The asymptotic behaviour of Sudler products

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For $\alpha \in \mathbb{R}, N \in \mathbb{N}$, we define the Sudler product at stage $N$ as

$$P_N(\alpha) := \prod_{r=1}^{N} 2|\sin \pi r \alpha|.$$ 

This product appears in many different areas of mathematics (restricted partition functions, KAM theory, Padé approximants, almost Mathieu operators, Kashaev invariants of hyperbolic knots, . . .).

$$\log P_N(\alpha) = \sum_{r=1}^{N} \log(2|\sin \pi r \alpha|)$$ is a Birkhoff sum for circle rotations with logarithmic singularities at the integers.

Question: What happens for $P_N(\alpha)$ when $N \to \infty$?

For rational $\alpha$, $P_N(\alpha) = 0$ eventually, 1-periodic in $\alpha$, so from now on $\alpha \in [0, 1) \setminus \mathbb{Q}$. 

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Different directions of analysis

- \( \lim_{N \to \infty} \sup_{\alpha \in [0,1]} P_N(\alpha)^{1/N} = C \approx 1.22 \) (Erdős, Szekeres, Sudler, Wright).

- Pointwise behaviour (Lubinsky, Saff): \( \lim_{N \to \infty} P_N(\alpha)^{1/N} = 1 \) for almost every \( \alpha \).

- Convergence in measure (Borda):
  \[
  \max_{1 \leq N \leq M} \frac{\log P_N(\alpha)}{\log M \log \log M} \to \frac{12}{\pi^2} \quad \text{as} \quad M \to \infty.
  \]

- Topic of the talk is the pointwise behaviour, i.e. \( \alpha \) is fixed and \( N \) varies.
Erdős + Szekeres proved that $\liminf_{N \to \infty} P_N(\alpha) = 0$ for almost all $\alpha$.

What is the speed of convergence for typical $\alpha$?

What is the asymptotic density of those $N$ where this speed of convergence is attained?

What can be said about the lim sup?

Can we characterize those $N$ where $P_N(\alpha)$ is large/resp. small?

Lubinsky: $\liminf_{N \to \infty} P_N(\alpha) = 0$ for all non-badly approximable numbers.

What happens for $\alpha$ being a badly approximable number?
Let $\alpha = [0; a_1, a_2, \ldots], a_i \in \mathbb{N}$ be the continued fraction expansion of $\alpha$ and $p_k/q_k$ the convergents. Morally, $1 \ll P_{q_k}(\alpha) \ll 1$ since $\{\{n\alpha\} : 1 \leq n \leq q_k\}$ equidistributes well in $[0, 1)$:

$$\log P_{q_k}(\alpha) = \sum_{r=1}^{q_k} \log \left( |2 \sin(\pi (r (\alpha - p_k/q_k) + rp_k/q_k)))| \right)$$

$$:= d_k \approx 1/a_{k+1}q_k^2$$

$$= \sum_{r=1}^{q_k} \log \left( |2 \sin(\pi (rp_k^{-1}d_k + r/q_k)))| \right)$$

$$\approx q_k \cdot \int_0^1 \log(2|\sin \pi x|) \, dx = 0,$$

with the size of the deviation from 0 determined by $(q_k \cdot p_k^{-1})d_k$ (and thus, of $a_{k+1}$).
Ostrowski decomposition

\[ P_{q_k+q_{k-1}}(\alpha) = P_{q_k}(\alpha) \cdot \prod_{r=1}^{q_{k-1}} 2|\sin(\pi(r\alpha + \{q_k\alpha\})))| \]

\[ \approx P_{q_k}(\alpha) \cdot P_{q_{k-1}}(\alpha). \]

- **Ostrowski numeration with respect to** \( \alpha = [0; a_1, a_2, \ldots] \):

  We can write every integer \( N < q_{n+1} \) uniquely as

  \[ N = \sum_{i=0}^{n} b_i q_i, \quad 0 \leq b_i \leq a_{i+1}, \text{if } b_i = a_{i+1} \Rightarrow b_{i-1} = 0. \]

  We use this numeration for the following decomposition:
Ostrowski decomposition

Proposition (Aistleitner, Technau, Zafeiropoulos)

