A strong version of Cobham's theorem

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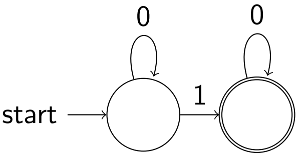

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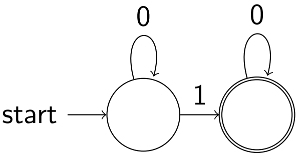

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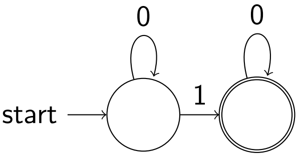

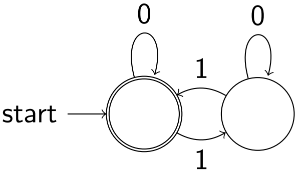

Let $n \geq 1$. A base k representation of a tuple $(x_1, \ldots, x_n) \in \mathbb{N}^n$ is a word over the alphabet $\{0, 1, \ldots, k-1\}^n$. For example, a base 2 representation of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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Definition. A subset of $\mathbb{N}^n$ is **semilinear** if it is finite union of the form

$$\Big\{ \boldsymbol{v} + \sum_{i=1}^{m} k_i \boldsymbol{v}_i \ : \ k_1, \ldots, k_m \in \mathbb{N} \Big\},$$

where $m \in \mathbb{N}$ and $\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \in \mathbb{N}^n$.

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- finite number of variables taking values in the given domain (here: in $\mathbb{N}$),
- equality and other given predicates (here: just $=$),
- functions (repeatedly) applied to variables (here: $+$),
- logical connectives such as $\mathrm{and}, \mathrm{or}$ and not,
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Example. The order relation is definable in $(\mathbb{N}, +)$:

$$\{(x, y) \in \mathbb{N}^2 \ : \ x < y\} = \{(x, y) \in \mathbb{N}^2 \ : \ \exists z \in \mathbb{N} \, \neg(z + z = z) \ \wedge \ y = x + z\}$$

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Semilinear sets are definable:

$$\{u \in \mathbb{N} \ : \ \exists x, y, z \in \mathbb{N} \ u = 6x + 9y + 20z\}$$

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Cobham-Semenov restated. Let $k, \ell \in \mathbb{N}_{\geq 2}$ be multiplicatively independent. A set $X \subseteq \mathbb{N}^n$ is definable in both $(\mathbb{N}, +, V_k)$ and $(\mathbb{N}, +, V_\ell)$ if and only if it is definable in $(\mathbb{N}, +)$.

H.-Schulz (2019-). Let $k, \ell \in \mathbb{N}_{\geq 2}$ be multiplicatively independent, and let $X \subseteq \mathbb{N}^m$ and $Y \subseteq \mathbb{N}^n$ be such that

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Proof of Cobham-Semenov.

Suppose $X \subseteq \mathbb{N}^n$ is definable in both $(\mathbb{N}, +, V_k)$ and $(\mathbb{N}, +, V_\ell)$, but not in $(\mathbb{N}, +)$. Then the theory of $(\mathbb{N}, +, X, X)$ is undecidable. However, then the theory of $(\mathbb{N}, +, V_k)$ is undecidable.

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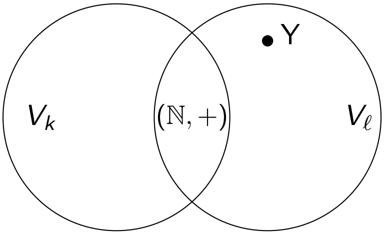

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This question is an old question. Bruyère, Cherlin and van den Dries asked this question as early as 1985, and it has been restated in the literature many times.

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Corollary of Baker's theorem on linear forms. For every $m \in \mathbb{N}$, there exists $C(m)$ such that if $n_1, n_2 \in \mathbb{N}$ with $2^{n_1} - 3^{n_2} = m$, then $n_1, n_2 \leq C$.

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Open question. What fragments of the theory of $(\mathbb{N}, +, 2^{\mathbb{N}}, 3^{\mathbb{N}})$ are decidable?

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For $y \in 2^{\mathbb{N}}$, define $S(y)$ to be the set of all $x \in 3^{\mathbb{N}}$ such that $\lambda(x - \lambda(x)) = y$.

In words: $S(y)$ is the set of all powers of 3 for which y is the second largest power of 2 that appears in the binary representation of x.

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For example: $27 = 16 + 8 + 2 + 1$. So $27 \in S(8)$.

For simplicity, let $k = 2$ and $\ell = 3$. So consider $(\mathbb{N}, +, 2^{\mathbb{N}}, 3^{\mathbb{N}})$.

Let $\lambda : \mathbb{N} \to 2^{\mathbb{N}}$ map x to the unique element $2^m \in 2^{\mathbb{N}}$ with $2^m \leq x < 2^{m+1}$.

