Zero measure Cantor spectrum for Schrödinger operators with quasi-periodic potentials

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Outline

QM in 1 dimension on a lattice

Describe a particle by a wave function $\psi \in \ell^2(\mathbb{Z})$.

- Probability of finding the particle in a region $A \subset \mathbb{Z}$:

$$P(A) = \sum_{n \in A} |\psi_n|^2.$$

External potential $V = (V_n)_{n \in \mathbb{Z}}$. How does ψ change over time?

$$\mathrm{i}\partial_t \psi(t) = H_V \psi(t)$$

$H_V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the Schrödinger operator

$$\boxed{(H_V \psi)_n = \psi_{n-1} + \psi_{n+1} + V_n \psi_n.}$$

Time evolution

Assume $(V_n)_{n\in\mathbb{Z}}$ is bounded.

$\Rightarrow$ H_V is a bounded, self-adjoint operator.

$\Rightarrow$ $\psi(t) = \mathrm{e}^{-\mathrm{i}tH_V}\psi(0)$.

Idea: "Diagonalize" the operator H_V.

$$\sigma(H_V) = \{E \in \mathbb{R} \mid (H_V - E)^{-1} \text{ does not exist}\},$$

is the *spectrum* of H_V.

$\sigma(H_V) \equiv$ frequencies in the expansion of $\mathrm{e}^{-\mathrm{i}tH_V}$.

The role of the potential

Reminder: $\sigma(H_V) = \{E \in \mathbb{R} \mid (H_V - E)^{-1}$ does not exist$\}$,

$$(H_V \psi)_n = \psi_{n-1} + \psi_{n+1} + V_n \psi_n.$$

Structural properties of $\sigma(H_V)$ highly depend on V!

- V periodic: $\sigma(H_V)$ is a finite union of closed intervals (electronic bands).
- V_n are iid random variables on a finite set $\mathcal{A} \subset \mathbb{R}$: again finite union of intervals a.s. (but different spectral type).
- V substitutive: $\sigma(H_V)$ is a Cantor set of 0-Lebesgue measure.

Substitutive potentials

Example: Fibonacci substitution.

$$\varrho\colon \begin{cases} 0 \mapsto 01, \\ 1 \mapsto 0, \end{cases} \qquad M_\varrho = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

M_ϱ^2 has only positive entries $\to$ ϱ is *primitive*.
Iterating the substitution

$$1|0 \mapsto 0|01 \mapsto 01|010 \mapsto 010|01001 \dots$$

Take V to be a fixed point of ϱ^2 in $\{0,1\}^{\mathbb{Z}}$.

- $\Rightarrow$ V is non-periodic as a sequence, has self-similar properties.
- $\Rightarrow$ 1-dim model of a quasicrystal.

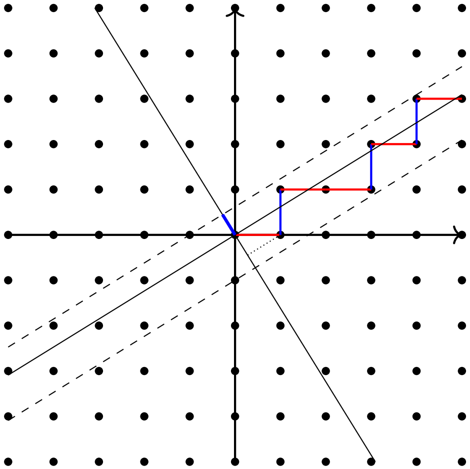
Substitutive potentials II

Theorem (Bovier–Ghez '93)

If V is the fixed point of a primitive substitution, then $\sigma(H_V)$ is a Cantor set of Lebesgue measure 0.

Fibonacci sequence from a torus translation

$V = \cdots | 01001010 \cdots$



Sampling potentials

More generally, $V_n = g(R_\alpha^n x)$ with

- $x, \alpha \in \mathbb{T}$.
- $R_\alpha \colon \mathbb{T} \to \mathbb{T}$, $x \mapsto x + \alpha \mod 1$.
- $g \colon \mathbb{T} \to \mathbb{R}$.

That is, we sample along the orbit of a torus translation.

