IFS with linear interval maps

Ale Jan Homburg

Korteweg-de Vries Institute for Mathematics
University of Amsterdam
&
Department of Mathematics
Free University Amsterdam

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Joint with Charlene Kalle (Leiden University)

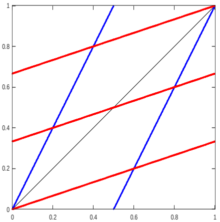
Linear maps on the unit interval

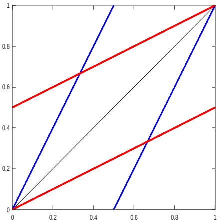
Given a pair (M, N) of integers $M, N \geq 2$, consider the N-adic map $f_0 : [0, 1) \to [0, 1)$,

$$f_0(x) = Nx \pmod 1$$

and the collection of M contracting maps $f_i : [0, 1) \to [0, 1)$,

$$f_i(x) = \frac{1}{M}x + i/M, \qquad 0 \leq i < M.$$





Probability vector

Take $(p_0, \ldots, p_M)$ with $0 < p_0 < 1$ and

$$p_j = \frac{1 - p_0}{M}, \qquad 1 \leq j \leq M.$$

Symbol space

$\Sigma = \{0, \ldots, M\}^{\mathbb{N}}$ with product topology, Borel σ-algebra, and Bernoulli measure ν;

$$\nu([a_0 \cdots a_k]) = \prod_{j=0}^{k} p_{a_j}$$

for

$$[a_0 \cdots a_k] = \{\omega \in \Sigma \; ; \; \omega_j = a_j, 0 \leq j \leq k\}.$$

Skew product system

Denote $\omega = (\omega_0\omega_1\cdots)$ for $\omega \in \Sigma$.

$F : \Sigma \times [0,1) \to \Sigma \times [0,1)$ is given by

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

Write

$$F^k(\omega, x) = (\sigma^k\omega, f_\omega^k(x)) = (\sigma^k\omega, f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}(x)).$$

Stationary measure

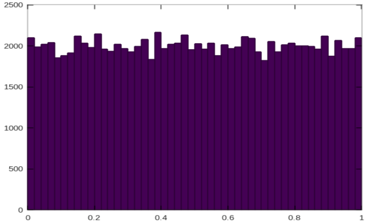
Write λ for Lebesgue measure on $[0,1)$ and write $\mu = \nu \times \lambda$.

Invariance

The product measure μ is an invariant probability measure for F (λ is a stationary measure for the IFS):

$$\mu(A) = \mu(F^{-1}(A)).$$

The measure μ is ergodic.



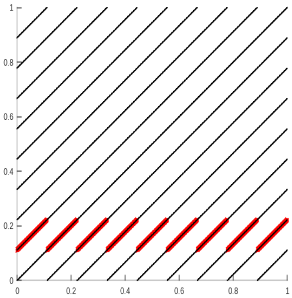
Uniform distribution

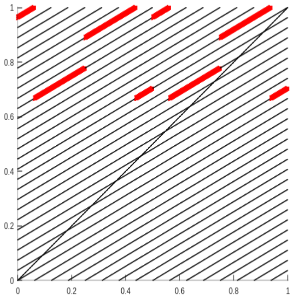
For μ-almost all (ω, x),

$$\{f_\omega^n(x)\ ;\ n \in \mathbb{N}\}$$

is uniformly distributed in $[0,1)$.

Graphs of iterates





Left picture: $(M, N) = (3, 3)$. The thick graph is the graph of f_ω^8 for $\omega = (12020020)$.
Right picture: $(M, N) = (3, 2)$. The thick graph is the graph of f_ω^7 for $\omega = (3020020)$.

Lyapunov exponent

The Lyapunov exponent L_p is given by

$$L_p = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left(f_{\omega_i}\right)' = p_0 \ln(N) - (1-p_0) \ln(M),$$

where by the strong law of large numbers the limit exists ν-almost surely and equals the given constant.

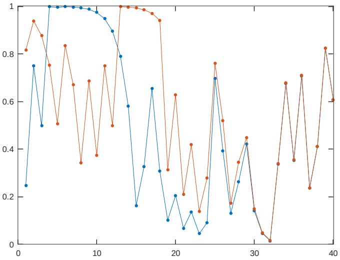
Two-point motion

Synchronization

Consider $L_p < 0$. For all $x, y \in [0,1)$,

$$\lim_{n\to\infty} |f_\omega^n(y) - f_\omega^n(x)| = 0$$

for ν-almost all $\omega \in \Sigma$.



