Good's Theorem for Hurwitz continued fractions

Gerardo González Robert

Universidad Nacional Autónoma de México

February 16, 2021

Outline

The Real Gauss Map

Define the following functions.

- $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ is the usual floor function,
- The real Gauss map is the function $T_{\mathbb{R}} : [0,1) \to [0,1)$ given by

$$\forall \alpha \in [0,1) \quad T_{\mathbb{R}}(\alpha) = \begin{cases} \alpha^{-1} - \lfloor \alpha^{-1} \rfloor, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

- $a_1 : (0,1) \to \mathbb{N}$ by $a_1(\alpha) = \lfloor \alpha^{-1} \rfloor$,
- For $\alpha \in (0,1)$ and $n \in \mathbb{N}$ such that $T_{\mathbb{R}}^n(\alpha) \neq 0$,

$$a_n(\alpha) = a_1(T_{\mathbb{R}}^n(\alpha)).$$

The Real Gauss Map

For any $\alpha \in (0,1)$ we have that

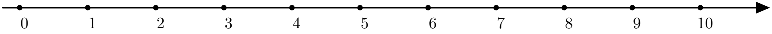
$$\alpha = [0; a_1(x), a_2(x), \ldots]_{\mathbb{R}} := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}}.$$

The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.

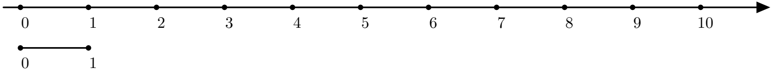
The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



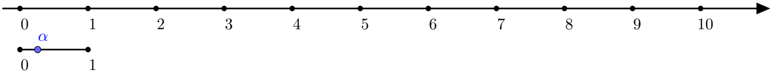
The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



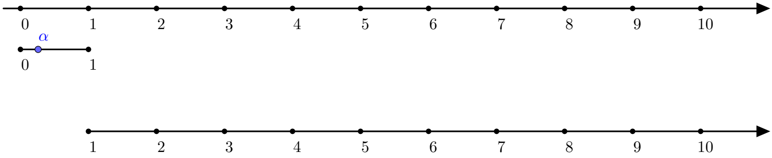
The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



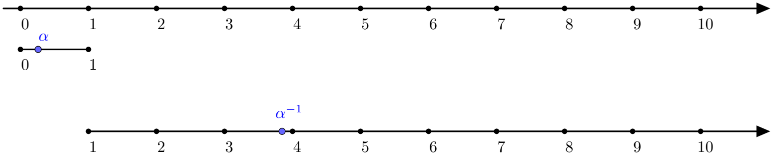
The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



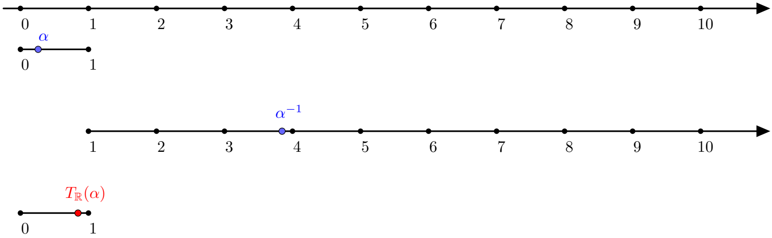
The Real Gauss Map

Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



The Real Gauss Map

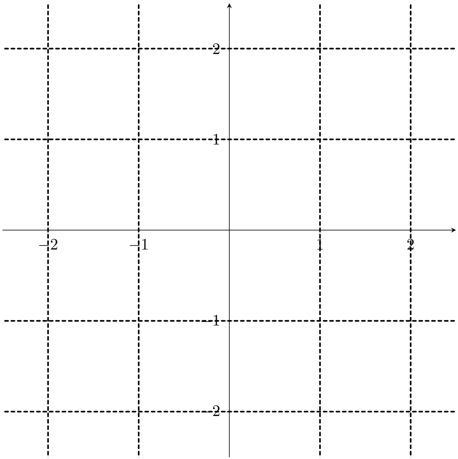
Take $0 < \alpha < 1$, $T_{\mathbb{R}}(\alpha) = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor = \frac{1}{\alpha} - a_1(\alpha)$.



Goal. Write any complex number z as a continued fraction of the form

$$z = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \qquad a_n \in \mathbb{Z}[i] \text{ for all } n.$$

A failed continued fraction expansion. The partition.



