Partitions and A Multi-dimensional Continued Fraction Algorithm

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Goal

Use the dynamics of the triangle map (a type of multi-dimensional continued fraction algorithm) to create an almost internal symmetry on the space of all partitions of a given integer N.

Outline

1. Partitions
2. The Farey Tree, Farey map and its links to partitions
3. The Triangle Map and its link to partitions
4. Method to Generate Many New Partition Identities
5. Why the triangle map? Questions.

Partitions

$p(n)$ is the number of ways of writing n as the sum of less than or equal t positive integers (ordering not mattering).
$p(7) = 15$ since

$$\begin{array}{ccc} 7 & 6+1 & 5+2 \\ 5+1+1 & 4+3 & 4+2+1 \\ 4+1+1+1 & 3+3+1 & 3+2+2 \\ 3+2+1+1 & 3+1+1+1+1 & 2+2+2+1 \\ 2+2+1+1+1 & 2+1+1+1+1+1 & 1+1+1+1+1+1+1. \end{array}$$

$$\begin{array}{cccc} (7) & (6,1) & (5,2) & (5,1^2) \\ (4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\ (3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\ (2^2,1^3) & (2,1^5) & (1^7). & \end{array}$$

Partitions

$$\begin{array}{cccc} (7) & (6,1) & (5,2) & (5,1^2) \\ (4,3) & (4,2,1) & (4,1^3) & (3^2,1) \\ (3,2^2) & (3,2,1^2) & (3,1^4) & (2^3,1) \\ (2^2,1^3) & (2,1^5) & (1^7). & \end{array}$$

$$\begin{array}{cccc} (7)\times[1] & (6,1)\times[1,1] & (5,2)\times[1,1] & (5,1)\times[1,2] \\ (4,3)\times[1,1] & (4,2,1)\times[1,1,1] & (4,1)\times[1,3] & (3,1)\times[2,1] \\ (3,2)\times[1,2] & (3,2,1)\times[1,1,2] & (3,1)\times[1,4] & (2,1)\times[3,1] \\ (2,1)\times[2,3] & (2,1)\times[1,5] & (1)\times[7]. & \end{array}$$

Partitions

$$\lambda = (\lambda_0, \ldots, \lambda_m) \times [k_0, \ldots, k_m] \vdash N$$

means

$$\begin{aligned} N &= k_0\lambda_0 + \ldots + k_m\lambda_m \\ &= (k_0, \ldots, k_m) \cdot \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_m \end{pmatrix} \end{aligned}$$

The λ_i are the parts and the k_i are the multiplicities.

Partitions

There are many remarkable identities.

For example, Andrew and Eriksson's *Integer Paritions* starts with discussing Euler's identity:

"*Every number has as many integer partitions into odd parts as into distinct parts.*"

Partitions

Two Questions

1. How to find possible identities
2. How to prove them

Partitions

To a given partition

$$(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m]$$

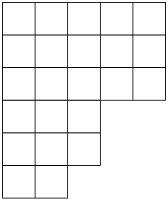
we associate the *Young shape*, a diagram $k_1 + \cdots + k_m$ rows such that there are k_1 rows with λ_1 squares on top of k_2 rows with λ_2 squares, and so on.

Partitions

For example, the Young shape for

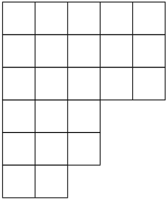
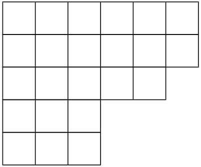
$$(5, 3, 2) \times [3, 2, 1] \vdash 23$$

is



Partitions

Flip a Young shape, turning the rows into columns, to get the *conjugate partition*
Flipping the Young shape of the partition $(5, 3, 2) \times [3, 2, 1] \vdash 23$ of the previous example gives us the Young shape