Two-point motion

Ideas in proof

We have

$$(f_\omega^n)' \leq C\zeta^n$$

for some $C > 1, \zeta < 1$ and ω from a set Ω_C of positive measure.
Let B_n denote the $1/n^2$-neighborhood of discontinuity points i/N.
Borel-Cantelli: for μ-almost all (ω, x),

$$f_\omega^n(x) \in B_n \text{ for at most finitely many n.}$$

Using ergodicity it suffices to construct $\Psi \subset \Sigma$ of positive measure with: for $\omega \in \Psi$, $f_\omega^n([0,1))$ is an interval and

$$\lim_{n\to\infty} \lambda\left(f_\omega^n([0,1))\right) = 0.$$

There are intervals J_ω so that $f_\omega^n(J_\omega)$ is an interval for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \lambda\left(f_\omega^n(J_\omega)\right) = 0.$$

Take $J \subset J_\omega$ for ω from a positive measure set. There exists $t \in \mathbb{N}$ and $\eta \in \{1, \ldots, M\}^t$ with $f_\eta^t([0,1)) \subset J$.

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Two-point motion

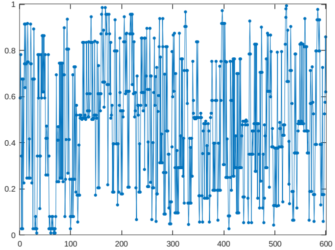
Intermittency

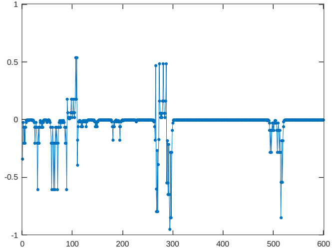
Consider $L_p = 0$. For every $\varepsilon > 0$, for all $x, y \in [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{0 \leq i < n \; ; \; |f_\omega^i(x) - f_\omega^i(y)| < \varepsilon\}\right| = 1$$

for ν-almost all $\omega \in \Sigma$.
Let β be a small positive number. Let $x, y \in [0, 1)$. Then for ν-almost all $\omega \in \Sigma$, either $|f_\omega^n(x) - f_\omega^n(y)| = 0$ for some n or $|f_\omega^n(x) - f_\omega^n(y)| > \beta$ for infinitely many values of n.

Two-point motion





Left panel: time series of $f_\omega^n(x)$ for $p_0 = 1/2$ and $(M, N) = (3, 3)$.

Right panel: signed difference with another time series with the same ω.

Two-point motion

Ideas in proof. $M = N$ and $p_0 = 1/2$.

Then $y_n = |f_\omega^n([0,1))|$ satisfies

$$y_{n+1} = \begin{cases} y_n/M, & \omega_n \in \{1,\dots,M\}, \\ \min\{y_n M, 1\}, & \omega_n = 0. \end{cases}$$

Consider $z_i = -\log_N(y_i)$. Then $z_0 = 0$ and

$$z_{n+1} = \begin{cases} z_n + 1, & \omega_n \in \{1,\dots,M\}, \\ \max\{z_n - 1, 0\}, & \omega_n = 0. \end{cases}$$

So z_n is a random walk on $\mathbb{N}$ with partially reflecting boundary at 0. It is null-recurrent.

Two-point motion

Ideas in proof. Remarks on the general case.

Let $\varepsilon > 0$ be small.
Let $(\zeta_1 \dots \zeta_D)$ be a fixed sequence in $\{1, \dots, M\}^D$ with D large so that $1/M^D < \varepsilon$. Start with $[0, 1)$ and iterate under f_ω^n until the final D symbols $\omega_{n-D} \dots \omega_{n-1}$ equal $\zeta_1 \dots \zeta_D$. Then

$$|f_\omega^n([0, 1))| < \varepsilon.$$

The expected stopping time is finite. We could continue iterating until $f_\omega^n([0, 1))$ is no longer contained in an interval of size ε. To control possible intersections with discontinuities, we modify. Iterate instead until

$$|f_\omega^n([0, 1))| > 1/n^2.$$

The expected stopping time is infinite.

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Two-point motion

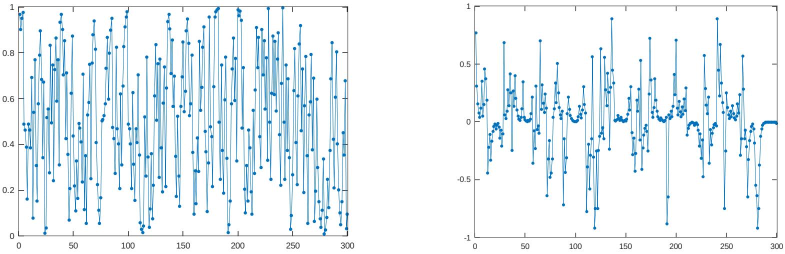
The two-point map $f_i^{(2)} : [0,1)^2 \to [0,1)^2$ given by

$$f_i^{(2)}(x,y) = (f_i(x), f_i(y)).$$

Expansion

Consider $L_p > 0$. The iterated function system on $[0,1)^2$ generated by $f_i^{(2)}$, $0 \leq i \leq M$, admits an absolutely continuous stationary probability measure $m^{(2)}$ of full support.