A failed continued fraction expansion

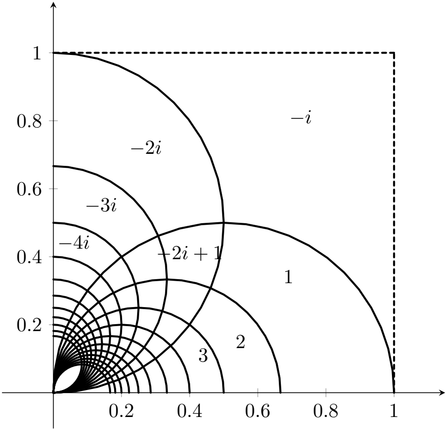
Define

$$\mathfrak{S} := \{z \in \mathbb{C} : 0 \le \Re(z) < 1,\ 0 \le \Im z < 1\}.$$

For any $z \in \mathfrak{S} \setminus \{0\}$ let $a_1(z) \in \mathbb{Z}[i]$ be the unique Gaussian integer satisfying

$$\frac{1}{z} - a_1\left(\frac{1}{z}\right) \in \mathfrak{S}.$$

A failed continued fraction expansion. Partition of $\mathfrak{S}$.

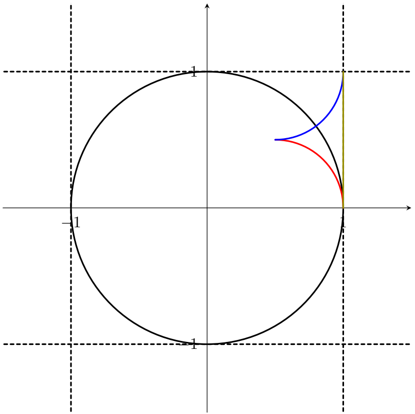


A failed continued fraction expansion

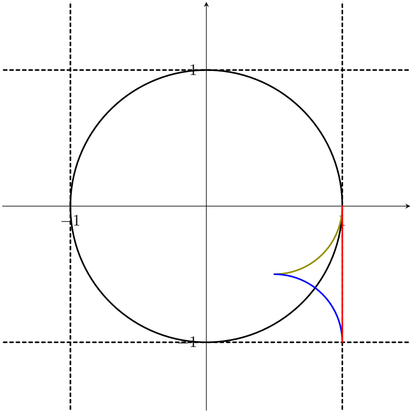
Let R the curves

i. The circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$.
ii. The circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}+i$.
iii. The line segment joining 1 with $1+i$.

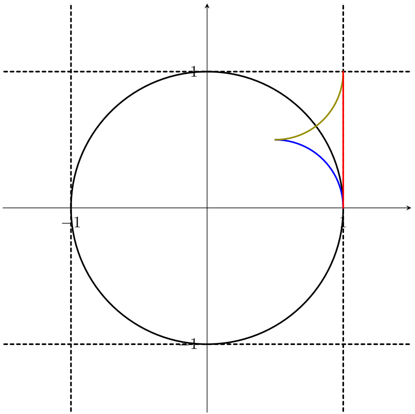
A failed continued fraction expansion.



A failed continued fraction expansion.



A failed continued fraction expansion.



A failed continued fraction expansion

We cannot have $a_n = -i$ for $n \in \mathbb{N}$ in

$$z = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}.$$

The **nearest Gaussian integer function** $[\cdot] : \mathbb{C} \to \mathbb{Z}[i]$ is given by

$$\forall z \in \mathbb{C} \quad [z] := \left\lfloor \Re(z) + \frac{1}{2} \right\rfloor + i \left\lfloor \Im(z) + \frac{1}{2} \right\rfloor.$$

Definition

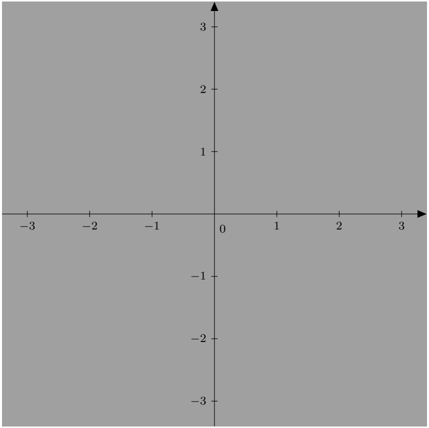
Define $\mathfrak{F} := \{z \in \mathbb{C} : [z] = 0\}$, $T : \mathfrak{F} \to \mathfrak{F}$ *by*

$$\forall z \in \mathfrak{F} \quad T(z) = \begin{cases} z^{-1} - [z^{-1}], & \textit{if } z \neq 0, \\ 0, & \textit{if } z = 0. \end{cases}$$