 $\sim_{\mathcal{C}}$ 

which represents the conjugate partition

$$(5, 3, 2) \times [3, 2, 1] \sim_{\mathcal{C}} (6, 5, 3) \times [2, 1, 21]$$

Partitions

$$(\lambda_1, \lambda_2) \times [k_1, k_2] \sim_{\mathcal{C}} (k_1 + k_2, k_1) \times [\lambda_2, \lambda_1 - \lambda_2]$$

and in general

$$\begin{gathered}(\lambda_1, \ldots, \lambda_m) \times [k_1, \ldots, k_m] \\ \sim_{\mathcal{C}} \\ (k_1 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots, k_1) \\ \times \\ [\lambda_m, \lambda_{m-1} - \lambda_m, \ldots, \lambda_1 - \lambda_2]\end{gathered}$$

Question

What do partitions have to do with multi-dimensional continued fractions algorithms?

What do partitions have to do with division algorithms in general?

Gauss map

For $\lambda_0 > \lambda_1 > 0$, the Gauss map is

$$G(\lambda_0, \lambda_1) = (\lambda_1, \lambda_0 - n\lambda_1)$$

if

$$\lambda_0 - n\lambda_1 \geq 0 > \lambda_0 - (n+1)\lambda_1.$$

Gauss map

Iterate the Gauss map, keeping track of relevant integer n. Its link to continued fractions is that

$$\frac{\lambda_1}{\lambda_0} = \frac{1}{n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \dots}}}$$

Farey map

For $\lambda_0 > \lambda_1 > 0$, the Farey map is

$$(\lambda_0, \lambda_1) \begin{array}{lll} \xrightarrow{F_0} & (\lambda_1, \lambda_0 - \lambda_1) & \text{if} \quad \lambda_0 < 2\lambda_1 \\ \xrightarrow{F_1} & (\lambda_0 - \lambda_1, \lambda_1) & \text{if} \quad \lambda_0 > 2\lambda_1 \end{array}$$

Via matrices

$$F_0 \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_0 - \lambda_1 \end{pmatrix}$$

$$F_1 \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 - \lambda_1 \\ \lambda_1 \end{pmatrix}$$

Farey and Gauss map

$$G(\lambda_0, \lambda_1) = F_0 \circ F_1^{n-1}(\lambda_0, \lambda_1)$$

The rhetoric is

$$\begin{aligned} \text{Gauss} &= \text{fast} &= \text{multiplicative} \\ \text{Farey} &= \text{slow} &= \text{additive} \end{aligned}$$

Farey map and partitions

Suppose

$$(\lambda_0, \lambda_1) \times [k_0, k_1) \vdash N.$$

We have

$$N = (k_0, k_1) \cdot \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = (k_0, k_1) F_i^{-1} F_i \cdot \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}$$

Farey map and partitions

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},\ F_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$F_0^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ F_1^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Key is that both F_0^{-1} and F_1^{-1} have non-zero entries.

Farey Tree and Map

The extended Farey map:

$$(\lambda_0,\lambda_1)\times[k_0,k_1] \begin{array}{ll} \xrightarrow{\tilde{F}_0} & (\lambda_1,\lambda_0-\lambda_1)\times[k_0+k_1,k_0] \quad \text{if} \quad \lambda_0<2\lambda_1 \\ \xrightarrow{\tilde{F}_1} & (\lambda_0-\lambda_1,\lambda_1)\times[k_0,k_0+k_1] \quad \text{if} \quad \lambda_0>2\lambda_1 \end{array}$$

This is actually the natural extension of the Farey map.

Farey Tree and Map

Via matrices

$$
\begin{aligned}
\tilde{F}_0 \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} &= \begin{pmatrix} F_0 & 0 \\ 0 & (F_0^{-1})^T \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 \\ \lambda_0 - \lambda_1 \\ k_0 + k_1 \\ k_0 \end{pmatrix}
\end{aligned}
$$

Farey Tree and Map

$$
\begin{aligned}
\tilde{F}_1 \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} &= \begin{pmatrix} F_1 & 0 \\ 0 & (F_1^{-1})^T \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ k_0 \\ k_1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_0 - \lambda_1 \\ \lambda_1 \\ k_0 \\ k_0 + k_1 \end{pmatrix}
\end{aligned}
$$

Farey Tree and Map

Paths:

$$\begin{aligned}
(19,8)\times[1,0] &\xrightarrow{\tilde{F}_1} (11,8)\times[1,1]\\
&\xrightarrow{\tilde{F}_0} (8,3)\times[2,1]\\
&\xrightarrow{\tilde{F}_1} (5,3)\times[2,3]\\
&\xrightarrow{\tilde{F}_0} (3,2)\times[5,2]\\
&\xrightarrow{\tilde{F}_0} (2,1)\times[7,5]
\end{aligned}$$

Farey Tree and Map

Thus there is a link between continued fractions and partitions of integers into two distinct parts.