Theorem (Bellissard–Iochum–Scoppola–Testard '89)

If $g = \lambda \chi_{[1-\alpha,1)}$, with $\lambda > 0$ and α irrational, the spectrum $\sigma(H_V)$ is a Cantor set of Lebesgue measure 0.

The Fibonacci substitution sequence is a special case.

The almost Mathieu operator

H_V, with $V_n = \lambda \cos(2\pi(x + n\alpha))$.
That is, $g(x) = \lambda \cos(2\pi x)$ is the sampling function.

Theorem (Avila–Jitomirskaya '09)

For every $\lambda \neq 0$, the spectrum of the almost Mathieu operator $\sigma(H_V)$ is a Cantor set.

Theorem (Jitormirskaya–Kravsovsky '02, Avila–Krikorian '05)

The Lebesgue measure of $\sigma(H_V)$ is given by $|4 - 2|\lambda||$. It is 0 precisely if $\lambda = 2$.

Torus-dimensions $d > 1$

What if we sample on a higher-dimensional torus?
$V_n = g(R_\alpha^n x)$, now $R_\alpha \colon \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto x + \alpha$.
Some remarks:

- $d > 1$ is much less understood than $d = 1$.
- Cantor spectrum is generic in $C(\mathbb{T}^d)$ for general $d \in \mathbb{N}$.
- How to create examples with Lebesgue measure 0?

The Tribonacci substitution

$$\varrho\colon \begin{cases} 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 1, \end{cases} \qquad M_\varrho = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

ϱ^3 has a fixed point $u = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$.
For a prefix $u_0 \cdots u_n$ take the abelianisation vector $v^n = (v_1^n, v_2^n, v_3^n) \in \mathbb{N}^3$. Here, $v_i^n = \#_i(u_0 \cdots u_n)$.
For all $n \in \mathbb{N}$, the vector v_n remains close to the line spanned by the expanding eigenvector of M_ϱ.

Rauzy fractal

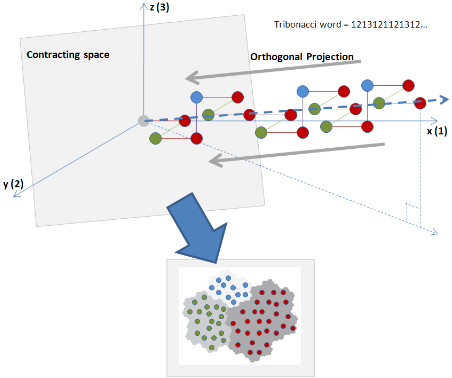
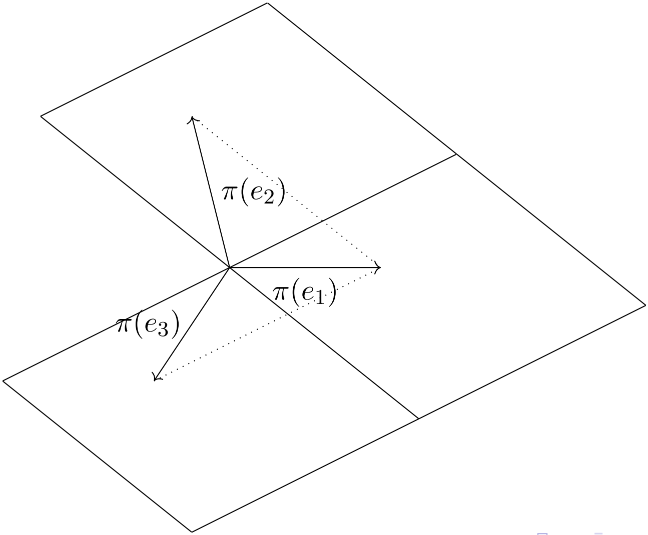


Figure: Image by Prokofiev, published under a CC-BY-SA-3.0 license.

Choice of the torus



Rauzy fractal II

The projection of the Rauzy fractal to $\mathbb{T}^2$ is $1:1$ almost everywhere.

- ⇒ Partitions $\mathbb{T}^2$ into smaller sets $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3\}$.
- ⇒ Gives sampling function to recover u from R_α.

$$g(x) = \sum_{j=1}^{3} j\chi_{\mathbb{T}_j}(x)$$

satisfies $g(R_\alpha^n x) = j$ iff $R_\alpha^n x \in \mathbb{T}_j$.