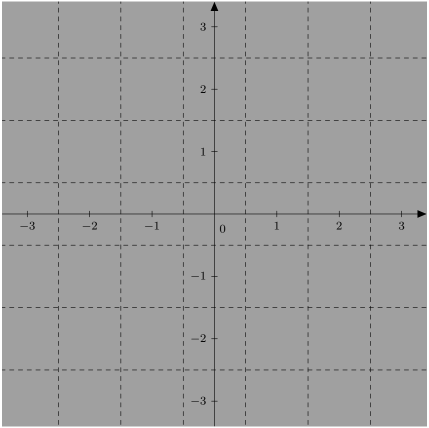
Define $a_1 : \mathfrak{F} \setminus \{0\} \to \mathbb{Z}[i]$ *by* $a_1(z) = [z^{-1}]$, $a_n(z) = a_1(T^n(z))$ *whenever* $T^n(z) \neq 0$, *and* $a_0 : \mathbb{C} \to \mathbb{Z}[i]$ *by* $a_0(z) = [z]$. *The* **Hurwitz continued fraction** *of a complex number* z *is*

$$a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{a_3(z) + \cfrac{1}{\ddots}}}}.$$

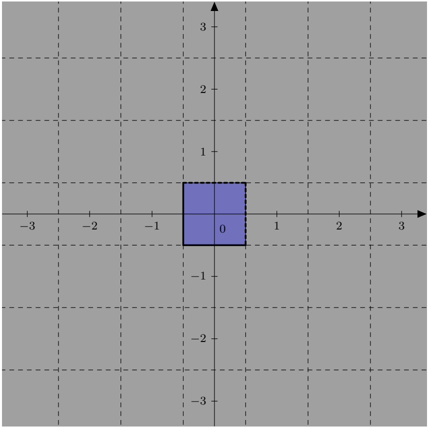
Hurwitz Continued Fraction Process



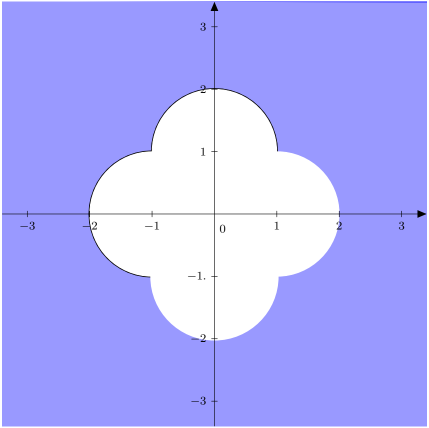
Hurwitz Continued Fraction Process



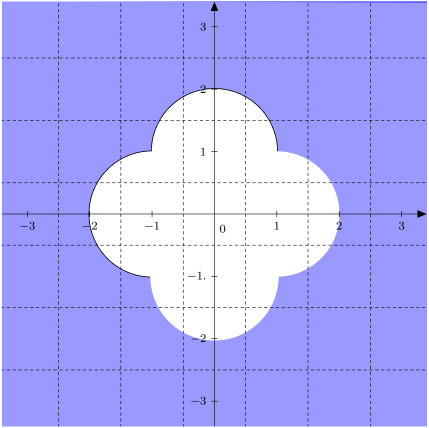
Hurwitz Continued Fraction Process



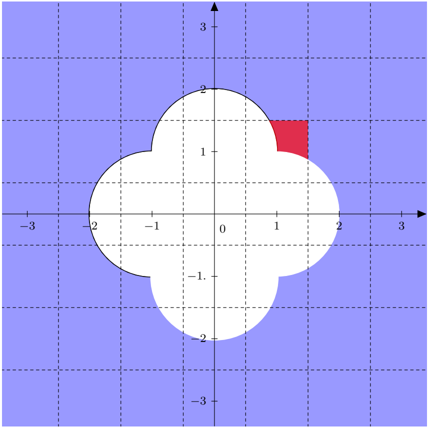
Hurwitz Continued Fraction Process



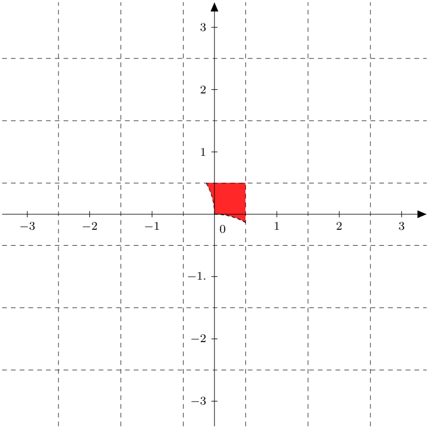
Hurwitz Continued Fraction Process



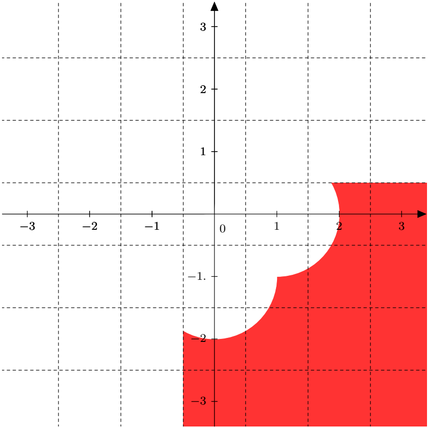
Hurwitz Continued Fraction Process



Hurwitz Continued Fraction Process



Hurwitz Continued Fraction Process



Shift space

Proposition

There is no function $M : \mathbb{Z}[i] \times \mathbb{Z}[i] \to \{0,1\}$ *such that a sequence in* $\mathbb{Z}[i]$, $(a_n)_{n\geq 1}$, *is the sequence of Hurwitz elements of some* $z \in \mathfrak{F}$ *if and only if*

$$\forall n \in \mathbb{N} \quad M(a_n, a_{n+1}) = 1.$$

Some aspects of the basic theory of Hurwitz continued fractions are discussed in [6], [8], [10], and [11].

Partition of $\mathfrak{F}$ by a_1

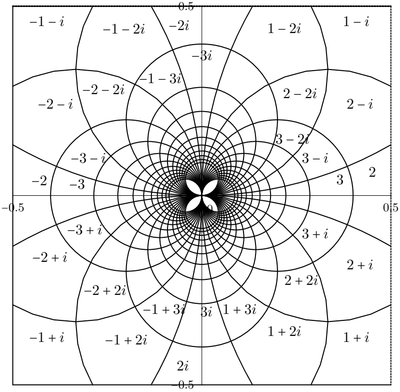


Figure 1: Partition of $\mathfrak{F}$.

A basic definition