Still seems of limited interest, though.

The Triangle Map

Multi-dimensional Continued Fractions are attempts to generalize the Euclidean algorithm for two numbers to the case of three or more numbers.

The Triangle Map

Roots of Multi-dimensional Continued Fractions:

1. Generalize the fact that a number has an eventually periodic continued fraction expansion if and only if it is a quadratic irrational.
2. Finding best Diophantine approximations of n-tuples of reals by n-tuples of rationals
3. As a rich source of dynamical systems, starting with Gauss on continued fractions all the way to the current work on interval exchange maps.

The Triangle Map

The difficulty:

$$\text{Big Number} \;>\; \text{Middle Number} \;> \text{Little Number} \;> 0$$

Each method for dividing middle number and little number into big number gives a different multi-dimensional continued fraction algorithm.

The Triangle Map

A dynamical system on simplices.
Earlier work
(TG) (2001)
S. Assaf, L. Chen, T. Cheslack-Postava, B. Cooper, A. Diesl, TG, M. Lepinski and A. Schuyler (2005)

A. Messaoudi, A. Nogueira, and F. Schweiger (2009)

V. Berthé, W. Steiner and J. Thuswaldner (2021)

Fougeron and A. Skripchenko (2021)

C.Bonanno, A. Del Vigna and S. Munday (2021)

C. Bonanno and A. Del Vigna (2021)

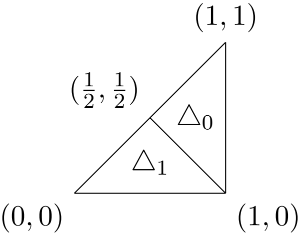
H. Ito (2023)

The Triangle Map

Set

$$
\begin{aligned}
\triangle &:= \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > \cdots > x_n > 0\} \\
\triangle_0 &:= \{x_0, \dots, x_n) \in \triangle : x_1 + x_n > x_0\} \\
\triangle_1 &:= \{x_0, \dots, x_n) \in \triangle : x_1 + x_n < x_0\} \\
\triangle_D &:= \{x_0, \dots, x_n) \in \triangle : x_1 + x_n = x_0\}
\end{aligned}
$$

When $n = 2$ and $x_0 = 1$, we have



The Triangle Map

$$\begin{aligned}
\triangle &:= \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > \cdots > x_n > 0\} \\
\triangle_0 &:= \{(x_0,\ldots,x_n) \in \triangle : x_1 + x_n > x_0\} \\
\triangle_1 &:= \{(x_0,\ldots,x_n) \in \triangle : x_1 + x_n < x_0\} \\
\triangle_D &:= \{(x_0,\ldots,x_n) \in \triangle : x_1 + x_n = x_0\}
\end{aligned}$$

and define the slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ by

$$\begin{aligned}
T(x_0,\ldots,x_n) &= \begin{cases} T_0(x_0,\ldots,x_n), & \text{if } x_1 + x_n > x_0 \\ T_1(x_0,\ldots,x_n), & \text{if } x_1 + x_n < x_0 \end{cases} \\
&= \begin{cases} (x_1, x_2, \ldots, x_n, x_0 - x_1), & \text{if } x_1 + x_n > x_0 \\ (x_0 - x_n, x_1, x_2, \ldots, x_n), & \text{if } x_1 + x_n < x_0 \end{cases}
\end{aligned}$$

The Triangle Map

$$T\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{cases} T_0\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, & \text{if } x_1 + x_n > x_0 \\ T_1\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, & \text{if } x_1 + x_n < x_0 \end{cases}$$

The Triangle Map

where

$$T_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Thus for $n = 2$, we have

$$T_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Triangle Map

$$T_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Both T_0^{-1} and T_1^{-1} have non-zero entries.

