Dynamics of simplicial systems

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INSTITUT
DE RECHERCHE
EN INFORMATIQUE
FONDAMENTALE

Continued fraction algorithm

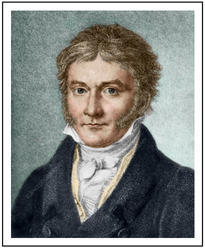
$x \in [0, 1]$

Rational approximation

$$|x - \frac{p}{q}| \leq \frac{1}{q^2}$$

Continued fraction algorithm

$x \in [0, 1]$



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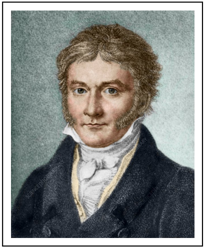
$$G : x \to \frac{1}{x} - \left[\frac{1}{x}\right]$$

Continued fraction algorithm



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Development

$$x = [a_1, a_2, \dots]$$

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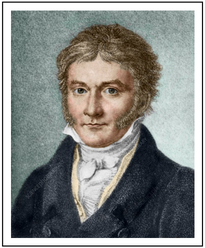

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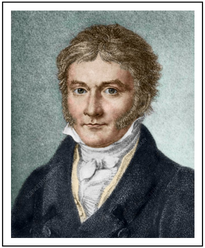

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Algorithm

$$T : [1 : x] \to [1 : x - [x]]$$

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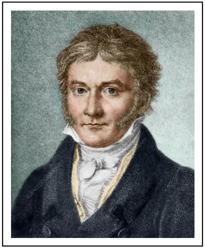

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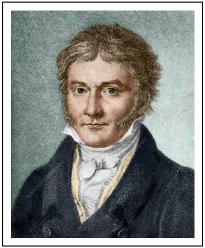
Algorithm

$$T : [1 : x] \to [1 : x - 1]$$
$$\text{for } 1 < x$$

Continued fraction algorithm

$x \in [0,1]$



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Algorithm

$$T : [x_1 : x_2] \to [x_1 : x_2 - x_1]$$

for $x_1 < x_2$

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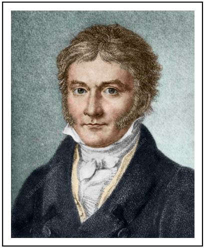

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$$|x - \frac{p}{q}| \leq \frac{1}{q^2}$$

Development

$$x \to \sigma_1^{a_1} \sigma_2^{a_2} \sigma_1^{a_3} \dots$$

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Algorithm

$$T : [x_1 : x_2] \to [x_1 : x_2 - x_1]$$

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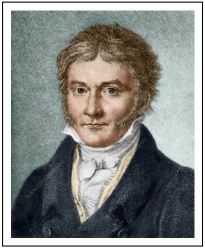
Continued fraction algorithm

$\mathbf{x} \in [0,1]^2$



Rational approximation

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$$x \to \sigma_1^{a_1}\sigma_2^{a_2}\sigma_1^{a_3}\dots$$

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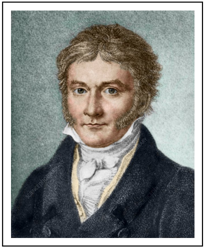

Rational approximation

$$|\mathbf{x} - \frac{\mathbf{p}}{q}| \leq \frac{1}{q^{1+1/2}}$$

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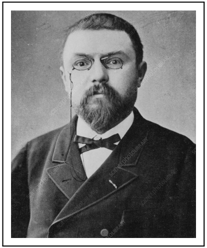

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Algorithm

$$T : [x_1 : x_2 : x_3] \to [x_1 : x_2 - x_1 : x_3 - x_2]$$

for $x_1 < x_2 < x_3$

Continued fraction algorithm

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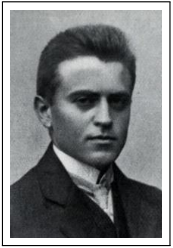

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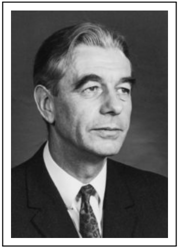

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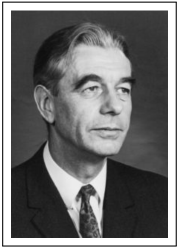

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$$|\mathbf{x} - \frac{\mathbf{p}}{q}| \leq \frac{1}{q^{1+1/2}}$$

Development

$$\mathbf{x} \to w_1 w_2 w_3 \ldots$$

$$\frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{\ddots}{a_{n-1} + \cfrac{1}{a_n}}}$$

Algorithm

$$T : [x_1 : x_2 : x_3] \to [x_1 : x_2 - x_1 : x_3 - x_2]$$
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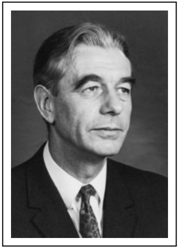

Development

$$\mathbf{x} \to w_1 w_2 w_3 \dots$$

- $\lim_{n\to\infty} \frac{p_n}{q_n} \stackrel{?}{=} \mathbf{x}$

Algorithm

$$T : [x_1 : x_2 : x_3] \to [x_1 : x_2 - x_1 : x_3 - x_2]$$
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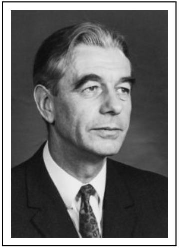

