The coincidence of Rényi-Parry measures for β-transformation

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Contents

β-expansions

The β-expansions were first introduced by Rényi in 1957.

Given $\beta > 1$, for any $x \in [0, 1]$ there is a non-negative integer sequence $(c_i)_{i\geq 1}$ such that

$$x = \sum_{i\geq 1} \frac{c_i}{\beta^i} \quad c_i \in \{0, ..., \lfloor\beta\rfloor\},$$

where $\lfloor\beta\rfloor := \max\{n \in \mathbb{N} : n \leq \beta\}$. We now call this the β-expansion of x.

β-transformations

Fixed a $\beta > 1$, the β-transformation $T_\beta : [0,1] \to [0,1]$ is defined by letting

$$T_\beta(x) := \beta x \pmod 1.$$

Then we have

$$T_\beta(x) = \begin{cases} \beta x - i, & x \in \left[\frac{i}{\beta}, \frac{i+1}{\beta}\right), \quad i \in \{0, \ldots, \lfloor \beta \rfloor - 1\}; \\ \beta x - \lfloor \beta \rfloor, & x \in \left[\frac{\lfloor \beta \rfloor}{\beta}, 1\right]. \end{cases}$$

β-transformations

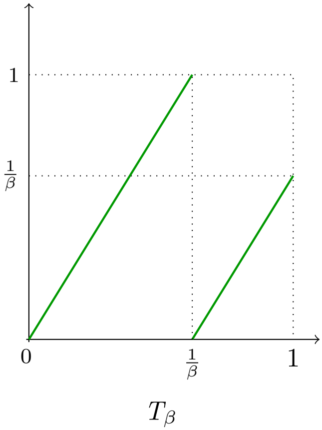

Figure: The β-transformation T_β with $\beta = \frac{1+\sqrt{5}}{2}$.

Rényi-Parry measure of β-transformations

- For any non-integer $\beta > 1$, the Lebesgue measure is not T_β-invariant.
- Rényi (1957) proved that there exists a unique T_β-invariant probability measure ν_β.
- Parry (1960) gave the explicit formula of the density function of ν_β.
- ν_β is called the *Rényi-Parry measure* of β-transformation.

Rényi-Parry measure of β-transformations

Theorem (Parry, 1960)

Let $T_\beta(x) = \beta x \pmod 1$ *for any* $\beta > 1$. *Then, there exists a unique* T_β*-invariant probability measure* ν_β *which is given by* $\nu_\beta(E) = \int_E \widetilde{h}_\beta \, d\lambda$ *for all Borel set* $E \subset [0,1]$ *with*

$$\widetilde{h}_\beta(x) = \frac{h_\beta(x)}{\int_{[0,1]} h_\beta \, d\lambda} \quad \text{and} \quad h_\beta(x) = \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}, \ x \in [0,1).$$

Density function of ν_β

- $h_\beta(x) \geq 0$ for λ-a.e. $x \in [0,1]$.
- T_β is ergodic with ν_β for all $\beta > 1$.
- We define the normalization constant by

$$K_\beta := \int_{[0,1]} h_\beta \, d\lambda = \sum_{n=0}^{\infty} \frac{T_\beta^n(1)}{\beta^n},$$

which implies that

$$\widetilde{h}_\beta(x) = \frac{h_\beta(x)}{K_\beta}.$$

Density function of ν_β

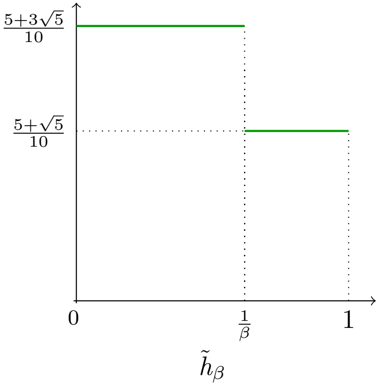

Figure: The density function $\widetilde{h}_\beta(x)$ with $\beta = \frac{1+\sqrt{5}}{2}$.

Question

For what pairs (β_1, β_2) the Rényi-Parry measures coincide ?