Write $\delta_i = \| q_i \alpha \| \approx 1/(q_i a_{i+1})$ and let $N = \sum_{i=0}^{n} b_i q_i$ be in its Ostrowski numeration. Then

$$P_N(\alpha) = \prod_{i=0}^{n} \prod_{k=0}^{b_i-1} P_{q_i}(\alpha, \varepsilon_i, k(N))$$

where

$$P_{q_n}(\alpha, \varepsilon) := \prod_{r=1}^{q_n} 2 \left| \sin \left( \pi \left( r \alpha + (-1)^n \frac{\varepsilon}{q_n} \right) \right) \right|$$

and

$$\varepsilon_i, k(N) := q_i \left( k \delta_i + \sum_{j=1}^{n-i} (-1)^j b_{i+j} \delta_{i+j} \right).$$
Reflection principle

\[ P_{q_k-1}(\alpha) \approx \prod_{r=1}^{q_k-1} 2|\sin(\pi r/q_k)| = q_k. \] So for \( 0 \leq N < q_k \), we have

\[ P_N(\alpha) = P_{q_k-1}(\alpha) / \left( \prod_{r=N+1}^{q_k-1} 2|\sin(\pi r\alpha)| \right) \]

\[ = P_{q_k-1}(\alpha) / \left( \prod_{r=1}^{q_k-N-1} 2|\sin(\pi (r\alpha - q_k\alpha))| \right) \approx \frac{q_k}{P_{q_k-N-1}(\alpha)}. \]

Thus,

\[ \limsup_{N \to \infty} \frac{P_N(\alpha)}{N} \approx \frac{1}{\liminf_{N \to \infty} P_N(\alpha)}. \]
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Theorem (Bernstein, 1912)

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be an increasing function. Then for almost every $\alpha = [0; a_1, a_2, \ldots]$, $\limsup_{k \to \infty} \frac{a_k}{\psi(k)} > 0 \iff \sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty$.

Theorem (Khintchine, 1924)

Let $\psi : \mathbb{N} \to \mathbb{R}$ be such that $q\psi(q)$ is decreasing. Then for a.e. $\alpha$

$$\# \left\{ (p, q) \in \mathbb{Z} : \left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q} \right\} = \infty \iff \sum_{q=1}^{\infty} \psi(q) = \infty.$$
Theorem (Lubinsky, 1999)

Let \( \psi : \mathbb{N} \to \mathbb{R} \) be a positive increasing function. Then for almost every \( \alpha \) we have

\[
\limsup_{N \to \infty} \log P_N(\alpha) \geq \psi(\log N), \quad \liminf_{N \to \infty} \log P_N(\alpha) \leq -\psi(\log N)
\]

if and only if \( \sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty. \)

So in particular,

\[
\log N \log \log N \ll \max_{M \leq N} \log P_M(\alpha) \ll \log N (\log \log N)^{1+\varepsilon}, \quad N \to \infty.
\]

Reflection principle \( \log P_N(\alpha) \approx \log N - \log P_{q_k-N-1}(\alpha) \) shows

\[
\max_{1 \leq N < q_k} \log P_N(\alpha) \approx - \min_{1 \leq N < q_k} \log P_N(\alpha).
\]
Recall: The upper density of a set $A \subseteq \mathbb{N}$ is defined as

$$\limsup_{M \to \infty} \frac{\#\{N \leq M : N \in A\}}{M}.$$ 

**Theorem (Borda)**

Let $\psi : \mathbb{N} \to \mathbb{R}$ be a positive non-decreasing function with

$$\sum_{n=1}^{N} \frac{1}{\psi(n)} = \infty.$$ 

For almost every $\alpha$, the sets

$$\{N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N)\},$$

$$\{N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N)\}$$

both have upper density at least $\pi^2/(1440V^2) \approx 0.2627$. 

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The asymptotic behaviour of Sudler products
Theorem (H., 2022+)

Let $\psi$ be an increasing, positive function such that
\[ \sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty. \]
Then for almost every $\alpha$, the set
\[ \{ N \in \mathbb{N} : \log P_N(\alpha) \leq -\psi(\log N) \} \]
has upper density 1. The set
\[ \{ N \in \mathbb{N} : \log P_N(\alpha) \geq \psi(\log N) \} \]
has upper density at least $1/2$, with equality if
\[ \liminf_{k \to \infty} \frac{\psi(k)}{k \log k} \geq C \]
for some absolute constant $C > 0$.