Definition

*Let z be a complex number and $(a_n)_{n\geq 0}$ its Hurwitz elements. The $\mathcal{Q}$-**pair** of z is the pair of sequences $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ given by*

$$\begin{pmatrix} p_{-2} & p_{-1} \\ q_{-2} & q_{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix},$$

as long as a_n is defined.

Theorem

Let z be any complex number, $(a_n)_{n\geq 0}$ its Hurwitz elements, and $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ its $\mathcal{Q}$-pair.

1. *(Hurwitz, [11]) $(q_n)_{n\geq 0}$ is strictly increasing,*
2. *(Dani, Nogueira, [5]) There is a constant $\kappa > 1$ such that $|q_n| \geq \kappa^n$ whenever q_n is defined.*
3. *(Dani, Nogueira, [5]) If $z \in \mathfrak{F}$, $n \in \mathbb{N}$ is such that $z_{n+1} := T^{n+1}(z) \neq 0$, then*
$$z = \frac{p_n z_{n+1}^{-1} + p_{n-1}}{q_n z_{n+1}^{-1} + q_{n-1}}.$$
4. *(Hurwitz, [11]) The sequence $(a_n)_{n\geq 0}$ is infinite if and only if $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ and in this case*
$$z = [a_0; a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

The Theorems of Lagrange and Galois

Theorem (S.G. Dani, A. Nogueira, 2014 [5])

A complex number z has a periodic Hurwitz continued fraction if and only if there exist $a, b, c \in \mathbb{Z}[i]$ such that $a \neq 0$ and

$$az^2 + bz + c = 0.$$

Theorem (G.R., 2018 [8])

Let $\xi = [a_0; a_1, a_2, \ldots]$ be quadratic over $\mathbb{Q}(i)$ and let $\eta \in \mathbb{C}$ be its conjugate over $\mathbb{Q}(i)$.