Conjecture (Bertrand-Mathis, 1998)

Given two real numbers $1 < \beta_1 < \beta_2$*, the Rényi-Parry measures coincide, i.e.,* $\nu_{\beta_1} = \nu_{\beta_2}$*, if and only if*

- $\beta_1, \beta_2 \in \mathbb{N}$*, in which the Rényi-Parry measures are the Lebesgue measure;*
- $\beta_2 = \beta_1 + 1$*,* β_1 *is the positive root of* $x^2 - qx - p = 0$*, where* $p, q \in \mathbb{N}$ *with* $p \leq q$*.*

It is easy to check that $\nu_{\beta_1} = \nu_{\beta_2} = \lambda$ for $\beta_1, \beta_2 \in \mathbb{N}$.

- $\beta > 1$ is called a *Pisot number* if β is an algebraic integer, whose algebraic conjugates are of modulus strictly smaller than 1.
- Two positive real numbers $a, b > 0$ are *multiplicatively independent*, denoted by $a \nsim b$, if $\log a / \log b \notin \mathbb{Q}$.

Theorem (Hochman-Shmerkin, 2015)

Let $\beta_1, \beta_2 > 1$ with $\beta_1 \nsim \beta_2$ and β_1 a Pisot number. If μ is jointly invariant under T_{β_1}, T_{β_2}, and if all ergodic components of μ under T_{β_2} have positive entropy, then μ is the common Rényi-Parry measure for T_{β_1} and T_{β_2}.

They also mentioned that they didn't know what non-integer pairs (β_1, β_2) have the same Rényi-Parry measures.

Main results

Theorem (H.-Wang, 2025)

Given two non-integers $1 < \beta_1 < \beta_2$, the Rényi-Parry measures coincide, i.e., $\nu_{\beta_1} = \nu_{\beta_2}$, if and only if

- β_1 is the root of $x^2 - qx - p = 0$, where $p, q \in \mathbb{N}$ with $p \leq q$,
- $\beta_2 = \beta_1 + 1$.

- This result proves the Bertrand-Mathis conjecture.
- If $\nu_{\beta_1} = \nu_{\beta_2}$, then β_1 and β_2 are Pisot numbers.

Main results

Corollary

Let $\beta_1 > 1$ be a Pisot number of degree ≥ 3 and $\beta_2 > 1$ with $\beta_1 \not\sim \beta_2$. Then there are no jointly invariant under T_{β_1}, T_{β_2} and ergodic Borel probability measures with positive entropy under T_{β_2}.

Corollary

For two non-integers $\beta_1, \beta_2 > 1$, $\nu_{\beta_1} = \nu_{\beta_2}$ and $\log \beta_1 / \log \beta_2 \in \mathbb{Q}$ if and any if

$$\{\beta_1, \beta_2\} = \left\{\frac{1+\sqrt{5}}{2}, \left(\frac{1+\sqrt{5}}{2}\right)^2\right\}.$$

Proof sketch

Recall that for all Borel set E, $\nu_\beta(E) = \int_E \widetilde{h}_\beta \, d\lambda$ and $\widetilde{h}_\beta(x) = h_\beta(x)/K_\beta$, where

$$h_\beta(x) = \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n} \quad \text{and} \quad K_\beta = \int_0^1 h_\beta(x) \mathrm{d}x, \; x \in [0,1).$$

For two non-integers $\beta_1, \beta_2 > 1$, then to prove that $\nu_{\beta_1} = \nu_{\beta_2}$ only need to show that $h_{\beta_1} = h_{\beta_2}$ for a.e $x \in [0,1]$.

Sufficiency

- Let $\beta_1 > 1$ be the root of $x^2 - qx - p = 0$, where $p, q \in \mathbb{N}$ with $p \leq q$. We have $T_{\beta_1}(1) = \beta_1 - q$ and $T^n_{\beta_1}(1) = 0$ for all $n \geq 2$. So,

$$h_{\beta_1}(x) = \mathbb{1}_{[0,1)}(x) + \frac{1}{\beta_1}\mathbb{1}_{[0,\beta_1 - q)}(x).$$

- Let $\beta_2 = \beta_1 + 1$, which implies that $T^n_{\beta_2}(1) = \beta_1 - q$ for all $n \geq 1$. Then

$$h_{\beta_2}(x) = \mathbb{1}_{[0,1)}(x) + \sum_{n=1}^{\infty} \frac{1}{\beta_2^n} \cdot \mathbb{1}_{[0,\beta_1 - q)}(x) = h_{\beta_1}(x).$$

Therefore, we conclude that $\nu_{\beta_1} = \nu_{\beta_2}$.