- Even the union of those two sets has lower density 0.
- The reflection principle suggests symmetry of the densities of the sets - this is not the case!
Almost sure continued fraction properties

- (Bernstein, 1912): For any non-negative function \( \psi : \mathbb{N} \rightarrow [0, \infty) \) we have

\[
\# \{ k \in \mathbb{N} : a_k > \psi(k) \} \begin{cases} \text{infinite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty, \\ \text{finite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} < \infty. \end{cases}
\]

- (Khintchine and Lévy, 1936):

\[
\log q_k \sim \frac{\pi^2}{12 \log 2} k \quad \text{as } k \to \infty.
\]

- (Diamond and Vaaler, 1986):

\[
\sum_{\ell \leq K} a_{\ell} - \max_{\ell \leq K} a_{\ell} \sim \frac{K \log K}{\log 2}, \quad K \to \infty.
\]
Almost sure continued fraction properties

**Corollary**

For almost every $\alpha$ and $\psi : \mathbb{N} \to \mathbb{R}$ increasing with

\[ \sum_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty, \]

there exist infinitely many $K \in \mathbb{N}$ such that:

- $\psi(K) < a_K < K^2$.
- $K-1 \sum_{\ell=1}^{K-1} a_\ell \ll K \log K$.
- Assuming $\psi(K) > C \cdot K \log K$, we have $\sum_{\ell=1}^{K-1} a_\ell = o(a_K)$.
- For most $1 \leq N \leq q_K$

\[ \psi(\log N) \asymp \psi(\log q_K) \asymp \psi(K) < a_K. \]
Proof Sketch

For almost every $\alpha$, $P_{q_\ell}(\alpha, x) \approx |2 \sin(\pi x)|$ (technical estimates on cotangent sums by Aistleitner and Borda), hence

$$\log P_N(\alpha) \approx \sum_{\ell=0}^{K-1} \sum_{b=1}^{b_\ell-1} \log \left| 2 \sin \left( \pi \left( bq_\ell \delta_\ell \right) \right) \right| \approx b/a_{\ell+1}$$

$$\lesssim a_K \int_0^{b_{K-1}/a_K} \log |2 \sin(\pi x)| \, dx + O \left( \log(2) \cdot \sum_{i=1}^{K-1} a_i \right)$$

$$= a_K \left( \int_0^{b_{K-1}/a_K} \log |2 \sin(\pi x)| \, dx + o(1) \right).$$

$$\log P_N(\alpha) \leq -\psi(\log N) \text{ if } \int_0^{b_{K-1}/a_K} \log|2 \sin(\pi x)| \, dx < -\varepsilon < 0$$

$$\iff \varepsilon < b_{K-1}(N)/a_K < 1/2 - \varepsilon.$$
Upper density 1

\[
\frac{\#\{1 \leq N \leq q_k/2 : \varepsilon < b_{K-1}(N)/a_K < 1/2 - \varepsilon\}}{q_k/2} \sim 1 - 4\varepsilon.
\]

Let \((K_j)_{j \in \mathbb{N}}\) be the sequence of those \(K\) where \(\sum_{\ell=1}^{K-1} a_\ell = o(a_K)\). Then

\[
\limsup_{j \to \infty} \frac{\#\{1 \leq N \leq q_{K_j}/2 : \log P_N(\alpha) \leq -\psi(\log N)\}}{q_{K_j}/2} > 1 - 4\varepsilon \to 1.
\]

By reflection principle: \(\log P_N(\alpha) \approx -\log P_{q_{K-N-1}}(\alpha)\), thus

\[
\lim_{j \to \infty} \frac{\#\{1 \leq N \leq q_{K_j} : \log P_N(\alpha) \geq \psi(\log N)\}}{q_{K_j}} = 1/2.
\]
$1/2$ is sharp if $\psi(K) > CK \log K$

