The asymptotic behaviour of Sudler products

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Introduction

- For $\alpha \in \mathbb{R}, N \in \mathbb{N}$, we define the Sudler product at stage N as $P_N(\alpha) := \prod_{r=1}^{N} 2|\sin \pi r\alpha|$.
- This product appears in many different areas of mathematics (restricted partition functions, KAM theory, Padé approximants, almost Mathieu operators, Kashaev invariants of hyperbolic knots, ...).
- $\log P_N(\alpha) = \sum_{r=1}^{N} \log(2|\sin \pi r\alpha|)$ is a Birkhoff sum for circle rotations with logarithmic singularities at the integers.
- Question: What happens for $P_N(\alpha)$ when $N \to \infty$?
- For rational α, $P_N(\alpha) = 0$ eventually, 1-periodic in α, so from now on $\alpha \in [0, 1) \setminus \mathbb{Q}$.

Different directions of analysis

- $\lim\limits_{N\to\infty} \sup\limits_{\alpha\in[0,1]} P_N(\alpha)^{1/N} = C \approx 1.22$ (Erdős, Szekeres, Sudler, Wright).
- Pointwise behaviour (Lubinsky, Saff): $\lim\limits_{N\to\infty} P_N(\alpha)^{1/N} = 1$ for almost every α.
- Convergence in measure (Borda): $\frac{\max_{1\leq N\leq M} \log P_N(\alpha)}{\log M \log\log M} \underset{M\to\infty}{\longrightarrow} \frac{12V}{\pi^2}$.
- Topic of the talk is the pointwise behaviour, i.e. α is fixed and N varies.

Directions of this talk

- Erdős + Szekeres proved that $\liminf_{N\to\infty} P_N(\alpha) = 0$ for almost all α.
- What is the speed of convergence for typical α?
- What is the asymptotic density of those N where this speed of convergence is attained?
- What can be said about the $\limsup$?
- Can we characterize those N where $P_N(\alpha)$ is large/resp. small?
- Lubinsky: $\liminf_{N\to\infty} P_N(\alpha) = 0$ for all non-badly approximable numbers.
- What happens for α being a badly approximable number?

Connection with Diophantine approximation

- Let $\alpha = [0; a_1, a_2, \ldots], a_i \in \mathbb{N}$ be the continued fraction expansion of α and p_k/q_k the convergents. Morally, $1 \ll P_{q_k}(\alpha) \ll 1$ since $\big\{\{n\alpha\} : 1 \leq n \leq q_k\big\}$ equidistributes well in $[0,1)$:

$$\log P_{q_k}(\alpha) = \sum_{r=1}^{q_k} \log \Big(\big|2 \sin(\pi(r \underbrace{(\alpha - p_k/q_k)}_{:= d_k \approx 1/a_{k+1} q_k^2} + r p_k/q_k))\big| \Big)$$

$$\underset{r \mapsto r p_k^{-1}}{\overset{\uparrow}{=}} \sum_{r=1}^{q_k} \log \Big(\big|2 \sin(\pi(r p_k^{-1} d_k + r/q_k))\big| \Big)$$

$$\approx q_k \cdot \int_0^1 \log(2|\sin \pi x|)\,\mathrm{d}x = 0,$$

with the size of the deviatiaton from 0 determined by $(q_k \cdot p_k^{-1}) d_k$ (and thus, of a_{k+1}).

Ostrowski decomposition

$$P_{q_k+q_{k-1}}(\alpha) = P_{q_k}(\alpha) \cdot \prod_{r=1}^{q_{k-1}} 2|\sin(\pi(r\alpha + \underbrace{\{q_k\alpha\}}_{\text{small}}))| \approx P_{q_k}(\alpha) \cdot P_{q_{k-1}}(\alpha).$$

- Ostrowski numeration with respect to $\alpha = [0; a_1, a_2, \ldots]$: We can write every integer $N < q_{n+1}$ uniquely as

$$N = \sum_{i=0}^{n} b_i q_i, \quad 0 \leq b_i \leq a_{i+1}, \text{if } b_i = a_{i+1} \Rightarrow b_{i-1} = 0.$$