The Triangle Map

The *extended slow-Triangle map* $\tilde{T}$ will act on

$$(\lambda_0, \dots, \lambda_m) \times [k_0, \dots, k_m]$$

as the action of two $2(m+1) \times 2(m+1)$ matrices on column vectors in $\mathbb{R}^{2(m+1)}$, with the matrices

$$\begin{pmatrix} T_0 & 0 \\ 0 & (T_0^{-1})^\top \end{pmatrix}, \begin{pmatrix} T_1 & 0 \\ 0 & (T_1^{-1})^\top \end{pmatrix}.$$

The Triangle Map

$$\begin{array}{c}(\lambda_0, \ldots, \lambda_m) \times [k_0, \ldots, k_m] \\ \tilde{T}_0 \downarrow \\ (\lambda_1, \lambda_2 \ldots, \lambda_m, \lambda_0 - \lambda_1) \times [k_0 + k_1, k_2, \ldots, k_m, k_0]\end{array}$$

if $\lambda_1 + \lambda_m > \lambda_0$ and

$$\begin{array}{c}(\lambda_0, \ldots, \lambda_m) \times [k_0, \ldots, k_m] \\ \tilde{T}_1 \downarrow \\ (\lambda_0 - \lambda_m, \lambda_1, \ldots, \lambda_m) \times [k_0, \ldots, k_{m-1}, k_0 + k_m]\end{array}$$

if $\lambda_1 + \lambda_m < \lambda_0$

The Triangle Map

$$\begin{aligned}(14,7,6,5)\times[1,0,0,0] \quad &\xrightarrow{\tilde{T}_1} \quad (9,7,6,5)\times[1,0,0,1]\\ &\xrightarrow{\tilde{T}_0} \quad (7,6,5,2)\times[1,0,1,1]\\ &\xrightarrow{\tilde{T}_0} \quad (6,5,2,1)\times[1,1,1,1]\end{aligned}$$

Key is that the matrices T_0^{-1} and T_1^{-1} have non-negative entries, which means that the "multiplicities" never become negative.

The Triangle Map

What if

$$\lambda_0 = \lambda_1 + \lambda_m$$

$$\begin{array}{c} (\lambda_0, \dots, \lambda_m) \times [k_0, \dots, k_m] \\ \tilde{T}_D \downarrow \\ (\lambda_1, \lambda_2 \dots, \lambda_m) \times [k_0 + k_1, k_2, \dots, k_{m-1}, k_0 + k_m] \end{array}$$

The dimension drops.

Triangle Map and Integer Partitions

$$
\begin{aligned}
(14,7,6,5)\times[1,0,0,0] \quad & \xrightarrow{\tilde{T}_1} \quad (9,7,6,5)\times[1,0,0,1] \\
& \xrightarrow{\tilde{T}_0} \quad (7,6,5,2)\times[1,0,1,1] \\
& \xrightarrow{\tilde{T}_0} \quad (6,5,2,1)\times[1,1,1,1] \\
& \xrightarrow{\tilde{T}_D} \quad (5,2,1)\times[2,1,2] \\
& \xrightarrow{\tilde{T}_1} \quad (4,2,1)\times[2,1,4] \\
& \xrightarrow{\tilde{T}_1} \quad (3,2,1)\times[2,1,6] \\
& \xrightarrow{\tilde{T}_D} \quad (2,1)\times[3,8] \\
& \xrightarrow{\tilde{T}_D} \quad (1)\times[14]
\end{aligned}
$$

Triangle Map

T_D, while weird in dynamics, is natural here.