Let $a_{K_0} := \max_{1 \leq \ell \leq K} a_{\ell}$ and $b_{K_0-1}(N)/a_{K_0} < 1/2$ (happens for at least 50% of all $N$). Recall $\sum_{\ell \neq K_0} a_{\ell} \sim \frac{K \log K}{\log 2}$, thus

$$\log P_N(\alpha) \lesssim \log(2) \cdot \sum_{\ell \neq K_0} a_{\ell} + a_{K_0} \left( \int_0^{b_{K_0-1}/a_{K_0}} \log |2 \sin(\pi x)| \, dx \right) \leq 0$$

$$\leq \frac{K \log K}{2} < \psi(\log N).$$
Open questions

1. How large can we choose $\psi$ such that $|\log P_N(\alpha)| \geq \psi(\log N)$ holds on a set of positive lower density?
2. What can be said about the distribution/growth of
$$\log \left( \prod_{r=1}^{N} 2|\sin(\pi(r\alpha + x_0))| \right)$$
where $x_0 \notin \mathbb{Z}$?
3. I expect its behaviour to be similar if $x_0 \in \mathbb{Q}$ (probably not with the asymmetry), but much more involved if $x_0 \notin \mathbb{Q}$ (inhomogeneous diophantine approximation).
Characterize $N$ such that $P_N$ is large/small

\[
\log P_N(\alpha) \approx \sum_{\ell=0}^{K-1} \sum_{b=1}^{b_{\ell}-1} \log \left| 2 \sin \left( \pi \frac{b q_{\ell} \delta_{\ell}}{b_{\ell-1}/a_{\ell}} \right) \right|
\]

\[
\approx \sum_{\ell=0}^{K-1} a_{\ell} \int_0^{b_{\ell-1}/a_{\ell}} \log \left| 2 \sin(\pi x) \right| \, dx
\]

\[
\leq \left( \sum_{\ell=0}^{K-1} a_{\ell} \right) \max_{0 \leq y \leq 1} \int_0^y \log \left| 2 \sin(\pi x) \right| \, dx = V
\]

typically $\frac{12}{\pi^2} \log N \log \log N$

where $V := \int_0^{5/6} \log \left| 2 \sin(\pi x) \right| \, dx \approx 0.1615$. 
Characterize $N$ such that $P_N$ is large/small

- Heuristic explanation behind Borda’s convergence result
  \[
  \max_{1 \leq N \leq M} \frac{\log P_N}{\log M \log \log M} \to \frac{12V}{\pi^2} \text{ in measure.}
  \]

- Aistleitner and Borda: Let $N_0 := \arg \max_{1 \leq N < q_K} P_N(\alpha)$ and $N^* := \sum_{k=0}^{K-1} \lceil (5/6)a_{k+1} \rceil q_k$.

  Under mild technical assumptions on $\alpha$, the Ostrowski expansions of $N_0, N^*$ do not deviate much from each other.

- By the reflection principle, the minimizers’ Ostrowski expansion is close to $\sum_{k=0}^{K-1} \lceil (1/6)a_{k+1} \rceil q_k$. 
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About $\lim \inf_{N \to \infty} P_N(\alpha)$ for exceptional $\alpha$

- Lubinsky proved that

$$\lim \inf_{N \to \infty} P_N(\alpha) = 0$$

holds for all $\alpha$ that have sufficiently large partial quotients ($\approx e^{800}$) infinitely often, conjecturing it to hold for every $\alpha$.

- Conjecture disproven by Grepstad, Kaltenböck, Neumüller: $\lim \inf_{N \to \infty} P_N(\phi) > 0$ ($\phi = [0; 1, 1, 1, \ldots]$).

- Behaviour depends delicately on the actual size of the partial quotients.
\[ \liminf_{N \to \infty} P_N(\alpha) \] for badly approximable \( \alpha \)

- Question: If \( \alpha = [0; a_1, a_2, \ldots] \) with \( \limsup_{m \to \infty} a_m < e^{800} \), when do we have \( \liminf_{N \to \infty} P_N(\alpha) = 0 \)?

**Proposition (Aistleitner, Borda)**

*For badly approximable \( \alpha \), we have*

\[
\liminf_{N \to \infty} P_N(\alpha) > 0 \iff \limsup_{N \to \infty} \frac{P_N(\alpha)}{N} < \infty.
\]

**Theorem (Aistleitner, Technau, Zafeiropoulos)**

*If \( \alpha = [0; a, a, \ldots] \), then* \( \liminf_{N \to \infty} P_N(\alpha) > 0 \iff a \leq 5 \).
Theorem (Grepstad, Neumüller, Zafeiropoulos)

If \( \alpha \) is a quadratic irrational and \( \limsup_{m \to \infty} a_m \geq 23 \), then
\[ \liminf_{N \to \infty} P_N(\alpha) = 0. \]

Conjecture: \( \limsup_{m \to \infty} a_m \geq 6 \) implies \( \liminf_{N \to \infty} P_N(\alpha) = 0 \).