We use this numeration for the following decomposition:

Ostrowski decomposition

Proposition (Aistleitner, Technau, Zafeiropoulos)

Write $\delta_i = \|q_i\alpha\| \approx 1/(q_i a_{i+1})$ *and let* $N = \sum_{i=0}^{n} b_i q_i$ *be in its Ostrowski numeration. Then*

$$P_N(\alpha) = \prod_{i=0}^{n} \prod_{k=0}^{b_i - 1} P_{q_i}(\alpha, \varepsilon_{i,k}(N))$$

where

$$P_{q_n}(\alpha, \varepsilon) := \prod_{r=1}^{q_n} 2 \left| \sin\left(\pi\left(r\alpha + (-1)^n \frac{\varepsilon}{q_n}\right)\right)\right|$$

and

$$\varepsilon_{i,k}(N) := q_i \left(k\delta_i + \sum_{j=1}^{n-i} (-1)^j b_{i+j}\delta_{i+j} \right).$$

Reflection principle

$P_{q_k-1}(\alpha) \approx \prod_{r=1}^{q_k-1} 2|\sin(\pi r/q_k)| = q_k$. So for $0 \leq N < q_k$, we have

$$P_N(\alpha) = P_{q_k-1}(\alpha) \Big/ \Big(\prod_{r=N+1}^{q_k-1} 2|\sin(\pi r\alpha)| \Big)$$

$$= P_{q_k-1}(\alpha) \Big/ \Big(\prod_{r=1}^{q_k-N-1} 2|\sin(\pi(r\alpha - \underbrace{q_k\alpha}_{\text{small}}))| \Big) \approx \frac{q_k}{P_{q_k-N-1}(\alpha)}.$$

Thus,

$$\limsup_{N\to\infty} \frac{P_N(\alpha)}{N} \approx \frac{1}{\liminf_{N\to\infty} P_N(\alpha)}.$$

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Classical theorems in metric Diophantine approximation

Theorem (Bernstein, 1912)

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *be an increasing function. Then for almost every* $\alpha = [0; a_1, a_2, \ldots]$, $\limsup_{k\to\infty} \frac{a_k}{\psi(k)} > 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty$.

Theorem (Khintchine, 1924)

Let $\psi : \mathbb{N} \to \mathbb{R}$ *be such that* $q\psi(q)$ *is decreasing. Then for a.e.* α

$$\#\left\{(p,q) \in \mathbb{Z} : \left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}\right\} = \infty \iff \sum_{q=1}^{\infty} \psi(q) = \infty.$$

Theorem (Lubinsky, 1999)

Let $\psi : \mathbb{N} \to \mathbb{R}$ be a positive increasing function. Then for almost every α we have

$$\limsup_{N\to\infty} \log P_N(\alpha) \geq \psi(\log N), \quad \liminf_{N\to\infty} \log P_N(\alpha) \leq -\psi(\log N)$$

if and only if $\sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty$.

So in particular,

$$\log N \log\log N \ll \max_{M\leq N} \log P_M(\alpha) \ll \log N(\log\log N)^{1+\varepsilon}, \quad N \to \infty.$$

Reflection principle $\log P_N(\alpha) \approx \log N - \log P_{q_k-N-1}(\alpha)$ shows

$$\max_{1\leq N<q_k} \log P_N(\alpha) \approx -\min_{1\leq N<q_k} \log P_N(\alpha).$$

Upper density

Recall: The upper density of a set $A \subseteq \mathbb{N}$ is defined as

$$\limsup_{M\to\infty} \frac{\#\{N \leq M : N \in A\}}{M}.$$

Theorem (Borda)

Let $\psi : \mathbb{N} \to \mathbb{R}$ be a positive non-decreasing function with $\sum_{n=1}^{N} \frac{1}{\psi(n)} = \infty$. For almost every α, the sets

$$\{N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N)\},$$
$$\{N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N)\}$$

both have upper density at least $\pi^2/(1440V^2) \approx 0.2627$.