Sliding block codes

Goal: approximate arbitrary $g \in C(\mathbb{T})$ by elementary functions, maintaining Cantor spectrum of Lebesgue measure 0 (CSL0).

$$
\begin{array}{ccccccccccccc}
\cdots & x_{-3} & x_{-2} & x_{-1} & x_0 & \boxed{x_1 \quad x_2 \quad x_3 \quad x_4} & x_5 & x_6 & \cdots \\
 & & & & & \downarrow h & & & \\
\cdots & y_{-3} & y_{-2} & y_{-1} & y_0 & \boxed{y_1} \quad y_2 \quad y_3 \quad y_4 & y_5 & y_6 & \cdots
\end{array}
$$

Sliding block code: $\phi_h \colon x \mapsto y$ with $y_n = h(x_{[n,n+3]})$, h injective.

- $\phi_h(V)$ still has CSL0, for V the Tribonacci sequence.
- $V_{[n,n+3]} = w$ iff $R_\alpha^n x \in \mathbb{T}_w$, for some $\mathbb{T}_w \subset \mathbb{T}$.

$\implies$ $\phi_h(V)$ from finer partition $\{\mathbb{T}_w \colon |w| = 4\}$.

S-adic sequences

Can we get such a result for *almost every* $\alpha \in \mathbb{T}^2$? Problem: there are only countably many substitutions!
S-adic system: directive sequence of substitutions $\tau = (\tau_n)_{n\in\mathbb{N}}$, drawn from a finite set $\{\gamma_1, \cdots, \gamma_n\}$. Consider

$$u_\tau = \lim_{n\to\infty} \tau_1 \circ \tau_2 \circ \ldots \circ \tau_n(a_n|b_n).$$

Steps:

1. Condition $(*)$ on $(\tau_n)_{n\in\mathbb{N}}$ such that H_V with $V = u_\tau$ has CSL0.
2. Relate (R_α, T) to an S-adic system for a.e. $\alpha \in \mathbb{T}^2$.

Boshernitzan's condition

Symbolic dynamical system from sequence $V \in \mathcal{A}^{\mathbb{Z}}$:

- $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ with $(SW)_n = W_{n+1}$.
- $\mathbb{X}_V = \overline{\{S^j V : j \in \mathbb{Z}\}}$.

Equip $(\mathbb{X}_V, S)$ with some S-invariant measure μ. Cylinder sets:

$$[w_1 \cdots w_n] = \{W \in \mathbb{X}_V : W_1 \cdots W_n = w_1 \cdots w_n\}.$$

Definition

$(\mathbb{X}_V, S)$ satisfies *Boshernitzan's condition* (B) if there is an S-invariant probability measure μ on $\mathbb{X}_V$ and $C > 0$ with

$$\inf\{\mu([w_1 \cdots w_n]) : [w_1 \cdots w_n] \neq \varnothing\} > \frac{C}{n},$$

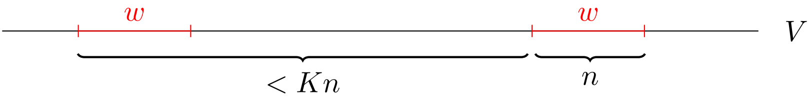
for infinitely many $n \in \mathbb{N}$.

$(B) \implies$ CSL0

Theorem (Damanik–Lenz '06)

If $(\mathbb{X}_V, S)$ is minimal and satisfies (B), H_W has CSL0 for every $W \in \mathbb{X}_V$, unless V is periodic.

This is satisfied if V is *linearly recurrent*:



Implies for $w = w_1 \cdots w_n$

$$\mu([w]) = \lim_{m \to \infty} \frac{\#_w(V_1 \cdots V_m)}{m} > \frac{1}{Kn}.$$

Property $(*)$

Substitution γ acting on $\mathcal{A}$.

- γ is *positive* if $\gamma(a)$ contains all letters for all $a \in \mathcal{A}$.
- γ is a *pair builder* if $\gamma(a)$ contains all words $v_1 v_2$ that occur in some $\gamma(w_1 w_2)$.

