K(\alpha) := \dim_{\mathrm{H}} \left\{x \in (0,1) : \mathrm{k}(x) = \alpha\right\}$$

is called the Khintchine spectrum.

Theorem (Fan, Liao, Wang & Wu, 2009)

The Khintchine spectrum satisfies the following properties:

- *it is real analytic in* $(0,\infty)$*;*
- *it has a unique maximum at point* γ*;*
- *it is strictly increasing in* $[0,\gamma)$ *and strictly decreasing in* $[\gamma,\infty)$*;*
- *it is neither convex nor concave;*
- $\lim_{\alpha\to\infty} K(\alpha) = 1/2$.

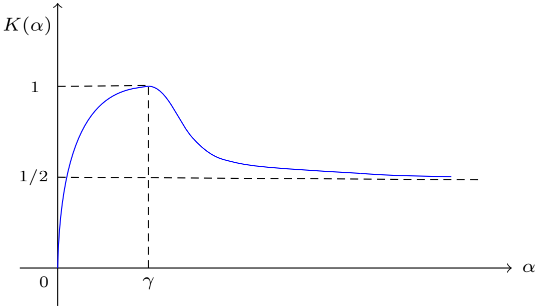

Figure 1. The graph of $K(\cdot)$

Proposition (Iommi & Jordan, 2015)

$$K(\infty) = \frac{1}{2}.$$

II. Fast Khintchine Spectrum

Fast Khintchine exponent

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\psi(n)/n \to \infty$ as $n \to \infty$. The fast Khintchine exponent of x, relative to ψ, is defined by

$$\mathrm{k}_\psi(x) := \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)}$$

if the limit exists. For example, let $\phi(n) := n \log n$, then $\mathrm{k}_\phi(e-2) = 1/3$. For Lebesgue almost all $x \in (0,1)$, $\mathrm{k}_\psi(x) = 0$.

The dimensional function

$$[0,\infty] \ni \alpha \mapsto K_\psi(\alpha) := \dim_{\mathrm{H}} \left\{x \in (0,1) : \mathrm{k}_\psi(x) = \alpha\right\}$$

is called the fast Khintchine spectrum relative to ψ. Clearly, $K_\psi(0) = 1$.

Theorem (Fan, Liao, Wang & Wu, 2013)

Let $0 < \alpha < \infty$. Assume that $\psi : \mathbb{N} \to \mathbb{R}^+$ is non-decreasing. Then

$$K_\psi(\alpha) = \frac{1}{\beta+1}, \qquad \textit{with} \;\; \beta := \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)}.$$

For the case $\alpha = \infty$, we will see below that

$$K_\psi(\infty) = \frac{1}{B+1}.$$

Upper bound of $K_\psi(\alpha)$

Define $q_n(x)$ by the the recursive formula:

$$q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x)$$

with the conventions $q_{-1} \equiv 0$ and $q_0 \equiv 1$. Then

$$\prod_{k=1}^{n} a_k(x) \leq q_n(x) < 2^n \prod_{k=1}^{n} a_k(x),$$

and so

$$\mathrm{k}_\psi(x) = \lim_{n\to\infty} \frac{\log q_n(x)}{\psi(n)}.$$

If $\mathrm{k}_\psi(x) = \alpha$, then

$$\limsup_{n\to\infty} \frac{\log q_{n+1}(x)}{\log q_n(x)} = \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)} = \beta.$$

Equivalently,

$$\mu(x) := 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}(x)}{\log q_n(x)} = \beta + 1,$$

where $\mu(x)$ is the irrationality exponent of x in Diophantine approximation. It follows form a result of Jarník (1929) that

$$K_\psi(\alpha) \leq \dim_{\mathrm{H}} \left\{x \in (0,1) : \mu(x) = \beta + 1\right\} = \frac{2}{\beta+1},$$

which is strictly greater than the desired upper bound of $K_\psi(\alpha)$.

Note that $\{x \in (0,1) : \mathrm{k}_\psi(x) = \alpha\} \subseteq \{x \in (0,1) : \mathrm{k}(x) = \infty\}$, so we guess that the Hausdorff dimension of the intersection of

$$\{x \in (0,1) : \mu(x) = \beta + 1\} \text{ and } \{x \in (0,1) : \mathrm{k}(x) = \infty\}$$

will approximate the required upper bound of $K_\psi(\alpha)$.

Lemma

For $1 \leq \beta \leq \infty$,

$$\dim_{\mathrm{H}} \{x \in (0,1) : \mu(x) = \beta + 1, \, \mathrm{k}(x) = \infty\} = \frac{1}{\beta + 1}.$$

III. Upper and Lower Fast Khintchine Spectra

Upper and Lower Fast Khintchine exponent

The upper and lower fast Khintchine exponent of x, relative to ψ, are defined by

$$\overline{\mathrm{k}}_{\psi}(x) := \limsup_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)}$$

and

$$\underline{\mathrm{k}}_{\psi}(x) := \liminf_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)}$$

respectively. Then for Lebesgue almost all $x \in (0,1)$, $\overline{\mathrm{k}}_{\psi}(x) = \underline{\mathrm{k}}_{\psi}(x) = 0$.