Theorem (H., 2022+)

Let ψ be an increasing, positive function such that $\sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty$. Then for almost every α, the set

$$\{N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N)\}$$

has upper density 1*.The set*

$$\{N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N)\}$$

has upper density at least $1/2$*, with equality if* $\liminf_{k\to\infty} \frac{\psi(k)}{k \log k} \geq C$ *for some absolute constant* $C > 0$*.*

- Even the union of those two sets has lower density 0.
- The reflection principle suggests symmetry of the densities of the sets - this is not the case!

Almost sure continued fraction properties

- (Bernstein, 1912): For any non-negative function $\psi : \mathbb{N} \to [0, \infty)$ we have

$$\# \left\{ k \in \mathbb{N} : a_k > \psi(k) \right\} \text{ is } \begin{cases} \text{infinite} & \text{if } \sum\limits_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty, \\ \text{finite} & \text{if } \sum\limits_{k=0}^{\infty} \frac{1}{\psi(k)} < \infty. \end{cases}$$

- (Khintchine and Lévy, 1936):

$$\log q_k \sim \frac{\pi^2}{12 \log 2} k \text{ as } k \to \infty.$$

- (Diamond and Vaaler, 1986):

$$\sum_{\ell \leq K} a_\ell - \max_{\ell \leq K} a_\ell \sim \frac{K \log K}{\log 2}, \quad K \to \infty.$$

Almost sure continued fraction properties

Corollary

For almost every α and $\psi : \mathbb{N} \to \mathbb{R}$ increasing with $\sum_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty$, there exist infinitely many $K \in \mathbb{N}$ such that:

- $\psi(K) < a_K < K^2$.
- $\sum_{\ell=1}^{K-1} a_\ell \ll K \log K$.
- *Assuming* $\psi(K) > C \cdot K \log K$, *we have* $\sum_{\ell=1}^{K-1} a_\ell = o(a_K)$.
- *For most* $1 \leq N \leq q_K$

$$\psi(\log N) \asymp \psi(\log q_K) \asymp \psi(K) < a_K.$$

Proof Sketch

For almost every α, $P_{q_\ell}(\alpha, x) \approx |2\sin(\pi x)|$ (technical estimates on cotangent sums by Aistleitner and Borda), hence

$$\log P_N(\alpha) \approx \sum_{\ell=0}^{K-1} \sum_{b=1}^{b_\ell - 1} \log \left| 2\sin\left(\pi \underbrace{bq_\ell \delta_\ell}_{\approx b/a_{\ell+1}}\right)\right|$$

$$\lesssim a_K \int_0^{b_{K-1}/a_K} \log\left|2\sin(\pi x)\right| \,\mathrm{d}x + \mathcal{O}\left(\log(2) \cdot \sum_{i=1}^{K-1} a_i\right)$$

$$= \underbrace{a_K}_{>1000\cdot\psi(\log N)} \left(\int_0^{b_{K-1}/a_K} \log\left|2\sin(\pi x)\right| \,\mathrm{d}x + o(1)\right).$$

$$\log P_N(\alpha) \leq -\psi(\log N) \text{ if } \int_0^{b_{K-1}/a_K} \log|2\sin(\pi x)| \,\mathrm{d}x < -\varepsilon < 0$$

$$\Leftrightarrow \varepsilon < b_{K-1}(N)/a_K < 1/2 - \varepsilon.$$

Upper density 1

$$\frac{\#\{1 \leq N \leq q_K/2 : \varepsilon < b_{K-1}(N)/a_K < 1/2 - \varepsilon\}}{q_k/2} \sim 1 - 4\varepsilon.$$

