Algebraic Automatic Continued Fractions in Characteristic 2

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1 Automatic Sequences

2 Automatic Continued Fractions
A sequence is said to be \textbf{\textit{\textbf{d-automatic}}} if it can be generated by a \textbf{\textit{d-DFAO}} (\textit{deterministic finite automaton with output}). For an integer \(d \geq 2\), a \(d\)-DFAO is defined to be a 6-tuple

\[ M = (Q, \Sigma, \delta, q_0, \Delta, \tau) \]

where \(Q\) is the set of states with \(q_0 \in Q\) being the initial state, \(\Sigma = \{0, 1, \ldots, d - 1\}\) the input alphabet, \(\delta : Q \times \Sigma \to Q\) the transition function, \(\Delta\) the output alphabet, and \(\tau : Q \to \Delta\) the output function.
The $d$-DFAO $M$ generates a sequence $(c_n)_{n \geq 0}$ in the following way: for each non-negative integer $n$, the base-$d$ expansion of $n$ is read by $M$ from right to left starting from the initial state $q_0$, and the automaton moves from state to state according to its transition function $\delta$. When the end of the string is reached, the automaton halts in a state $q$, and the automaton outputs the symbol $c_n = \tau(q)$.

\[
\begin{array}{c}
0 \\
\tau(i) = 0, \tau(a) = 1.
\end{array}
\]

The sequence $(t(n))_n$ generated by this automaton is the *Thue-Morse* sequence whose first terms are 0110100110010110 ···.
Theorem 1 (Christol, Kamae, Mendès France, Rauzy 1979)

The formal power series \( \sum_{n=0}^{\infty} u_n x^n \) in \( \mathbb{F}_p[x] \) is algebraic over \( \mathbb{F}_p(x) \) if and only if \( (u_n)_n \) is \( p \)-automatic.

Theorem 2 (Adamczewski, Bugeaud 2007)

An automatic real number is either rational or transcendental.

Theorem 3 (Bugeaud 2013)

The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a finite automaton.
Let $K$ be a field. Let

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0 + c_{-1} z^{-1} + \cdots$$

be an arbitrary element of $K((1/z))$. Define the integer part of $f(z)$ as

$$[f(z)] = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0.$$  \hspace{1cm} (2)

Set $f_0 = f$, $a_0 = [f_0]$, $f_0 = a_0 + 1/f_1$, $a_1 = [f_1]$, $f_1 = a_1 + 1/f_2$, $a_2 = [f_1]$, ... Then $a_0 \in K[z]$ and $a_j \in K[z] \setminus K$ for $j \geq 1$, and $f(z)$ admits the following continued fraction expansion

$$f(z) = a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \frac{1}{\cdots}}}.$$  \hspace{1cm} (3)
Hu and Han studied continued fractions defined by the Thue-Morse sequence and the period-doubling sequence, and proved the algebraicity for some cases. Bugeaud and Han proved the algebraicity of Thue-Morse continued fractions. Hu and Lasjaunias proved the algebraicity of period-doubling continued fractions.

Recall that the Thue-Morse sequence can be defined by the substitution $0 \rightarrow 01$, $1 \rightarrow 10$, And the period-doubling sequence can be defined by the substitution $0 \rightarrow 01$, $1 \rightarrow 00$. 
More sequences

We notice that the period-doubling sequence \( p = 01000101 \ldots \) is the limit of the following sequence of finite words:

\[
\begin{align*}
0 \\
010 \\
0100010 \\
010001010100010 \\
\ldots
\end{align*}
\]

More precisely, let \( W_0 \) be the empty word, let \( \varepsilon = (01)^\infty \), and \( W_{n+1} = W_n, \varepsilon_n, W_n \) for \( n \geq 0 \). Then \( p = \lim W_n \). Therefore it is natural to consider sequences defined in the same way but with \( W_0 \) any finite word and \( \varepsilon \) any periodic sequence. We let \( P(W_0, \varepsilon) \) denote the sequence defined this way and let \( P \) denote the family of such sequences.
We notice that the Thue-Morse sequence \( t = 01101001 \ldots \) is the sum of the period-doubling sequence \( p = 10111010 \ldots \). More precisely, if we define \( \sigma \) to be the operator that maps a binary sequence \((u_n)\) to \((\sum_{j=0}^{n-1} u_j \pmod{2})\), then \( t = \sigma(p) \). It is therefore natural to consider continued fractions defined by \( \sigma^n(p) \) for \( n \geq 2 \).