Necessity

Write

$$\mathcal{O}_\beta := \left\{ T_\beta^n(1) : n \geq 1 \right\}.$$

Suppose that $\beta_1, \beta_2 > 1$ are two different non-integers with $\nu_{\beta_1} = \nu_{\beta_2}$. So, we have $\widetilde{h}_{\beta_1}(x) = \widetilde{h}_{\beta_2}(x)$ for λ-a.e. $x \in [0, 1)$.

- STEP 1: $h_{\beta_1}(x) = h_{\beta_2}(x)$ for all $x \in [0, 1)$.
- STEP 2: Both $\mathcal{O}_{\beta_1}$ and $\mathcal{O}_{\beta_2}$ are finite, and $\mathcal{O}_{\beta_1} \setminus \{0\} = \mathcal{O}_{\beta_2} \setminus \{0\}$.
- STEP 3: 0 is in exactly one of the sets $\mathcal{O}_{\beta_1}$ and $\mathcal{O}_{\beta_2}$.
- STEP 4: We prove the necessity by analysing $h_{\beta_1} = h_{\beta_2}$.

Necessity

Proposition

Let $\beta > 1$ be a non-integer. Then we have

(i) $\lim_{x \to 1^-} h_\beta(x) = 1$;

(ii) $h_\beta(x)$ *is decreasing and right continuous on* $[0, 1)$;

(iii) $h_\beta(x)$ *is constant on an open interval* $(a, b) \subset (0, 1)$ *if and only if* $(a, b) \cap \mathcal{O}_\beta = \emptyset$.

Necessity

Lemma

Let $f, g : (0,1) \to \mathbb{R}$ be two functions satisfying $f(x) = g(x)$ for Lebesgue almost everywhere $x \in (0,1)$. If the left limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^-} g(x)$ exist for some $0 < x_0 \leq 1$, then we have

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^-} g(x).$$

Similarly, if the right limits $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^+} g(x)$ exist for some $0 \leq x_0 < 1$, then

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^+} g(x).$$

STEP 1

- Proposition (i) + Lemma $\Longrightarrow$

$$\frac{1}{K_{\beta_1}} = \lim_{x \to 1^-} \widetilde{h}_{\beta_1}(x) = \lim_{x \to 1^-} \widetilde{h}_{\beta_2}(x) = \frac{1}{K_{\beta_2}} \Longrightarrow K_{\beta_1} = K_{\beta_2}.$$

- $h_{\beta_1}(x) = h_{\beta_2}(x)$ for λ-a.e. $x \in [0, 1)$.

- Proposition (ii) (right continuity) + Lemma $\Longrightarrow$

$$h_{\beta_1}(x) = h_{\beta_2}(x) \text{ for all } x \in [0, 1).$$

STEP 2

Assume that $\mathcal{O}_{\beta_1}$ and $\mathcal{O}_{\beta_2}$ are both infinite.

- $T^n_{\beta_1}(1) > 0$ and $T^n_{\beta_2}(1) > 0$ for all $n \geq 1$.
- $$h_{\beta_i}(0) = \sum_{n=0}^{\infty} \frac{1}{\beta_i^n} = \frac{\beta_i}{\beta_i - 1} \quad \text{for } i = 1, 2.$$
- $h_{\beta_1}(0) = h_{\beta_2}(0) \Longrightarrow \beta_1 = \beta_2$, a contradiction.

WOLG, we assume that $\mathcal{O}_{\beta_1}$ is finite. Then by Proposition (iii), we can show that $\mathcal{O}_{\beta_2}$ is finite, and $\mathcal{O}_{\beta_1} \setminus \{0\} = \mathcal{O}_{\beta_2} \setminus \{0\}$.

STEP 3

Suppose that $0 \notin \mathcal{O}_{\beta_1} \cup \mathcal{O}_{\beta_2}$.