Let $(K_j)_{j \in \mathbb{N}}$ be the sequence of those K where $\sum_{\ell=1}^{K-1} a_\ell = o(a_K)$. Then

$$\limsup_{j \to \infty} \frac{\#\{1 \leq N \leq q_{K_j}/2 : \log P_N(\alpha) \leq -\psi(\log N)\}}{q_{K_j}/2} > 1 - 4\varepsilon \underset{\varepsilon \to 0}{\to} 1.$$

By reflection principle: $\log P_N(\alpha) \approx -\log P_{q_K - N - 1}(\alpha)$, thus

$$\lim_{j \to \infty} \frac{\#\{1 \leq N \leq q_{K_j} : \log P_N(\alpha) \geq \psi(\log N)\}}{q_{K_j}} = 1/2.$$

$1/2$ is sharp if $\psi(K) > CK \log K$

Let $a_{K_0} := \max_{1 \leq \ell \leq K} a_K$ and $b_{K_0 - 1}(N)/a_{K_0} < 1/2$ (happens for at least 50% of all N). Recall $\sum\limits_{\ell \neq K_0} a_\ell \sim \frac{K \log K}{\log 2}$, thus

$$\log P_N(\alpha) \lesssim \log(2) \cdot \sum_{\ell \neq K_0}^{K-1} a_\ell + \underbrace{a_{K_0} \left(\int_0^{b_{K_0-1}/a_{K_0}} \log \left| 2 \sin(\pi x) \right| \, \mathrm{d}x \right)}_{\leq 0}$$

$$\leq \frac{K \log K}{2} < \psi(\log N).$$

Open questions

- How large can we choose ψ such that $|\log P_N(\alpha)| \geq \psi(\log N)$ holds on a set of positive **lower** density?
- What can be said about the distribution/growth of $\log\left(\prod_{r=1}^{N} 2|\sin(\pi(r\alpha + x_0))|\right)$ where $x_0 \notin \mathbb{Z}$?
- I expect its behaviour to be similiar if $x_0 \in \mathbb{Q}$ (probably not with the asymmetry), but much more involved if $x_0 \notin \mathbb{Q}$ (inhomogeneous diophantine approximation).

Characterize N such that P_N is large/small

$$\log P_N(\alpha) \approx \sum_{\ell=0}^{K-1} \sum_{b=1}^{b_\ell - 1} \log \left| 2 \sin \left(\pi \underbrace{b q_\ell \delta_\ell}_{\approx b/a_{\ell+1}} \right) \right|$$

$$\approx \sum_{\ell=0}^{K-1} a_\ell \int_0^{b_{\ell-1}/a_\ell} \log \left| 2 \sin(\pi x) \right| \, \mathrm{d}x$$

$$\leq \underbrace{\left(\sum_{\ell=0}^{K-1} a_\ell \right)}_{\text{typically} \frac{12}{\pi^2} \log N \log \log N} \underbrace{\max_{0 \leq y \leq 1} \int_0^y \log \left| 2 \sin(\pi x) \right| \, \mathrm{d}x}_{=V}$$

where $V := \int_0^{5/6} \log \left| 2 \sin(\pi x) \right| \, \mathrm{d}x \approx 0.1615.$

Characterize N such that P_N is large/small

- Heuristic explanation behind Borda's convergence result $\frac{\max_{1 \leq N \leq M} \log P_N}{\log M \log\log M} \to \frac{12V}{\pi^2}$ in measure.
- Aistleitner and Borda: Let $N_0 := \mathit{arg\ max}_{1 \leq N < q_K} P_N(\alpha)$ and $N^* := \sum_{k=0}^{K-1} \lfloor (5/6) a_{k+1} \rfloor q_k$.
  Under mild technical assumptions on α, the Ostrowski expansions of N_0, N^* do not deviate much from each other.
- By the reflection principle, the minimizers' Ostrowski expansion is close to $\sum\limits_{k=0}^{K-1} \lceil (1/6) a_{k+1} \rceil q_k$.