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_0}} (5,2,1,1) \times [2,1,1,1]$$

and

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_1}} (5,5,2,1) \times [1,1,1,2]$$

If you concatenate, you get

$$(6,5,2,1) \times [1,1,1,1] \xrightarrow{\tilde{T_D}} (5,2,1) \times [2,1,2]$$

New Partition Identities

$\mathcal{P}(N)$ = all partitions of N.

$$\begin{aligned}
\triangle &:= \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > x_n > 0\} \\
\triangle_0 &:= \{(x_0, \dots, x_n) \in \triangle : x_1 + x_n > x_0\} \\
\triangle_1 &:= \{(x_0, \dots, x_n) \in \triangle : x_1 + x_n < x_0\} \\
\triangle_D &:= \{(x_0, \dots, x_n) \in \triangle : x_1 + x_n = x_0\}
\end{aligned}$$

New Partition Identities

$\tilde{T}_0$ is one-to one on $\mathcal{P}(N) \cap \triangle_0$.

$\tilde{T}_1$ is one-to one on $\mathcal{P}(N) \cap \triangle_1$.

$\tilde{T}_D$ is not one-to one on $\mathcal{P}(N) \cap \triangle_D$.

New Partition Identities

Idea:

1. Start with an interesting subset of $\mathcal{P}(N)$
2. Apply $\tilde{T}$
3. Count image

New Partition Identities

Theorem

Every number has as many integer partitions into partitions with $\lambda_0 < \lambda_1 + \lambda_m$ as into partitions with $k_0 > k_m$. Similarly, every number has as many integer partitions into partitions with $\lambda_0 > \lambda_1 + \lambda_m$ as into partitions with $k_0 < k_m$.

There are many others.

All have generating function (q-series) interpretations.

Inverse Triangle Map

We now want to look at inverse maps.

T_0^{-1}
domain $\nabla_0 = \{k_0 > k_m\}$
$(x_0, \ldots, x_m) \times [k_0, \ldots, k_m]$ $\downarrow T_0^{-1}$ $(x_0 + x_m, x_0, x_1, \ldots, x_{m-1}) \times [k_m, k_0 - k_m, k_1, k_2, \ldots, k_{m-1}]$

Inverse Triangle Map

T_1^{-1}
domain $\nabla_1 = \{k_0 < k_m\}$
$(x_0, \ldots, x_m) \times [k_0, \ldots, k_m]$ $\downarrow T_1^{-1}$ $(x_0 + x_m, x_1, x_2, \ldots, x_m) \times [k_0, k_1, k_2, \ldots, k_{m-1}, k_m - k_0]$

Inverse Triangle Map

$$
\begin{array}{c}
T_D^{-1}(L) \\
\text{with } 1 \le L < \min\{k_0, k_m\} \\
\hline
\text{domain } \ \{\min\{k_0, k_m\} > 1\} \\
\hline
(x_0, \dots, x_m) \times [k_0, \dots, k_m] \\
\downarrow T_D^{-1}(L) \\
(x_0 + x_m, x_0, x_1, \dots, x_m) \\
\times [L, k_0 - L, k_1, k_2, \dots, k_{m-1}, k_m - L]
\end{array}
$$

Inverse Triangle Map

Now let $m = 2$.

T_0^{-1}
domain $\nabla_0 = \{k_0 > k_1\}$
$(x_0, x_1) \times [k_0, k_1]$ $\downarrow T_0^{-1}$ $(x_0 + x_1, x_0) \times [k_1, k_0 - k_1]$

Inverse Triangle Map

$$\begin{array}{c} T_1^{-1} \\ \hline \text{domain} \quad \nabla_1 = \{k_0 < k_1\} \\ \hline (x_0, x_1) \times [k_0, k_1] \\ \downarrow T_1^{-1} \\ (x_0 + x_1, x_1) \times [k_0, k_1 - k_0] \end{array}$$

Inverse Triangle Map

$$\begin{array}{c}
T_D^{-1}(L) \\
\text{with } 1 \leq L < \min\{k_0, k_1\} \\
\hline
\text{domain } \ \{\min\{k_0, k_1\} > 1\} \\
\hline
(x_0, x_1) \times [k_0, k_1] \\
\downarrow T_D^{-1}(L) \\
(x_0 + x_1, x_0, x_1) \times [L, k_0 - L, , k_1 - L] \\
\hline\hline
\end{array}$$