Definition

We say that $(\tau_n)_{n\in\mathbb{N}}$ satisfies $(*)$ if there are infinitely many $n \in \mathbb{N}$ and $j_n, k_n \leqslant N \in \mathbb{N}$ such that

1. $\tau_{n+1} \circ \cdots \circ \tau_{n+j_n}$ is positive.
2. $\tau_{n+j_n+1} \circ \cdots \circ \tau_{n+j_n+k_n}$ is a pair builder.

Weak linear recurrence

Property (∗) gives “linear recurrence” on infinitely many scales.

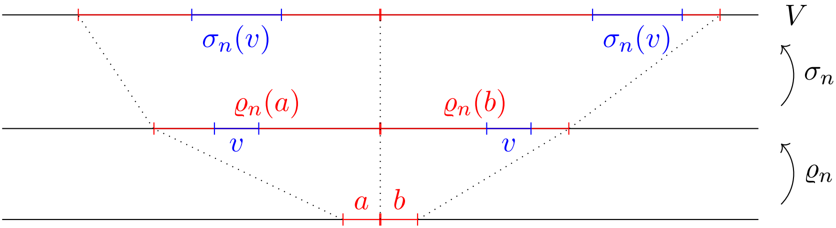


Figure: $\sigma_n = \tau_1 \circ \cdots \tau_{n+j_n}$ and $\varrho_n = \tau_{n+j_n+1} \circ \cdots \circ \tau_{n+j_n+k_n}$.

- Positivity: σ_n inflates sizes almost uniformly.
- Pair builder: $v = v_1 v_2$ in every $\varrho_n(a)$ with $a \in \mathcal{A}$.

$(*) \implies$ CSL0

Proposition (Chaika–Damanik–Fillman–G)

Assume $(\tau_n)_{n\in\mathbb{N}}$ satisfies property $(*)$ and $V = u_\tau$. Then, for all $W \in \mathbb{X}_V$, $\sigma(H_W)$ is a Cantor set of Lebesgue measure 0.

Chain of implications:

$$(*) \implies \text{weak LR} \implies (B) \implies \text{CSL0}.$$

Cassaigne–Selmer substitutions

$$\gamma_1\colon \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 13, \\ 3 \mapsto 2, \end{cases} \qquad \gamma_2\colon \begin{cases} 1 \mapsto 2, \\ 2 \mapsto 13, \\ 3 \mapsto 3, \end{cases}$$

and $\tau = (\tau_n)_{n\in\mathbb{N}}$, with $\tau_n \in \{\gamma_1, \gamma_2\}$.

Theorem (Berthé–Steiner–Thuswaldner '20)

For almost every $\alpha \in \mathbb{T}^2$ there exists a directive sequence $\tau = \tau(\alpha)$ such that u_τ is a coding of the torus translation R_α.

Remark: the sampling function is an elementary function on sets that are (generalized) Rauzy fractals.

Main result

Recall that $V_n = g(R_\alpha^n x)$.

Theorem (Chaika–Damanik–Fillman–G)

For almost every $\alpha \in \mathbb{T}^2$, the corresponding directive sequence $\tau(\alpha)$ satisfies property $(*)$. In particular, for almost every $\alpha \in \mathbb{T}^2$ there exists a sampling function g such that $\sigma(H_V)$ is a Cantor set of Lebesgue measure 0 (irrespective of x).

Remark: The $|\cdot|_\infty$-closure of such functions g contains $\mathcal{C}(\mathbb{T})$.

Summary

$$(H_V\psi)_n = \psi_{n-1} + \psi_{n+1} + V_n\psi_n.$$

where $V_n = g(R_\alpha^n x)$ is quasi-periodic.
The spectrum $\sigma(H_V) = \{E \in \mathbb{R} \mid (H_V - E)^{-1} \text{ does not exist}\}$ is a Cantor set of Lebesgue measure 0 if

- $d = 1$ and $\alpha \in \mathbb{T}^1$ is irrational, g appropriate step function.
- $d = 1$, $g(x) = 2\cos(2\pi x)$ and α irrational. Here, the factor 2 is crucial!
- $d = 2$, for almost every $\alpha \in \mathbb{T}^2$ and appropriate g.
- **Q**: Can we choose $g \in \mathcal{C}(\mathbb{T}^2)$?
- **Q**: What about $d > 2$? At least $d = 3$ seems within reach.

Background on aperiodic order and Schrödinger operators

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Thank you for your attention!