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About $\liminf_{N\to\infty} P_N(\alpha)$ for exceptional α

- Lubinsky proved that

$$\liminf_{N\to\infty} P_N(\alpha) = 0$$

  holds for all α that have sufficiently large partial quotients ($\approx e^{800}$) infinitely often, conjecturing it to hold for every α.
- Conjecture disproven by Grepstad, Kaltenböck, Neumüller: $\liminf_{N\to\infty} P_N(\phi) > 0$ ($\phi = [0; 1, 1, 1, \ldots]$).
- Behaviour depends delicately on the actual size of the partial quotients.

$\liminf_{N\to\infty} P_N(\alpha)$ for badly approximable α

- Question: If $\alpha = [0; a_1, a_2, \ldots]$ with $\limsup_{m\to\infty} a_m < e^{800}$, when do we have $\liminf_{N\to\infty} P_N(\alpha) = 0$?

Proposition (Aistleitner, Borda)

For badly approximable α, we have

$$\liminf_{N\to\infty} P_N(\alpha) > 0 \Leftrightarrow \limsup_{N\to\infty} \frac{P_N(\alpha)}{N} < \infty.$$

Theorem (Aistleitner, Technau, Zafeiropoulos)

If $\alpha = [0; a, a, \ldots]$, then $\liminf_{N\to\infty} P_N(\alpha) > 0 \Leftrightarrow a \leq 5$.

Theorem (Grepstad, Neumüller, Zafeiropoulos)

If α *is a quadratic irrational and* $\limsup_{m\to\infty} a_m \geq 23$*, then* $\liminf_{N\to\infty} P_N(\alpha) = 0$.

Conjecture: $\limsup_{m\to\infty} a_m \geq 6$ *implies* $\liminf_{N\to\infty} P_N(\alpha) = 0$.

Theorem (H., 2022+)

Let $\alpha = [0; a_1, a_2, \ldots]$ *with* $\limsup_{m\to\infty} a_m \geq 7$*, then* $\liminf_{N\to\infty} P_N(\alpha) = 0$.

For any $1 \leq a \leq 6$*, there exists a (quadratic) irrational* α *with* $\limsup_{m\to\infty} a_m = a$ *and* $\liminf_{N\to\infty} P_N(\alpha) > 0$.

- No restriction to quadratic irrationals.
- Sharp threshold value, disproving the conjecture.
- Not an equivalence as in the case of quadratic irrationals with period length 1: behaviour of $\liminf_{N\to\infty} P_N(\alpha)$ is not completely determined by the value of $\limsup_{m\to\infty} a_m$.

Proof idea

- Recall
$$P_N(\alpha) = \prod_{i=0}^{n} \prod_{k=0}^{b_i-1} P_{q_i}(\alpha, \varepsilon_{i,k}(N)).$$

- Idea: If there is a subsequence $(N_j)_{j\in\mathbb{N}}$ such that $P_{q_i}(\alpha, \varepsilon_{i,k}(N_j)) < 1$ holds for almost all (i,k), then $\lim_{j\to\infty} P_{N_j}(\alpha) = 0$.

- Special case: Consider $N_j = \sum\limits_{k=0}^{K_j} b_k q_k$ with $(b_0, \ldots, b_{k_j}) = (1, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, 1 \ldots, 0, 0, 1)$. Then
$$P_{N_j}(\alpha) = \prod_{i=0}^{j} P_{q_{m_i}}(\alpha, \varepsilon_{m_i,0}(N)).$$

Proof idea (cont.)

- $\varepsilon_{i,0}(N) \approx -\frac{b_{i+1}}{a_{i+1}a_{i+2}} + \frac{b_{i+2}}{a_{i+1}a_{i+2}a_{i+3}} - \frac{b_{i+3}}{a_{i+1}a_{i+2}a_{i+3}a_{i+4}} + \ldots$, so if $b_{i+1} = b_{i+2} = \ldots = b_{i+j} = 0$, then $\varepsilon_{i,0}(N) \approx 0$.
- $P_{q_n}(\alpha, \varepsilon)$ is smooth as a function in ε, so $P_{q_n}(\alpha, \varepsilon_{i,0}(N)) \approx P_{q_n}(\alpha, 0) = P_{q_n}(\alpha)$.
- So it suffices to show that $P_{q_m}(\alpha) < 1$ holds for infinitely many m.