1. *If $(a_n)_{n\geq 1}$ is purely periodic, then $|\eta| < 1$.*
2. *If $|\xi| > 1$, $\eta \in \mathfrak{F}$, and $|a_n| \geq \sqrt{8}$ for all $n \in \mathbb{N}_0$, then ξ has a purely periodic Hurwitz continued fraction. The conditions $\eta \in \mathfrak{F}$ and $|a_n| \geq \sqrt{8}$ for all $n \in \mathbb{N}_0$ cannot be removed.*

Theorem (J.A. Serret, 1866)

Two real numbers $\alpha = [a_0; a_1, \ldots]_{\mathbb{R}}, \beta = [b_0; b_1, \ldots]_{\mathbb{R}}$ *are equivalent under the action of* $\mathrm{PGL}(2, \mathbb{Z})$ *if and only if* $\alpha, \beta \in \mathbb{Q}$ *or if* $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ *and for some* $j, k \in \mathbb{N}$ *we have* $a_{j+n} = b_{k+n}$ *for all* $n \in \mathbb{N}$.

Theorem (R. Lakein, [15], A. Lukyanenko, J. Vandehey, [16])

The complex numbers

$$\Xi = \frac{i + (43 + 28i)^{\frac{1}{2}}}{2}, \quad A = \frac{5 - i - \Xi}{4 - i}, \quad B = \frac{3 + 2i + \Xi}{4}$$

satisfy $A = (2B - i)/(B - i)$, *but*

$$A = [\overline{2 + i, 3i, -1 + 2i, -1 + 2i, 3, -2 - i}],$$
$$B = [\overline{2 + i, -2 + i, -2 + i, 1 - 2i, -1 - 2i, 1 + 2i}].$$

However, for almost every pair $w, z \in \mathfrak{F}$, *the numbers are* $PGL(2, \mathbb{Z}[i])$ *equivalent if and only if their tails eventually coincide.*

The exponent of repetition $\operatorname{rep}(\mathbf{a})$ is defined in [3].

Theorem (Y. Bugeaud, 2013, [2])

Let $\mathbf{a} = (a_n)_{n\geq 1}$ be a bounded and non-periodic sequence of natural numbers such that

$$\operatorname{rep}(\mathbf{a}) < +\infty.$$

Then, the number $[0; a_1, a_2, \ldots]_{\mathbb{R}}$ is transcendental.

Theorem (G.R., 2018 [8])

Let $\mathbf{a} = (a_n)_{n\geq 1}$ be a bounded and non-periodic sequence of Gaussian integers such that

$$\operatorname{rep}(\mathbf{a}) < +\infty, \quad \min_{n\in\mathbb{N}} |a_n| \geq \sqrt{8}.$$

Then, the complex number $[0; a_1, a_2, \ldots]$ is transcendental.

Conjecture (Folklore Conjecture)

An algebraic real number has a bounded regular continued fraction if and only if it is quadratic over $\mathbb{Q}$.

Theorem (W. Bosma, D. Gruenewald, 2012 [1])

For each $n \in \mathbb{N}$ *there exists* $\zeta \in \mathbb{C} \setminus \mathbb{R}$ *with a bounded Hurwitz continued fraction satisfying* $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 2n$.

Theorem (R. Lakein, 1973, [14])

For any $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ with $\mathcal{Q}$-pair $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$, and any $n \in \mathbb{N}$ define $m_n(z) \in \mathbb{C}$ by

$$z - \frac{p_n}{q_n} = \frac{1}{m_n(z) q_n^2}.$$

Then, $\inf\{|m_n(z)| : n \in \mathbb{N}, z \in \mathbb{C} \setminus \mathbb{Q}(i)\} = 1$.