Development

$$\mathbf{x} \to w_1 w_2 w_3 \ldots$$

- $\lim_{n\to\infty} \frac{p_n}{q_n} \stackrel{?}{=} \mathbf{x}$
- Ergodicity of T

Algorithm

$$T : [x_1 : x_2 : x_3] \to [x_1 : x_2 - x_1 : x_3 - x_2]$$
$$[x_1 : x_2 : x_3 - x_2]$$
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$$\text{for } x_1 < x_2 < x_3$$

Continued fraction algorithms

Maps on $\mathbb{PR}^3_+$
for $x_1 < x_2 < x_3$

Poincaré

$$T_P(\mathbf{x}) = [x_1 : x_2 - x_1 : x_3 - x_2]$$

Brun

$$T_B(\mathbf{x}) = [x_1 : x_2 : x_3 - x_2]$$

Selmer

$$T_S(\mathbf{x}) = [x_1 : x_2 : x_3 - x_1]$$

Continued fraction algorithms

Maps on $\mathbb{PR}^3_+$
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Poincaré

Theorem (Nogueira 1995)

For almost all $\mathbf{x}$*, approximations do not converge to* $\mathbf{x}$*.*

Brun

$$T_B(\mathbf{x}) = [x_1 : x_2 : x_3 - x_2]$$

Selmer

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Continued fraction algorithms

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Theorem (Nogueira 1995)

For almost all $\mathbf{x}$, approximations do not converge to $\mathbf{x}$.

Selmer

Theorem (Schweiger 2000)

Approximations converge and T_S is ergodic.

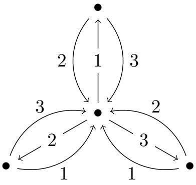
Brun

Theorem (Broise 1994)

Approximations converge and T_B is ergodic.

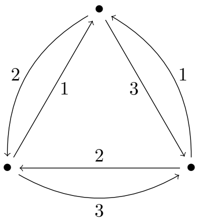
Continued fraction algorithms

Poincaré

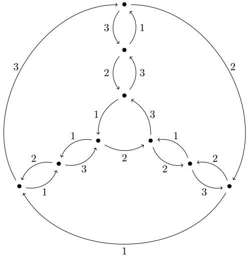


Finite subshift
Law of random walk with memory

Selmer

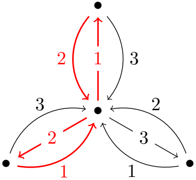


Brun



Continued fraction algorithms

Poincaré



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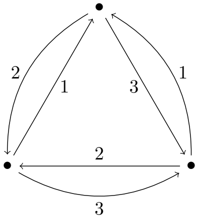

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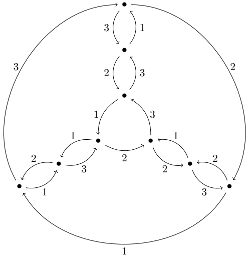


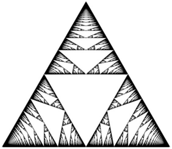
Ergodicity criterion

Theorem (F.)

If an algortihm does not contains a stable subgraph then it is ergodic and its canonical suspension admits a unique measure of maximal entropy.

Applications

- Fractal dimensions
- Numeration and Normality
- Teichmüller dynamics

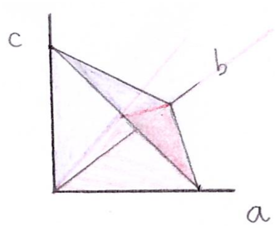


Farey algorithm

From simplex to simplicial system

Elementary operations

$$T : \mathbb{R}^3_+ \to \mathbb{R}^3_+ \text{ and } T(x_a, x_b, x_c) = \begin{cases} (x_a - x_c, x_b, x_c) & \text{if } x_a > x_c \\ (x_a, x_b, x_c - x_a) & \text{if } x_c > x_a \end{cases}$$



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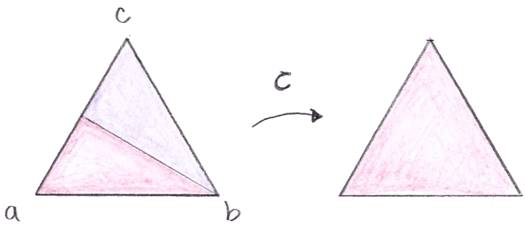

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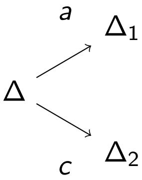