In fact, we also want to consider sequences of the form \( \sigma^n(u) \) where \( n \geq 1 \) and \( u \in \mathcal{P} \). It turns out more convenient to consider yet a larger family \( \mathcal{G} \) defined as follows: Let \( \Upsilon \) be a periodic \((0, 1)\)-sequence. Let \( u_0 \) and \( v_0 \) be two finite words. For \( n \geq 0 \), define

\[
\begin{align*}
  u_{n+1} &= u_n, u_n \\
  v_{n+1} &= v_n, v_n
\end{align*}
\]

if \( \Upsilon_n = 0 \), and

\[
\begin{align*}
  u_{n+1} &= u_n, v_n \\
  v_{n+1} &= v_n, u_n
\end{align*}
\]

otherwise. We let \( \mathcal{G}(u_0, v_0, \Upsilon) \) denote \( \lim_{n \to \infty} u_n \) and let \( \mathcal{G} \) denote the family of such sequences.
Let $A = \{a_0, \ldots, a_k\}$ be a finite alphabet. We treat $a_j$ as formal variables and define

$$\mathbb{F}_2[A] := \mathbb{F}_2[a_0, \ldots, a_k]$$

$$\mathbb{F}_2(A) := \mathbb{F}_2(a_0, \ldots, a_k)$$

$$\mathbb{F}_2((A)) := \mathbb{F}_2((\frac{1}{a_0}, \ldots, \frac{1}{a_k})).$$

Here $\mathbb{F}_2((\frac{1}{a_0}, \ldots, \frac{1}{a_k}))$ denotes the ring of power series of the form

$$\varphi = \sum_{n_0, \ldots, n_k \geq N} c_{n_0,n_1,\ldots,n_k} a_0^{-n_0} \cdots a_k^{-n_k}, \quad (4)$$

where $N$ is an integer and $c_{n_0,n_1,\ldots,n_k} \in \mathbb{F}_2$.

Let $(u_n)_{n \geq 0}$ be a sequence taking values in $A$. It defines a formal power series $\sum_{n \geq 0} u_n z^n$ in $\mathbb{F}_2[A][[z]]$.

We define a norm on $\mathbb{F}_2((A))$ by assigning a series of the form (4) the number $2^{-m}$, where $m = \min\{n_0 + n_1 \cdots + n_k \mid c_{n_0,n_1,\ldots,n_k} \neq 0\}$ (with the convention that $\min \emptyset = \infty$) is the valuation of the series $\varphi$. 


The continued fraction $CF(u) = [u_0, u_1, \ldots]$ is defined as the limit of the sequence $([u_0, u_1, \ldots, u_n])_n$:

$$
[u_0] = u_0,

[u_0, u_1, \ldots, u_n] = u_0 + \frac{1}{[u_1, \ldots, u_n]} \in F_2((A)),
$$

for $n \geq 1$. For example,

$$
[u_0, u_1, u_2] = u_0 + \frac{1}{u_1 + \frac{1}{u_2}}

= u_0 + \frac{u_1^{-1}}{1 + (u_1 u_2)^{-1}}

= u_0 + u_1^{-1} + u_1^{-2} u_2^{-1} + u_1^{-3} u_2^{-2} + \ldots
$$
Define

$$M_n = \left( \frac{1}{u_n} \frac{1}{u_n} \right) \left( \frac{1}{u_{n-1}} \frac{1}{u_{n-1}} \right) \cdots \left( \frac{1}{u_0} \frac{1}{u_0} \right)$$ \hspace{1cm} (5)$$

then

$$[u_0, u_1, \ldots, u_n] = \frac{M_{n,0,1}}{M_{n,0,0}}.$$ \hspace{1cm} (6)

In general, we do not have the convergence of $$(M_{n,0,1})_n$$ and $$(M_{n,0,0})_n$$, but we do have the convergence of $$\left( \frac{M_{n,0,1}}{M_{n,0,0}} \right)_n$$, which is proved in the same way as in the case of classical continued fraction for real numbers.
Main result

Theorem 4 (H.)

If \( u = P(W_0, \varepsilon) \) is a sequence in \( P \), where \( \varepsilon \) has period \( n \), then \( \text{CF}(u) \) is algebraic over \( \mathbb{F}_2(A) \) of degree at most \( 2^n \).

Theorem 5 (H.)

If \( u = G(u_0, v_0, \Upsilon) \) is a sequence in \( G \), where \( \Upsilon \) has period \( k \), and contains an even number of 1's in one period, then \( \text{CF}(u) \) is algebraic over \( \mathbb{F}_2(A) \) of degree at most \( 2^k \).