Corollary

For every $z \in \mathbb{C} \setminus \mathbb{Q}(i)$ with $\mathcal{Q}$-pair $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ and any $n \in \mathbb{N}$

$$\left| z - \frac{p_n}{q_n} \right| < \frac{1}{|q_n|^2}.$$

Let z be a complex number. A rational complex number p/q is a **good approximation** to ζ if

$$\forall q' \in \mathbb{Z}[i] \quad \forall p' \in \mathbb{Z}[i] \quad |q'| \leq |q| \implies |qz - p| \leq |q'z - p'|. \qquad (1)$$

and p/q is a **best approximation** to ζ if

$$\forall q' \in \mathbb{Z}[i] \quad \forall p' \in \mathbb{Z}[i] \quad |q'| < |q| \implies |qz - p| < |q'z - p'|. \qquad (2)$$

Theorem (R. Lakein, 1973 [14])

Every HCF convergent of any $z \in \mathbb{C}$ is a good approximation to z. For Lebesgue almost all $z \in \mathbb{C}$ every HCF convergent is a best approximation.

Theorem (H. Nakada, 1976 [17])

There exists a measure μ which is equivalent to the Lebesgue measure on $\mathfrak{F}$ and such that the system $(\mathfrak{F}, T, \mu)$ ergodic.

Theorem (G.R., 2021)

Let $\mathbf{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers and define

$$E = \{z = [0; a_1, a_2, \ldots] \in \mathfrak{F} : |a_n| \geq u_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

Then, if $\mathfrak{m}$ is the Lebesgue measure on $\mathfrak{F}$, we have that

$$\mathfrak{m}(E) = \begin{cases} 0, & \text{if } \sum_{n\in\mathbb{N}} u_n^{-2} < +\infty, \\ 1, & \text{if } \sum_{n\in\mathbb{N}} u_n^{-2} = +\infty. \end{cases}$$

Theorem (I.J. Good, 1941 [9])

The following equality holds

$$\dim_H \left\{ z = [a_0; a_1, a_2, \ldots]_{\mathbb{R}} \in \mathbb{R} : \lim_{n\to\infty} a_n = +\infty \right\} = \frac{1}{2}.$$

Theorem (G.R., 2018 [7])

The following equality holds

$$\dim_H \left\{ z = [a_0; a_1, a_2, \ldots] \in \mathbb{C} : \lim_{n\to\infty} |a_n| = +\infty \right\} = 1.$$

Let (X, d) be a complete metric space. Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n$ be a family of compact sets such that each $\mathcal{A}_n$ is finite, $\#\mathcal{A}_0 = 1$, and

i. $\forall A \in \mathcal{A} \quad |A| > 0$,
ii. $\forall n \in \mathbb{N} \quad \forall A, B \in \mathcal{A}_n \quad (A = B) \vee (A \cap B = \varnothing)$,
iii. $\forall n \in \mathbb{N} \quad \forall B \in \mathcal{A}_n \quad \exists A \in \mathcal{A}_{n-1} \quad B \subseteq A$,
iv. $\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_{n-1} \quad \exists B \in \mathcal{A}_n \quad B \subseteq A$,
v. $d_n(\mathcal{A}) := \max\{|A| : A \in \mathcal{A}_n\} \to 0$ as $n \to \infty$.

For each $n \in \mathbb{N}_0$ and each $A \in \mathcal{A}_n$, put

$$D(A) = \{B \in \mathcal{A}_{n+1} : B \subseteq A\}.$$

The **limit set** of $\mathcal{A}$, $\mathbf{A}_\infty$, is

$$\mathbf{A}_\infty := \bigcap_{n=0}^{\infty} \bigcup_{A\in\mathcal{A}_n} A.$$

(See strongly tree-like structure in [13]).

Lemma (First Generalized Jarník Lemma)

Let $\mathcal{A}$ be as above and assume that it satisfies the following conditions:

1. $\liminf_{n\to\infty} \frac{\log(d_n(\mathcal{A})^{-1})}{n} > 0$,
2. *There is a sequence of positive numbers $(B_n)_{n\geq 1}$ such that*

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \quad Y, Z \in D(A) \quad Y \neq Z \implies d(Y,Z) \geq B_n|A|$$

$$\limsup_{n\to\infty} \frac{\log\log(B_n^{-1})}{n} < 1$$

If $s > 0$ satisfies

$$\forall n \in \mathbb{N} \quad \forall A \in \mathcal{A}_n \quad \sum_{B \in D(A)} |B|^s \geq |A|^s. \tag{3}$$

then $\dim_H \mathbf{A}_\infty \geq s$.