Theorem (H., 2022+)

Let \( \alpha = [0; a_1, a_2, \ldots] \) with \( \limsup_{m \to \infty} a_m \geq 7 \), then
\[ \liminf_{N \to \infty} P_N(\alpha) = 0. \]

For any \( 1 \leq a \leq 6 \), there exists a (quadratic) irrational \( \alpha \) with \( \limsup_{m \to \infty} a_m = a \) and \( \liminf_{N \to \infty} P_N(\alpha) > 0 \).

- No restriction to quadratic irrationals.
- Sharp threshold value, disproving the conjecture.
- Not an equivalence as in the case of quadratic irrationals with period length 1: behaviour of \( \liminf_{N \to \infty} P_N(\alpha) \) is not completely determined by the value of \( \limsup_{m \to \infty} a_m \).
Proof idea

- Recall

\[ P_N(\alpha) = \prod_{i=0}^{n} \prod_{k=0}^{b_i-1} P_{q_i}(\alpha, \varepsilon_i, k(N)). \]

- Idea: If there is a subsequence \((N_j)_{j \in \mathbb{N}}\) such that \(P_{q_i}(\alpha, \varepsilon_i, k(N_j)) < 1\) holds for almost all \((i, k)\), then \(\lim_{j \to \infty} P_{N_j}(\alpha) = 0\).

- Special case: Consider \(N_j = \sum_{k=0}^{K_j} b_k q_k\) with

\[(b_0, \ldots, b_{K_j}) = (1, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, 1 \ldots, 0, 0, 1).\]

Then

\[ P_{N_j}(\alpha) = \prod_{i=0}^{j} P_{q_{m_i}}(\alpha, \varepsilon_{m_i}, 0(N)). \]
Proof idea (cont.)

- $\varepsilon_{i,0}(N) \approx -\frac{b_{i+1}}{a_{i+1}a_{i+2}} + \frac{b_{i+2}}{a_{i+1}a_{i+2}a_{i+3}} - \frac{b_{i+3}}{a_{i+1}a_{i+2}a_{i+3}a_{i+4}} + \ldots$, so if $b_{i+1} = b_{i+2} = \ldots = b_{i+j} = 0$, then $\varepsilon_{i,0}(N) \approx 0$.

- $P_{qn}(\alpha, \varepsilon)$ is smooth as a function in $\varepsilon$, so $P_{qn}(\alpha, \varepsilon_{i,0}(N)) \approx P_{qn}(\alpha, 0) = P_{qn}(\alpha)$.

- So it suffices to show that $P_{qm}(\alpha) < 1$ holds for infinitely many $m$. 
Proposition (H., 2022+)

Let \( \alpha_k = \{q_{k-1}/q_k\} = [0; a_k, a_{k-1}, \ldots, a_1] \), \( \delta_k := \|q_k \alpha\| \). Writing

\[
H_k(\alpha, \varepsilon) := 2\pi |\varepsilon + q_k \delta_k| \prod_{n=1}^{\lfloor q_k/2 \rfloor} h_{n,k}(\alpha, \varepsilon),
\]

where

\[
h_{n,k}(\alpha, \varepsilon) := \left| \left( 1 - q_k \delta_k \frac{n \alpha_k}{n} - \frac{1}{2} \right)^2 - \left( \frac{\varepsilon + q_k \delta_k}{n} \right)^2 \right|,
\]

we have for any badly approximable \( \alpha \) that

\[
\lim_{k \to \infty} |P_{q_k}(\alpha, \varepsilon) - H_k(\alpha, \varepsilon)| = 0,
\]

with the convergence being locally uniform.
Proof idea (cont.)