Proposition (H., 2022+)

Let $\alpha_k = \{q_{k-1}/q_k\} = [0; a_k, a_{k-1}, \ldots, a_1], \delta_k := \|q_k\alpha\|$. *Writing*

$$H_k(\alpha, \varepsilon) := 2\pi|\varepsilon + q_k\delta_k| \prod_{n=1}^{\lfloor q_k/2 \rfloor} h_{n,k}(\alpha, \varepsilon),$$

where

$$h_{n,k}(\alpha, \varepsilon) := \left| \left(1 - q_k\delta_k \frac{\{n\alpha_k\} - \frac{1}{2}}{n}\right)^2 - \frac{\left(\varepsilon + \frac{q_k\delta_k}{2}\right)^2}{n^2} \right|,$$

we have for any badly approximable α *that*

$$\lim_{k\to\infty} |P_{q_k}(\alpha, \varepsilon) - H_k(\alpha, \varepsilon)| = 0,$$

with the convergence being locally uniform.

Proof idea (cont.)

- We are left to show $H_k(\alpha, 0) < 1$ infinitely often.
- Elementary estimates reduce the problem to showing

$$\frac{\log\left(\frac{2\pi}{a_{k+1}+1/a_{k+2}+\alpha_k}\right)}{\frac{2}{a_{k+1}+1/a_{k+2}+\alpha_k}} < \sum_{n=1}^{q_k/2} \frac{\{n\alpha_k\} - 1/2}{n}.$$

- The left-hand-side is decreasing in a_{k+1}, thus choose $a_{k+1} = 7$ (maximal).
- Partial summation: For T a large constant and E an error term,

$$\sum_{n=1}^{q_k/2} \frac{\{n\alpha_k\} - 1/2}{n} = \sum_{\ell=1}^{T} \frac{\sum_{n=1}^{\ell} \{n\alpha_k\} - 1/2}{\ell^2} + E(T, \limsup_{r\to\infty} a_r).$$

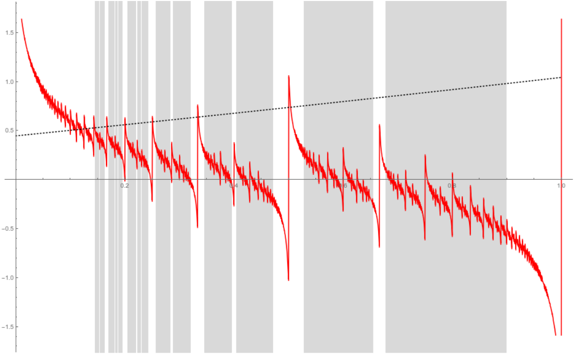
Proof idea (cont.)

- We are left to show $H_k(\alpha, 0) < 1$ infinitely often.
- Morally, for large T, this is equivalent to

$$\frac{\log\left(\frac{2\pi}{a_{k+1}+1/a_{k+2}+\alpha_k}\right)}{\frac{2}{a_{k+1}+1/a_{k+2}+\alpha_k}} < \sum_{\ell=1}^{T} \frac{\sum_{n=1}^{\ell}\{n\alpha_k\} - 1/2}{\ell^2}. \qquad (*)$$

- Now $(*)$ is almost continuous in α_k!

Visualization & Conclusion



Visualization of $(*)$. The red line function is the right-hand side in α_k, the dotted line the left-hand side for $a_{k+1} = 7$. As the first partial quotients of α_k are ≤ 7, α_k lies inside the grey area. This shows $H_k(\alpha, 0) < 1$ if $a_{k+1} \geq 7$, hence the statement follows.