Inverse Triangle Map

There is one more case we need for the inverse, namely when $m = 1$. There is no inverse for T_0, T_1 but there will be for T_D.

$$\begin{array}{c}
T_D^{-1}(L) \\
\hline
\text{domain } \{k_0 > 1\} \\
\hline
(x_0) \times [k_0] \\
\downarrow T_D^{-1}(L) \text{ with } 1 \leq L < (1/2)k_1 \\
(2x_0, x_0) \times [L, k_0 - 2L] \\
\hline\hline
\end{array}$$

Tree Structure

If N is prime, start with

$$(1) \times [N]$$

and look at all inverse images of $\tilde{T}_i^{-1}$

Tree Structure

$$
\begin{tikzcd}
 & (1) \times [5] \arrow[dl] \arrow[dr] & (5) \times [1] \\
(2,1) \times [1,3] \arrow[d] & & (2,1) \times [2,1] \arrow[d] \\
(3,1) \times [1,2] \arrow[d] & & (3,2) \times [1,1] \\
(4,1) \times [1,1] & &
\end{tikzcd}
$$

Tree Structure

If $1 = d_1, \ldots, d_m = N$ are the factors of N, then there will be trees with roots

$$(d) \times [N/d].$$

For $N = 12$, we would have six trees, with roots

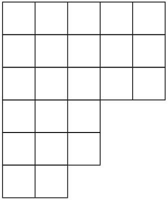
$$(1) \times [12], (2) \times [6], (3) \times [4], (4) \times [3], (6) \times [2], (12) \times [1].$$

Tree Structure

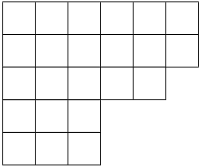
$$
\begin{tikzcd}
 & (5,2,1) \times [11,4,3] & \\
 & (6,5,2) \times [3,8,4] \arrow[u, "T_0"'] & \\
(8,5,2) \times [3,8,1] \arrow[ur, "T_1"] & (8,6,5,2) \times [1,2,8,3] \arrow[u, "T_D"'] & (8,6,5,2) \times [2,1,8,2] \arrow[ul, "T_D"']
\end{tikzcd}
$$

Young Conjugation

Recall conjugation, which is flipping a Young shape, turning the rows into columns, to get the *conjugate partition*
Flipping the Young shape of the partition $(5, 3, 2) \times [3, 2, 1] \vdash 23$ of the previous example gives us the Young shape



$\sim_{\mathcal{C}}$



which represents the conjugate partition

$$(5, 3, 2) \times [3, 2, 1] \sim_{\mathcal{C}} (6, 5, 3) \times [2, 1, 2]$$

Young Conjugation

Respects conjugation (is *Young compatible*):

Theorem

The diagram

$$\begin{array}{rcl} (\bar{\lambda}) \times [\bar{k}] & \sim_{\mathcal{C}} & \tilde{T}_0((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_0 \downarrow & & \uparrow \tilde{T}_0 \\ \tilde{T}_0((\bar{\lambda} \times [\bar{k}])) & \sim_{\mathcal{C}} & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m > \lambda_1$ *and*

$$\begin{array}{rcl} (\bar{\lambda}) \times [\bar{k}] & \sim_{\mathcal{C}} & \tilde{T}_0 1((\bar{\mu}) \times [\bar{l}]) \\ \tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\ \tilde{T}_1((\bar{\lambda} \times [\bar{k}])) & \sim_{\mathcal{C}} & (\bar{\mu}) \times [\bar{l}] \end{array}$$

when $\lambda_2 + \lambda_m < \lambda_1$ *are both commutative.*

Young Conjugation

$$\begin{array}{rcl}
(15,4)\times[1,1] & \sim_{\mathcal{C}} & (2,1)\times[4,11] \\
\tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\
(11,4)\times[1,2] & \sim_{\mathcal{C}} & (3,1)\times[4,7] \\
\tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\
(7,4)\times[1,3] & \sim_{\mathcal{C}} & (4,1)\times[4,3] \\
\tilde{T}_0 \downarrow & & \uparrow \tilde{T}_0 \\
(4,3)\times[4,1] & \sim_{\mathcal{C}} & (5,4)\times[3,1] \\
\tilde{T}_0 \downarrow & & \uparrow \tilde{T}_0 \\
(3,1)\times[5,4] & \sim_{\mathcal{C}} & (9,5)\times[1,2] \\
\tilde{T}_1 \downarrow & & \uparrow \tilde{T}_1 \\
(2,1)\times[5,9] & \sim_{\mathcal{C}} & (14,5)\times[1,1]
\end{array}$$