Lemma (Second Generalized Jarník Lemma)

Let $\mathcal{A}$ be as above. If $s > 0$ is such that

$$\forall k \in \mathbb{N} \quad \forall A \in \mathcal{A}_k \qquad \sum_{B \in D(A)} |B|^s \leq |A|^s, \tag{4}$$

then $\dim_H \mathbf{A}_\infty \leq s$.

"Proof" of the first lemma

1. We show that there exists $c > 0$ such that any finite collection $\mathfrak{X} \subseteq \mathcal{A}$ that covers $\mathbf{A}_\infty$ satisfies
$$\sum_{A \in \mathfrak{X}} |A|^s \geq c.$$
2. For each open covering $\mathcal{G}$ of $\mathbf{A}_\infty$ and any $\varepsilon > 0$ we find some finite $\mathfrak{X} \subseteq \mathcal{A}$ that covers $\mathbf{A}_\infty$ and
$$c < \sum_{A \in \mathfrak{X}} |A|^s \leq \sum_{G \in \mathcal{G}} |G|^{s-\varepsilon}.$$
3. Conclude.

"Proof" of the upper bound

Write

$$E := \{z = [0; a_1, a_2, \dots] \in \mathfrak{F} : \lim_{n\to\infty} |a_n| = +\infty\}.$$

For any $L > 0$ define

$$E'_L := \left\{[0; a_1, a_2, \dots] \in \mathfrak{F} : \liminf_{n\to\infty} |a_n| \geq L\right\},$$
$$E_L := \left\{[0; a_1, a_2, \dots] \in \mathfrak{F} : \inf_{n\to\infty} |a_n| \geq L\right\}.$$

then $E_L \subseteq E'_L$ and $\dim_H E_L \leq \dim_H E'_L$. Some basic properties of Hausdorff dimension yield

$$\forall L \in \mathbb{R}_{>0} \quad \dim_H E_L = \dim_H E'_L.$$

Using the First Generalized Jarník Lemma we get that

$$\lim_{L\to\infty} \dim_H E_L \leq 1.$$

Since $E \subseteq E'_L$ for all $L > 0$,

$$\dim_H E \leq 1.$$

For any $z \in \mathbb{C}$ write $\|z\| = \max\{|\mathfrak{R}(z)|, |\mathfrak{I}(z)|\}$.

Lemma

Let $f, g; \mathbb{N} \to \mathbb{R}_{>0}$ be to functions with the following properties:

1. *There is some $c' \in (0,1)$ such that*
$$\forall n \in \mathbb{N} \quad \sqrt{8} \leq f(n) \leq c' g(n),$$
2. $\lim_{n\to\infty} f(n) = +\infty$,
3. $\limsup_{n\to\infty} \frac{\log \log g(n)}{\log n} < 1.$

Then, the set

$$E_{f,g} := \left\{ [0; a_1, a_2, \ldots] \in \mathfrak{F}' : \forall n \in \mathbb{N} \quad f(n) \leq \|a_n\| \leq g(n) \right\}$$

satisfies $\dim_H E_{f,g} \geq 1$.

Proof of the Lemma.

Notation. For $n \in \mathbb{N}$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}[i]^n$ the **cylinder** determined by $\mathbf{a}$ is

$$\mathcal{C}_n(\mathbf{a}) := \{z = [0; a_1(z), a_2(z), \ldots] \in \mathfrak{F} : a_1(z) = a_1, \ldots, a_n(z) = a_n\},$$

and $\overline{\mathcal{C}_n}(\mathbf{a})$ is its closure. Note that for some absolute constant $\kappa > 1$ such that $|\mathcal{C}_n(\mathbf{a})| < \kappa^{-n}$.

Proof of the Lemma. Define $\mathcal{A}_0 = \{\overline{\mathfrak{F}}\}$ and for any $n \in \mathbb{N}$

$$\mathcal{A}_n := \left\{\overline{\mathcal{C}_n}(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}[i]^n, \ \forall j \in \{1, \ldots, n\} \quad f(j) \leq \|a_j\| \leq g(j)\right\}.$$

Apply the First Generalized Jarník Lemma to $\mathcal{A} = \bigcup_{n \in \mathbb{N}_0} \mathcal{A}_n$ and to an arbitrary $0 < s < 1$ and conclude.