Show that 7 is optimal

- We prove that $\alpha = [0; \overline{6,5,5}]$ fulfills $\liminf_{N\to\infty} P_N(\alpha) > 0$.
- Regrouping the shifted products (due to Aistleitner, Technau, Zafeiropoulos) shows

$$P_N(\alpha) = P_{q_n}(\alpha) \cdot \prod_{i=1}^{n} K_i(N)$$

where

$$K_i(N) := \prod_{k=1}^{b_i(N)-1} P_{q_i}(\alpha, \varepsilon_{i,k}(N)) \cdot P_{q_{i-1}}(\alpha, \varepsilon_{i-1,0}(N))^{\mathbb{1}_{[b_{i-1}(N)\neq 0]}}.$$

- The value of $K_i(N)$ is almost determined by the Ostrowski coefficients $b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}$.

Lemma (H.,2022+)

Let $(b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}, b_{i+5})$ be possible Ostrowski coefficients for $\alpha = [0; \overline{6, 5, 5}]$, not all 0, i sufficiently large and

$$K_i(b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}) \cdot K_{i+1}(b_i, b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}) \cdot K_{i+2}(b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}, b_{i+5}) < 1.001.$$

Then

$$K_i \cdot K_{i+1} \cdot K_{i+2} \cdot K_{i+3} \cdot K_{i+4} \cdot K_{i+5} > 1.001.$$

- Proof contains many case distinctions and computational assistance.
- With $P_N(\alpha) = P_{q_n}(\alpha) \cdot \prod_{i=1}^{n} K_i(N)$, the result follows.

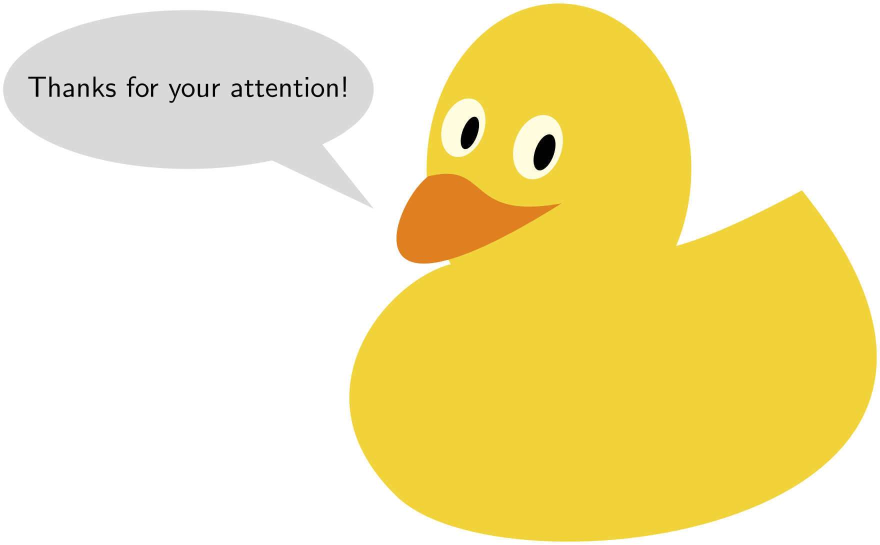
Extremal values and open questions

Theorem (H.,2022)

Let $1 \leq a \leq 5$, $\alpha = [0; a, a, \ldots]$.

- *For $q_n \leq N < q_{n+1}$ we have $P_{q_n}(\alpha) \leq P_N(\alpha) \leq P_{q_{n+1}-1}(\alpha)$.*
- $\min_{N \geq 1} P_N(\alpha) = P_1(\alpha)$.

- Does a similar behaviour hold for any quadratic irrational without preperiod where $\liminf_{N\to\infty} P_N(\alpha) > 0$?
- For $\limsup_{k\to\infty} a_k \leq 6$, it is in general open whether $\liminf_{N\to\infty} P_N(\alpha) > 0$.
- Conjecture: $\limsup_{k\to\infty} a_k \leq 3$ implies $\liminf_{N\to\infty} P_N(\alpha) > 0$ (this would be sharp since $\liminf_{N\to\infty} P_N([0; \overline{4,1}]) = 0$).


Thanks for your attention!