- We are left to show $H_k(\alpha, 0) < 1$ infinitely often.
- Elementary estimates reduce the problem to showing

$$\log \left( \frac{2\pi}{a_{k+1} + 1/a_{k+2} + \alpha_k} \right) \cdot \frac{q_k/2}{a_{k+1} + 1/a_{k+2} + \alpha_k} < \sum_{n=1}^{q_k/2} \frac{\{n\alpha_k\} - 1/2}{n}.$$

- The left-hand-side is decreasing in $a_{k+1}$, thus choose $a_{k+1} = 7$ (maximal).
- Partial summation: For $T$ a large constant and $E$ an error term,

$$\sum_{n=1}^{q_k/2} \frac{\{n\alpha_k\} - 1/2}{n} = \sum_{\ell=1}^T \frac{\sum_{n=1}^{\ell} \{n\alpha_k\} - 1/2}{\ell^2} + E(T, \limsup_{r \to \infty} a_r).$$
We are left to show $H_k(\alpha, 0) < 1$ infinitely often.

Morally, for large $T$, this is equivalent to

\[
\log \left( \frac{2\pi}{a_{k+1}+1/a_{k+2}+\alpha_k} \right) < \sum_{\ell=1}^{T} \frac{\ell}{\ell^2} \sum_{n=1}^{\ell} \{n\alpha_k\} - 1/2. \tag{\ast}
\]

Now (\ast) is almost continuous in $\alpha_k$!
Visualization of \((*)\). The red line function is the right-hand side in \(\alpha_k\), the dotted line the left-hand side for \(a_{k+1} = 7\). As the first partial quotients of \(\alpha_k\) are \(\leq 7\), \(\alpha_k\) lies inside the grey area. This shows \(H_k(\alpha, 0) < 1\) if \(a_{k+1} \geq 7\), hence the statement follows.
Show that 7 is optimal

- We prove that $\alpha = [0; 6, 5, 5]$ fulfills $\liminf_{N \to \infty} P_N(\alpha) > 0$.
- Regrouping the shifted products (due to Aistleitner, Technau, Zafeiropoulos) shows

$$P_N(\alpha) = P_{q_n}(\alpha) \cdot \prod_{i=1}^{n} K_i(N)$$

where

$$K_i(N) := \prod_{k=1}^{b_i(N)-1} P_{q_i}(\alpha, \varepsilon_i, k(N)) \cdot P_{q_{i-1}}(\alpha, \varepsilon_{i-1}, 0(N)) \mathbb{1}_{[b_{i-1}(N) \neq 0]}.$$ 

- The value of $K_i(N)$ is almost determined by the Ostrowski coefficients $b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}$. 

Lemma (H., 2022+)

Let \((b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}, b_{i+5})\) be possible Ostrowski coefficients for \(\alpha = [0; 6, 5, 5]\), not all 0, \(i\) sufficiently large and

\[
K_i(b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}) \cdot K_{i+1}(b_i, b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}) \cdot K_{i+2}(b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4}, b_{i+5}) < 1.001.
\]

Then

\[
K_i \cdot K_{i+1} \cdot K_{i+2} \cdot K_{i+3} \cdot K_{i+4} \cdot K_{i+5} > 1.001.
\]

- Proof contains many case distinctions and computational assistance.
- With \(P_N(\alpha) = P_{q_n}(\alpha) \cdot \prod_{i=1}^{n} K_i(N)\), the result follows.
Extremal values and open questions

**Theorem (H., 2022)**

Let $1 \leq a \leq 5$, $\alpha = [0; a, a, \ldots]$. 

- For $q_n \leq N < q_{n+1}$ we have $P_{q_n}(\alpha) \leq P_N(\alpha) \leq P_{q_{n+1}-1}(\alpha)$.
- $\min_{N \geq 1} P_N(\alpha) = P_1(\alpha)$.

Does a similar behaviour hold for any quadratic irrational without preperiod where $\lim \inf_{N \to \infty} P_N(\alpha) > 0$?

For $\lim \sup_{k \to \infty} a_k \leq 6$, it is in general open whether $\lim \inf_{N \to \infty} P_N(\alpha) > 0$.

- Conjecture: $\lim \sup_{k \to \infty} a_k \leq 3$ implies $\lim \inf_{N \to \infty} P_N(\alpha) > 0$ (this would be sharp since $\lim \inf_{N \to \infty} P_N([0; 4, 1]) = 0$).
Thanks for your attention!