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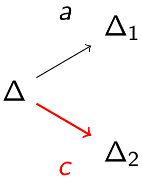
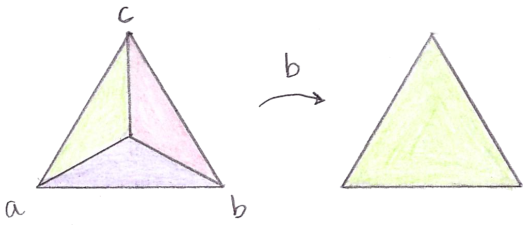


$$x_c < x_a$$

We say that c **loses** and a **wins**

Elementary operations

$$T(x_a, x_b, x_c) = \begin{cases} (x_a - x_c, x_b - x_c, x_c) & \textit{if } \min(x_a, x_b) > x_c \\ (x_a - x_b, x_b, x_c - x_b) & \textit{if } \min(x_a, x_c) > x_b \\ (x_a, x_b - x_a, x_c - x_a) & \textit{if } \min(x_b, x_c) > x_a \end{cases}$$



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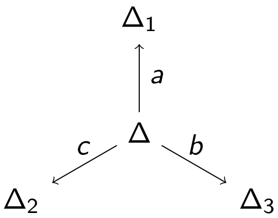

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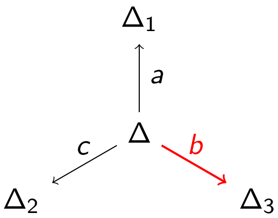


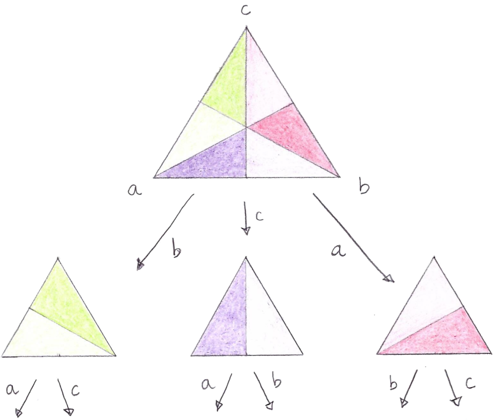
$$x_b < x_a, x_c$$

We say that b **loses** and a, c **win**

Poincaré algorithm

$$T(x_a, x_b, x_c) = \begin{cases} (x_a, x_b - x_a, x_c - x_b) & \textit{if } x_a < x_b < x_c \\ (x_a, x_b - x_c, x_c - x_a) & \textit{if } x_a < x_c < x_b \\ (x_a - x_c, x_b, x_c - x_b) & \textit{if } x_b < x_c < x_a \\ \dots \end{cases}$$

Poincaré algorithm



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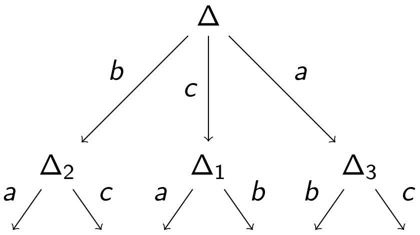

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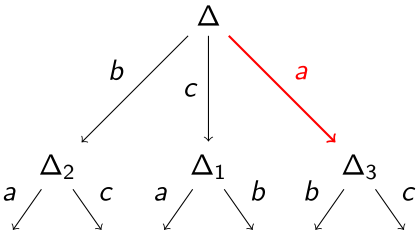

$x_a < x_b,\ x_c$

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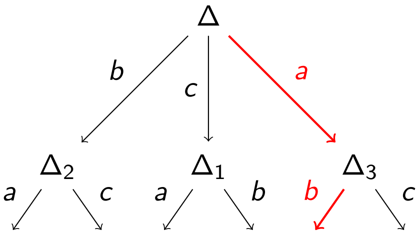

$$x_a < x_b < x_c$$

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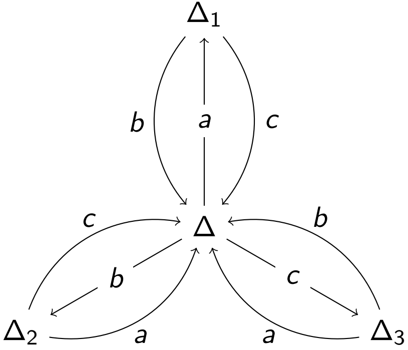
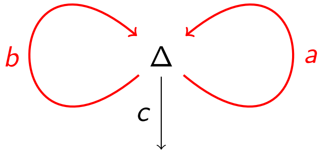


Properties of random walks

Subgraph stability

Assume that $q_a, q_b \gg q_c$



Consider the stopping time $\mathcal{L}_c$ corresponding to the length of the path before the letter c *loses*.

Lemma

For all $q \in \mathbb{R}_+^3$

$$\mathbb{P}_q(\mathcal{L}_c < \infty) \leq \frac{Kq_c}{\min(q_a, q_b)}.$$

Ergodicity criterion

Theorem

A simplicial system without stable subgraphs admits an unique ergodic measure absolutely continued w.r.t. Lebesgue and this measure induces the unique mesure of maximal entropy on the canonical suspension.

An application to normality

The continued fraction expansion of a number $x = [a_1, a_2, \ldots]$ is called *normal* if for all finite word $w \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{\|x[1\ldots n]\|_w}{n} = \mu(w).$$

Theorem (Vandehey)

Normality of the continued fraction expansion of a number is preserved by homography.

Work in progress (with Carton, Berthé)

Generalisation to multidimensional continued fractions.

Thank you !