For $y \in 2^{\mathbb{N}}$, define $S(y)$ to be the set of all $x \in 3^{\mathbb{N}}$ such that $\lambda(x - \lambda(x)) = y$.

In words: $S(y)$ is the set of all powers of 3 for which y is the second largest power of 2 that appears in the binary representation of x.

For example: $27 = 16 + 8 + 2 + 1$. So $27 \in S(8)$.

Fact. For all $y \in 2^{\mathbb{N}}$, $S(y)$ is finite. However, for all $m, n \in \mathbb{N}$ there is $y \in 2^{\mathbb{N}}$ such that $y > m$ and $|S(y)| > n$.

Main Lemma. Let $m, n \in \mathbb{N}$, let $Z_1, \ldots, Z_m$ be a partition of $\{1, \ldots, n\}$. Then there are $s \in \mathbb{N}_{>0}$ and $t_1 < \cdots < t_n$ such that for $i = 1, \ldots, m$

$$S(2^{s+i}) \cap [3^{t_1}, 3^{t_n}] = \{3^{t_j} \ : j \in Z_i\}.$$

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$Z_1 = 1, 3, 5,$

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- Proof of Main Lemma just uses density of $2^{-\mathbb{N}}3^{\mathbb{N}}$ in $\mathbb{R}_{>0}$.
- This allows us to code/interpret arbitrary large finite subsets of $\mathbb{N}^2$.
- Such theories are known to be undecidable, as the halting problem or the tiling problem can be encoded in such theories.

Remark 1. The proof does not dependent on $\mathbb{N}$. The theory of $(\mathbb{R}, <, +, k^{\mathbb{N}}, \ell^{\mathbb{N}})$ is undecidable whenever k and ℓ are multiplicatively independent.

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Remark 3. We expect that this method can be extended to prove other variants of Cobham's theorem.

Combinatorics on words.

We observed that $\{n \in \mathbb{N} \ : \ s_2(n) \text{ is even } \}$ is 2-recognizable, where $s_2(n)$ is the binary digit sum. Thus the function $f : \mathbb{N} \to \{0, 1\}$ given by

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Example. To check that the Thue-Morse sequence in not eventually periodic, we have to decide

$$(\mathbb{N}, +, V_2) \models \forall p \ (p > 0) \to \Big(\forall i \ \exists j \ j > i \ \wedge \ f(j) \neq f(j + p)\Big)$$

A **continued fraction expansion** $[a_0; a_1, \ldots, a_k, \ldots]$ is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

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Ostrowski (1918). Every natural number N can be written uniquely as

$$N = \sum_{k=0}^{n} b_{k+1} q_k,$$

where $b_k \in \mathbb{N}$ such that $b_1 < a_1$, $b_k \leq a_k$ and, if $b_k = a_k$, $b_{k-1} = 0$.

Let $V_a : \mathbb{N} \to \mathbb{N}$ be the function that maps $x \geq 1$ with Ostrowski representation $b_n \dots b_1$ to the least q_k with $b_{k+1} \neq 0$, and 0 to 1.

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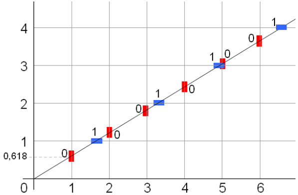
The **characteristic Sturmian word with slope** a is the infinite $\{0,1\}$-word $\mathrm{c_a} = c_a(0)c_a(1)c_a(2)\dots$ such that for all $n \in \mathbb{N}$

$$c_a(n) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor - \lfloor a \rfloor.$$



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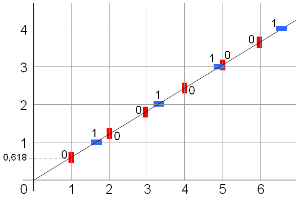

Fact. Let $n \in \mathbb{N}_{\geq 1}$. Then the following are equivalent:

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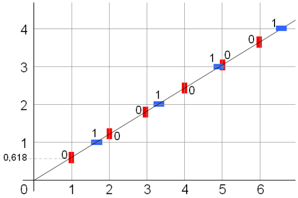

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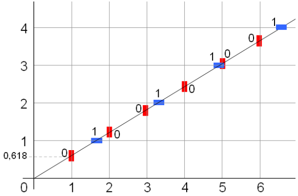

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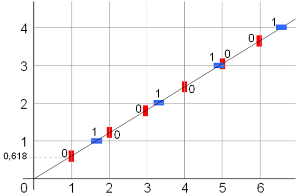

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The decision procedure for T_{Sturmian} allows us to check that no Sturmian word is eventually periodic.

An implementation: **Pecan**

- Try Pecan at `http://reedoei.com/pecan`
- Git: `https://github.com/ReedOei/Pecan`

Pecan improves on **Walnut** by Mousavi, another automated theorem prover for deciding combinatorial properties of automatic words, by using Büchi automata instead of finite automata.

This difference enables Pecan to handle uncountable families of sequences, allowing us quantify over all Sturmian words.

Thank you!