Why triangle map

There are many different multi-dimensional continued fraction algorithms.

Why use the triangle map?

Most multi-dimensional continued fraction algorithms seem to be not “partition friendly”.

For example, for both Mönkemeyer and Cassaigne, the multiplicities k start becoming negative numbers.

Why triangle map

From work of Cassaigne, Labbé and Leroy, there is the *extended slow-Cassaigne map* $\tilde{C}$, which is

$$\tilde{C}((n_1, n_2, n_3) \times [k_1, k_2, k_3])$$

is

$$(n_2, n_3, n_2 + n_3 - n_1) \times [k_1 + k_2, k_1 + k_3, -k_1],$$

if $n_2 + n_3 > n_1$ and is

$$(n_1 - n_3, n_2, n_2 - n_3) \times [k_1, k_1 + k_2 + k_3, -k_1 - k_3],$$

if $n_2 + n_3 < n_1$

Why triangle map

Consider the partition

$$(7, 5, 4) \times [3, 2, 4] \vdash 47$$

We have

$$\tilde{C}((7, 5, 4) \times [3, 2, 4]) = (5, 4, 2) \times [5, 7, -3].$$

That -3 for one of the multiplicities means that this dynamical system will also not generate partitions.

Why triangle map

Of course, the creation of this map $\mathcal{C}$ was motivated to find a multidimensional continued fraction algorithm that produces infinite words on three letters whose linear complexity is exactly $2n+1$. (And thus to find the "right" analog of the link between Sturmian sequenceses and low complexity.

TG and Osterman have recently shown, in part conjecturally, part numerically and part via proofs, that this algorithm is the MCF with lowest linear complexity. (The triangle map has linear complexity at most $3n$.)

Why triangle map

Dasaratha, Flapan, TG, C. Lee, C. Mihaila, N. Neumann-Chun, S. Peluse, M. Stoffregen (2014) set up a conceptual framework for all possible multi-dimensional continued fraction algorithms (call Triangle Partition Maps).

For each dimension n, the Gauss version paramerized by an element of

$$S_{n-1} \times S_{n-1} \times S_{n-1}$$

Half this number for the corresponding Farey version.

Why triangle map

n	$\frac{\lvert S_{n-1} \times S_{n-1} \times S_{n-1} \rvert}{2}$	number that are partition friendly
2	4	1
3	108	2
4	6912	3
5	864000	4
6	186624000	5
7	64012032000	6
8	32774160384000	7

Why the triangle map

Matthew Phang has shown that the Selmer and the Brun algorithms are partition friendly

Neither respect conjugation of the Young shape.

Neither do the other few examples that are partition friendly

Partitions and the Natural Extension

So far we used the triangle map to understand partitions.

Can partitions be used to understand anything new about the triangle map?

Partitions and the Natural Extension

The natural extension, for each m, is

$$\mathcal{P}(N)$$

is all

$$(\lambda_0, \ldots, \lambda_m) \times [k_0, \ldots, k_m]$$

with each $\lambda_i, k_j \in \mathbb{R}$,

$$\lambda_0 > \cdots > \lambda_m > 0, k_i > 0$$

and

$$\begin{aligned} N &= k_0\lambda_0 + \ldots + k_m\lambda_m \\ &= (k_0, \ldots, k_m) \cdot \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_m \end{pmatrix} \end{aligned}$$