- $T^n_{\beta_1}(1) > 0$ and $T^n_{\beta_2}(1) > 0$ for all $n \geq 1$.
- $$h_{\beta_i}(0) = \sum_{n=0}^{\infty} \frac{1}{\beta_i^n} = \frac{\beta_i}{\beta_i - 1} \quad \text{for } i = 1, 2.$$
- $h_{\beta_1}(0) = h_{\beta_2}(0) \Longrightarrow \beta_1 = \beta_2$, a contradiction.

STEP 3

Suppose that $0 \in \mathcal{O}_{\beta_1} \cap \mathcal{O}_{\beta_2}$.

- Let $n_i := \min\{n \geq 1 : T_{\beta_i}^{n+1}(1) = 0\}$ for $i = 1, 2$.
- $$\mathcal{O}_{\beta_i} \setminus \{0\} = \left\{ T_{\beta_i}(1), T_{\beta_i}^2(1), \ldots, T_{\beta_i}^{n_i}(1) \right\} \quad \text{and} \quad h_{\beta_i}(0) = \sum_{n=0}^{n_i} \frac{1}{\beta_i^n}.$$
- $\mathcal{O}_{\beta_1} \setminus \{0\} = \mathcal{O}_{\beta_2} \setminus \{0\} \implies n_1 = n_2$.
- $h_{\beta_1}(0) = h_{\beta_2}(0) \implies \beta_1 = \beta_2$, a contradiction.

Therefore, we conclude that 0 is in exactly one of the sets $\mathcal{O}_{\beta_1}$ and $\mathcal{O}_{\beta_2}$.

STEP 4

WLOG, assume that $0 \in \mathcal{O}_{\beta_1}$ and $0 \notin \mathcal{O}_{\beta_2}$.

- Let $m = \min\{n \geq 1 : T_{\beta_1}^{n+1}(1) = 0\}$. Then $m \geq 1$ and $T_{\beta_1}^n(1) = 0$ for all $n > m$.
- Write $x_n = T_{\beta_1}^n(1)$ for $1 \leq n \leq m$.
- $\mathcal{O}_{\beta_1} \setminus \{0\} = \{x_1, x_2, \ldots, x_m\}$ and

$$h_{\beta_1}(x) = \mathbb{1}_{[0,1)}(x) + \sum_{k=1}^{m} \frac{1}{\beta_1^k} \cdot \mathbb{1}_{[0,x_k)}(x).$$

STEP 4

- Note that $0 \notin \mathcal{O}_{\beta_2}$ and $\mathcal{O}_{\beta_2} = \mathcal{O}_{\beta_1} \setminus \{0\} = \{x_1, x_2, \ldots, x_m\}$.
- $\left\{T^n_{\beta_2}(1)\right\}_{n=1}^{\infty} = \{x_1, x_2, \ldots, x_m\}$ implies that $T^{m+1}_{\beta_2}(1) = T^{\ell}_{\beta_2}(1)$ for some $1 \le \ell \le m$.
- Write $y_n = T^n_{\beta_2}(1)$ for $1 \le n \le m$. Then $\left\{T^n_{\beta_2}(1)\right\}_{n=1}^{\infty}$ is

$$y_1, \ldots, y_\ell, \ldots, y_m, y_\ell, \ldots, y_m, y_\ell, \ldots, y_m, \cdots$$

and

$$h_{\beta_2}(x) = \mathbb{1}_{[0,1)}(x) + \sum_{k=1}^{\ell-1} \frac{1}{\beta_2^k} \cdot \mathbb{1}_{[0,y_k)}(x) + \frac{\beta_2^{m+1-\ell}}{\beta_2^{m+1-\ell} - 1} \sum_{k=\ell}^{m} \frac{1}{\beta_2^k} \cdot \mathbb{1}_{[0,y_k)}(x).$$

STEP 4

- For $m = 1$, $h_{\beta_1}(x) = h_{\beta_2}(x)$ if $\beta_2 = \beta_1 + 1$ and $\beta_1^2 - q\beta_1 - p = 0$ with $p < p$ and $p, q \in \mathbb{N}$.
- For $m \geq 2$, $h_{\beta_1}(x) \neq h_{\beta_2}(x)$ for all different non-integer pairs (β_1, β_2).

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Thank you for your attention!