The dimensional functions

$$[0, \infty] \ni \alpha \mapsto \overline{K}_\psi(\alpha) := \dim_{\mathrm{H}} \left\{x \in (0,1) : \overline{\mathrm{k}}_\psi(x) = \alpha\right\}$$

and

$$[0, \infty] \ni \alpha \mapsto \underline{K}_\psi(\alpha) := \dim_{\mathrm{H}} \left\{x \in (0,1) : \underline{\mathrm{k}}_\psi(x) = \alpha\right\}$$

are respectively called upper and lower fast Khintchine spectra relative to ψ. Then $\overline{K}_\psi(0) = \underline{K}_\psi(0) = 1$. For a positive and finite α, these two spectra have been studied by Liao and Rams (2016) under some restrictions on the growth rate of ψ.

Theorem (F., Shang & Wu, 2021)

Let $0 < \alpha \leq \infty$. *Then*

$$\overline{K}_\psi(\alpha) = \frac{1}{b+1} \quad \textit{and} \quad \underline{K}_\psi(\alpha) = \frac{1}{B+1},$$

where $b, B \in [1, \infty]$ *are given by*

$$\log b := \liminf_{n\to\infty} \frac{\log \psi(n)}{n} \quad \textit{and} \quad \log B := \limsup_{n\to\infty} \frac{\log \psi(n)}{n}.$$

Remarks:

(i). For the case $0 < \alpha < \infty$, Liao & Rams (2016) obtained the same result when $b, B > 1$.

(ii). $K_\psi(\infty) = \underline{K}_\psi(\infty) = 1/(B+1)$.

Upper bounds of $\overline{K}_\psi(\alpha)$ and $\underline{K}_\psi(\alpha)$

Write $\Pi_n(x) := \prod_{k=1}^n a_k(x)$. If $\overline{\mathrm{k}}_\psi(x) = \alpha \in (0, \infty)$, that is,

$$\limsup_{n\to\infty} \frac{\log \Pi_n(x)}{\psi(n)} = \alpha,$$

then $\Pi_n(x) \geq e^{\alpha\psi(n)/2}$ for infinitely many n's. If $\overline{\mathrm{k}}_\psi(x) = \infty$, then $\Pi_n(x) \geq e^{\psi(n)}$ for infinitely many n's. Hence

$$\{x \in (0,1) : \overline{\mathrm{k}}_\psi(x) = \alpha\} \subseteq \left\{x \in (0,1) : \Pi_n(x) \geq A^{\psi(n)},\ \text{i.m. } n \in \mathbb{N}\right\}$$

for some $A > 1$, where "i.m." denotes "infinitely many". Similarly,

$$\{x \in (0,1) : \underline{\mathrm{k}}_\psi(x) = \alpha\} \subseteq \left\{x \in (0,1) : \Pi_n(x) \geq A^{\psi(n)},\ \forall n \gg 1\right\},$$

where "$\forall n \gg 1$" denotes "for n sufficiently large".

Lemma

Let $A \in (1, \infty)$*. Then*

$$\dim_{\mathrm{H}} \left\{ x \in (0,1) : \Pi_n(x) \geq A^{\psi(n)},\ \textit{i.m. } n \in \mathbb{N} \right\} = \frac{1}{b+1}$$

and

$$\dim_{\mathrm{H}} \left\{ x \in (0,1) : \Pi_n(x) \geq A^{\psi(n)},\ \forall n \gg 1 \right\} = \frac{1}{B+1}.$$

- Łuczak-type result: for $a, c \in (1, \infty)$,

$$\dim_{\mathrm{H}} \left\{x \in (0,1) : \Pi_n(x) \geq a^{c^n},\ \text{i.m. } n \in \mathbb{N}\right\} = \frac{1}{c+1}$$

and

$$\dim_{\mathrm{H}} \left\{x \in (0,1) : \Pi_n(x) \geq a^{c^n},\ \forall n \gg 1\right\} = \frac{1}{c+1}.$$

A concrete example showing $b < B < \beta$

For $k \geq 1$, let $n_k := 1! + 2! + \cdots + k!$ and

$$\psi(n) := \begin{cases} \left(\frac{5}{3}\right)^{k-1} 4^{n-(1!+3!+\cdots+(2k-1)!)} 3^{1!+3!+\cdots+(2k-1)!}, & n_{2k-1} < n \leq n_{2k}, \\ \left(\frac{5}{3}\right)^{k} 4^{2!+4!+\cdots+(2k)!} 3^{n-(2!+4!+\cdots+(2k)!)}, & n_{2k} < n \leq n_{2k+1}. \end{cases}$$

Then $b = 3, B = 4$ and $\beta = 5$.

Recall that

$$\log b := \liminf_{n\to\infty} \frac{\log\psi(n)}{n},\ \log B := \limsup_{n\to\infty} \frac{\log\psi(n)}{n},\ \beta := \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)}.$$

Since

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k}(2j)!}{\sum_{j=1}^{2k} j!} = \lim_{k\to\infty} \frac{\sum_{j=1}^{k}(2j+1)!}{\sum_{j=1}^{2k+1} j!} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{\psi(n_{2k}+1)}{\psi(n_{2k})} = 5,$$

we have $b = 3, B = 4$ and $\beta = 5$.

Thank You for Your Attention !