Triangle Map and Integer Partitions

$$
\begin{array}{rcl}
\mathrm{Domain}(\tilde{T}_0) \cap \mathrm{Image}(\tilde{T}_0) & \xrightarrow{\mathcal{C}} & \mathrm{Domain}(\tilde{T}_0) \cap \mathrm{Image}(\tilde{T}_0) \\
\mathrm{Domain}(\tilde{T}_0) \cap \mathrm{Image}(\tilde{T}_1) & \xrightarrow{\mathcal{C}} & \mathrm{Domain}(\tilde{T}_1) \cap \mathrm{Image}(\tilde{T}_0) \\
\mathrm{Domain}(\tilde{T}_1) \cap \mathrm{Image}(\tilde{T}_0) & \xrightarrow{\mathcal{C}} & \mathrm{Domain}(\tilde{T}_0) \cap \mathrm{Image}(\tilde{T}_1) \\
\mathrm{Domain}(\tilde{T}_1) \cap \mathrm{Image}(\tilde{T}_1) & \xrightarrow{\mathcal{C}} & \mathrm{Domain}(\tilde{T}_1) \cap \mathrm{Image}(\tilde{T}_1)
\end{array}
$$

Thus Young conjugation gives us an involution of the natural extension. This seems new.

Natural Extension

Straightforward to prove a natural extension must exist, abstractly. Usually hard to make it concrete.

This is easy in the context of the extended triangle map. The natural extension can be described quite cleanly.

Tree Structure and the Natural Extenstion

The structure of the trees reflect the underlying dynamics of the various natural extensions.

There are far, far more paths involving $\tilde{T}_1^{-1}$ than $\tilde{T}_0^{-1}$ reflects that in all the natural extensions, the origin is an indifferent fixed point, and the corresponding invariant measures have infinite volume near the origin.

Tree Structure and the Natural Extenstion

For $n = 2$, $\tilde{T}$ is ergodic is classical.

For $n = 3$, Messaoudi, Nogueira, and Schweiger (2009) showed $\tilde{T}$ is ergodic.

For $n > 3$, TG and Lehmann Duke (2024) showed $\tilde{T}$ is ergodic.

Tree Structure and the Natural Extenstion

After de-homogenizing, the invariant measure is

$$\frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_{n-1}(1 + x_n)}$$

The origin is an indifferent fixed point. This is the underlying reason that there are far, far more paths involving $\tilde{T}_1^{-1}$ than $\tilde{T}_0^{-1}$

Sources

1. "On integer partitions and continued fraction type algorithms" by Baalbaki, Bonanno, Del Vigna, TG,Isola in *Ramanujan J.* (2024)
2. "Generating new partition identities via a generalized continued fraction algorithm" by Baalbaki and TG in *Electron. J. Combinatorics* (2024),
3. "Ergodicity and Algebraticity of the Fast and Slow Triangle Maps" by TG and Lehmann Duke, https://arxiv.org/abs/2409.05822
4. "Methods for Obtaining Partition Identities Using the Selmer and Brun Algorithm", Phang, Senior Thesis,, Williams College, 2023.
5. "Tree Structures on Partitions Shaped by the Dynamics of the Triangle Map", Fox, Senior Thesis,, Williams College, 2024.

Questions

1. Is it true that the triangle map is the only multi-dimensional continued fraction algorithm that is both partition friendly and Young compatible?
2. Understand the nature of the tree structure
3. Direct proofs of generating function identities.
4. Find more identities
5. Can you put "q" into this language. (Maybe link with work of Sophie Morier-Genoud, Valentin Ovsienko and collaborators)
6. Use integer partitions to understand the dynamics
7. Multi-dimensional continued fractions can be linked to billards, translations surfaces, automata theory, etc. Can integer partition theory be used?

Homework

1. Find the path under $\tilde{T}$ of
$$(12, 7, 3, 2) \times [2, 3, 1, 5]$$
2. Find the tree structure for all $\mathcal{P}(N)$, for
$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$
3. For $\triangle = \{1 > \lambda_2 > \lambda_3 > 0\}$, find the values of λ_2, λ_3 whose triangle sequence is
$$(0, 1, 0, 1, 0, 1, \dots).$$

THANKS