Two-point motion

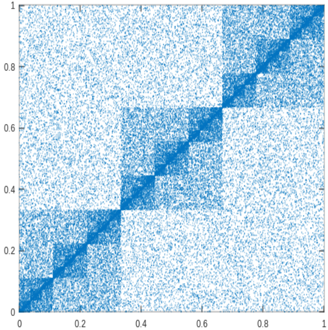


Left panel: time series of $f^n_\omega(x)$ for $p_0 = 1/2$ and $(M, N) = (2, 3)$.

Right panel: signed difference with another time series with the same ω.

Two-point motion

Stationary distribution
for two-point motion.
Example: $(M, N) = (3, 9)$.
Picture: $p_0 = 1/2$.



Stationary measure

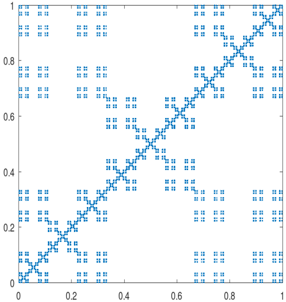
$$m^{(2)} = \sum_{\ell=0}^{\infty} \zeta^\ell \sum_{j=0}^{M^\ell - 1} M^\ell \, \lambda|_{[j/M^\ell,(j+1)/M^\ell)^2}, \; \zeta = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{4-3p_0}{p_0}}.$$

Density function

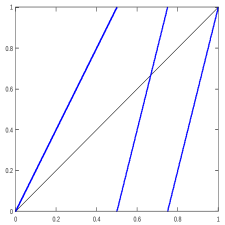
In this example: $L_p > 0$ for $p_0 > 1/3$. The density of the stationary measure for the two-point motion is bounded for $p_0 > 9/13$.

Two-point motion

Stationary distribution
for two-point motion.
Example: $(M, N) = (3, 3)$.
Supported on fractal invariant set.



Extensions



The skew product map F is measurably isomorphic to $G : [0,1)^2 \to [0,1)^2$ given by

$$G(w,x) = \begin{cases} \left(\frac{1}{p_0} w \pmod 1, Nx \pmod 1\right), & 0 \leq w < p_0, \\ \\ \left(\frac{M}{1-p_0}(w - r_i) \pmod 1, \frac{1}{M}x + \frac{i}{M}\right), & r_i \leq w < r_{i+1}, 1 \leq i \leq M \end{cases}$$

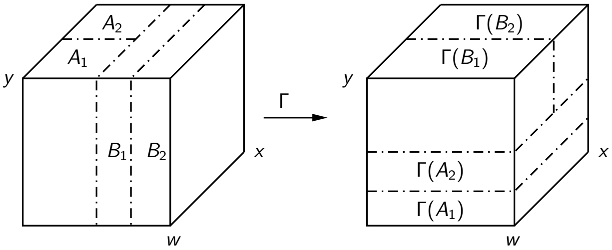
with two-dimensional Lebesgue measure as invariant measure.

Extensions

An invertible extension $\Gamma : [0,1)^3 \to [0,1)^3$ is given by

$$\Gamma(w,x,y) = \begin{cases} \left(\frac{1}{p_0}w \pmod 1, Nx \pmod 1, \frac{p_0}{N}y + \frac{jp_0}{N}\right), \\ \qquad 0 \le w < p_0 \\ \qquad \frac{j}{N} \le x < \frac{j+1}{N}, 0 \le j < N, \\ \left(\frac{M}{1-p_0}(w - r_i) \pmod 1, \frac{1}{M}x + \frac{i}{M}, (1-p_0)y + p_0\right), \\ \qquad r_i \le w < r_{i+1}, 1 \le i \le M. \end{cases}$$

Extensions



The action of Γ on the cube $[0,1)^3$, $(M,N) = (2,2)$.

Extensions

Consider subsets of periodic points for Γ in $[0,1)^3$:

$$\Omega_s = \left\{ (w,x,y) \; ; \; \Gamma^q(w,x,y) = (w,x,y), \left| \frac{\partial}{\partial x} \Gamma^q(w,x,y) \right| < 1 \right\},$$

$$\Omega_c = \left\{ (w,x,y) \; ; \; \Gamma^q(w,x,y) = (w,x,y), \left| \frac{\partial}{\partial x} \Gamma^q(w,x,y) \right| = 1 \right\},$$

$$\Omega_u = \left\{ (w,x,y) \; ; \; \Gamma^q(w,x,y) = (w,x,y), \left| \frac{\partial}{\partial x} \Gamma^q(w,x,y) \right| > 1 \right\}.$$

Extensions

Heterodimensional chaos (inspired by Saiki, Takahasi, Yorke)

Consider (M, N) with $M, N \geq 2$.

The sets Ω_s, Ω_u are dense in $[0, 1)^3$.
If $\ln(N)/\ln(M) \in \mathbb{Q}$ then Ω_c is dense in $[0, 1)^3$.
If $\ln(N)/\ln(M) \notin \mathbb{Q}$, then Ω_c is empty.