Remarks on the proof

1. When we assume that $z = [0; a_1, a_2, \dots] \in \mathfrak{F}$ satisfies $|a_n| \geq \sqrt{8}$ for all n, we get rid of the complicated structure of the associated shift space.
2. The computations with $\max\{|\mathfrak{R}(z)|, |\mathfrak{I}(z)|\}$ are simpler than with $|z|$. This is also used in the proof of the complex Borel-Bernstein Theorem.
3. In an ongoing collaboration with Rafael Alcaraz Barrera, we adapted the strategy to the context of distal sets for the Lüroth map.

Fix $t_0 \geq 1$. Given $\psi : [t_0, +\infty) \to \mathbb{R}_{>0}$, the set $D(\psi)$ of ψ-Dirichlet numbers is the collection of $x \in \mathbb{R}$ such that for every large $t > 0$ there are $p, q \in \mathbb{Z}$ such that

$$|qx - p| < \psi(t), \quad 0 < |q| < t.$$

Theorem (D. Kleinbock, N. Wadleigh, 2018 [12])

If $\psi : [t_0, \infty) \to \mathbb{R}_{\geq 0}$ is non-increasing and $\psi(t) < t^{-1}$ for every large t, then $D(\psi) \neq \mathbb{R}$.

Theorem (N. Chevallier, 2021 [4])

Let z be any complex number. For every $Q \geq 1$ there exist $p, q \in \mathbb{Z}[i]$ such that

$$0 < |q| < Q, \quad |qz - p| \leq \frac{\sqrt{2}}{3 - \sqrt{3}} \frac{1}{Q}.$$

Problem

Is there a complex analogue of the Kleinbock-Wadleigh Theorem?

Thank you for your attention.

Wieb Bosma and David Gruenewald.
Complex numbers with bounded partial quotients.
J. Aust. Math. Soc., 93(1-2):9–20, 2012.

Yann Bugeaud.
Automatic continued fractions are transcendental or quadratic.
Ann. Sci. Éc. Norm. Supér. (4), 46(6):1005–1022, 2013.

Yann Bugeaud and Dong Han Kim.
A new complexity function, repetitions in Sturmian words, and irrationality exponents of Sturmian numbers.
Trans. Amer. Math. Soc., 371(5):3281–3308, 2019.

Nicolas Chevallier.
Gauss lattices and complex continued fractions, 2021.

S. G. Dani and Arnaldo Nogueira.
Continued fractions for complex numbers and values of binary quadratic forms.
Trans. Amer. Math. Soc., 366(7):3553–3583, 2014.

Hiromi Ei, Shunji Ito, Hitoshi Nakada, and Rie Natsui.
On the construction of the natural extension of the Hurwitz complex continued fraction map.
Monatsh. Math., 188(1):37–86, 2019.

Gerardo González Robert.
Good's theorem for Hurwitz continued fractions.
Int. J. Number Theory, 16(7):1433–1447, 2020.

Gerardo González Robert.
Purely periodic and transcendental complex continued fractions.
Acta Arith., 194(3):241–265, 2020.

I. J. Good.
The fractional dimensional theory of continued fractions.
Proc. Cambridge Philos. Soc., 37:199–228, 1941.

Doug Hensley.
Continued fractions.
World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

A. Hurwitz.
Über die Entwicklung complexer Grössen in Kettenbrüche.
Acta Math., 11(1-4):187–200, 1887.

Dmitry Kleinbock and Nick Wadleigh.
A zero-one law for improvements to Dirichlet's Theorem.
Proc. Amer. Math. Soc., 146(5):1833–1844, 2018.

Dmitry Kleinbock and Barak Weiss.
Modified Schmidt games and Diophantine approximation with weights.
Adv. Math., 223(4):1276–1298, 2010.

Richard B. Lakein.
Approximation properties of some complex continued fractions.
Monatsh. Math., 77:396–403, 1973.

Richard B. Lakein.
Continued fractions and equivalent complex numbers.
Proc. Amer. Math. Soc., 42:641–642, 1974.

Anton Lukyanenko and Joseph Vandehey.
Ergodicity of iwasawa continued fractions via markable hyperbolic geodesics, 2018.

Hitoshi Nakada.
On the Kuzmin's theorem for the complex continued fractions.
Keio Engrg. Rep., 29(9):93–108, 1976.