Remark For a sequence \( u \in P \cup G \), \( \sigma(u) \in G \).
Example

Let $u_0$ and $v_0$ be single letters $a$ and $b$. Let $\Upsilon = 1^\infty$. Then $G(u_0, v_0, \Upsilon)$ is the Thue-Morse sequence $t = abbabaab \ldots$ Define

$$m_0 = \begin{pmatrix} 1 & 1/a \\ 1/a & 0 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 1 & 1/b \\ 1/b & 0 \end{pmatrix}$$

and for $k \geq 0$,

$$m_{k+1} = w_k m_k \quad w_{k+1} = m_k w_k.$$ 

By (5) and (6),

$$\text{CF}(t) = \lim_{k} \frac{m_{2k+1,0,1}}{m_{2k+1,0,0}}.$$ 

We will show that every entry of $(m_{2k+1})_k$ converges, and the limit is algebraic. The key to the proof is the relation

$$m_3 = (m_1 + d/co + d^2/r/co^2) \cdot l,$$ 

where

$$d = \det(m_1), \quad r = \text{tr}(m_1), \quad co = m_1 + w_1 + r, \quad l = r \cdot co^2.$$
Example

In fact, the relation \((\star)\) holds regardless of the particular form of \(m_0\) and \(w_0\); it remains true if we had taken \(m_0\) and \(w_0\) to be any \(2 \times 2\) matrix in characteristic 2. Therefore, it actually gives us the relation between \(m_{2k+1}\) and \(m_{2k+3}\) for all \(k \geq 0\). For example, if we define \(m'_0 = m_2\), and \(w'_0 = w_2\), and define \(m'_k\), \(w'_k\), \(d'\), \(r'\), \(co'\), \(l'\) accordingly, then by \((\star)\),

\[
m'_3 = \left( m'_1 + d'/co' + d'^2/r'/co'^2 \right) \cdot l'.
\]

We verify directly

\[
d' = d^4, \quad r' = r \cdot l, \quad co' = co \cdot l, \quad l' = l^4.
\]

Therefore

\[
m_5 = m'_3 = \left( m_3 + d^4/co/l + d^8/r/co^2/l^3 \right) \cdot l^4
= l^{1+4} \cdot \left( m_1 + (d + d^4/l^2)/co + (d^2 + d^8/l^4)/r/co^2 \right)
\]
Example

We continue in this way to find

\[ m_7 = l^{1+4+16} \cdot (m_1 + (d + d^4/l^2 + d^4/l^{2+8})/co + (d^2 + d^8/l^4 + d^{32}/l^{4+16})/r/co^2) \]

etc. Define

\[ f = l^{1+4+16+64+\cdots} \]

\[ H = d + d^4/l^2 + d^{16}/l^{2+8} + d^{64}/l^{2+8+32} + \cdots. \]

It is easy to prove by induction that

\[ \lim_{k} m_{2k+1} = f \cdot (m_1 + H/co + H^2/r/co^2). \]

Both \( f \) and \( H \) are algebraic:

\[ f^4 = f/l \]

\[ H^4/l^2 = H + d. \]

Therefore \( \text{CF}(t) \in \mathbb{F}_2(a, b)[H] \), and is algebraic of degree at most 4.
Idea of the Proof

Let $m_0, w_0$ be two $2 \times 2$ matrices with entries in a field of characteristic 2. Let $m_1 = w_0 m_0$, $w_1 = m_0 w_0$. We define $m_{1s}$ and $w_{1s}$ for all binary word $s$ inductively as follows:

\[
\begin{align*}
    m_{1s0} &= m_{1s}^2 \\
    m_{1s1} &= w_{1s} m_{1s} \\
    w_{1s0} &= w_{1s}^2 \\
    w_{1s1} &= m_{1s} w_{1s}.
\end{align*}
\]

Define $d = \det(m_1)$, $r = \text{tr}(m_1)$ (the trace of $m_1$), and $co = m_1 + w_1 + r$. For all binary word $s$, define $t(s)$ as the sum of digits of $s$ modulo 2.

Let $s$ be an arbitrary non-empty binary word. For $j < |s|$, define $s(j)$ to be $s_0 \ldots s_{j-1}$, the prefix of length $j$ of $s$. Define $e(s)$ inductively as follows: If $s$ is the empty word, then $e(s) = 0$; define $e(s0) = 2e(s) + t(s0)$, $e(s1) = 2e(s) + t(s1)$. Define and $c_j(s) = d^{2^{j-1}} / r^{2^{j-1} - 1} - e(s(j)) / co^{e(s(j))}$. We write $c_j$ for short when there is no ambiguity.
Idea of the Proof

Proposition 6

Let $s$ be a binary word of length $k \geq 1$. If $t(s) = 0$, then

$$m_{1s} = (m_1 + c_1 + \cdots + c_k) \cdot \frac{d^{2k-1}}{c_k}, \quad (7)$$
$$w_{1s} = (w_1 + c_1 + \cdots + c_k) \cdot \frac{d^{2k-1}}{c_k}; \quad (8)$$

If $t(s) = 1$, then

$$m_{1s} = (w_1 + c_1 + \cdots + c_k) \cdot \frac{d^{2k-1}}{c_k}, \quad (9)$$
$$w_{1s} = (m_1 + c_1 + \cdots + c_k) \cdot \frac{d^{2k-1}}{c_k}. \quad (10)$$
Thank You!