Arithmetic averages and normality in continued fractions

Godofredo Iommi

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Joint work with Thomas Jordan and Anibal Velozo

An irrational number $x \in (0,1)$ can be written in a unique way as a continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1 a_2 a_3 \dots],$$

where $a_i \in \mathbb{N}$.

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where $a_i \in \mathbb{N}$. The Gauss map, $T : (0,1] \to [0,1]$, is defined by

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Note that $T([a_1 a_2 \dots]) = [a_2 a_3 \dots]$.

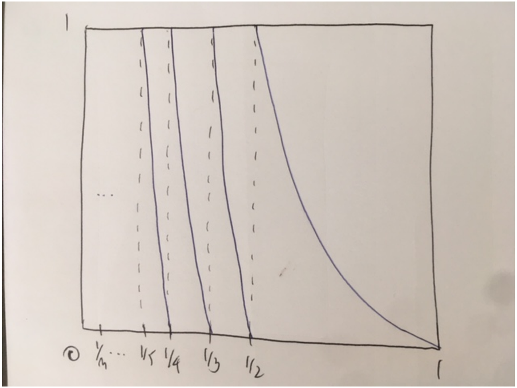

Figure: Gauss map

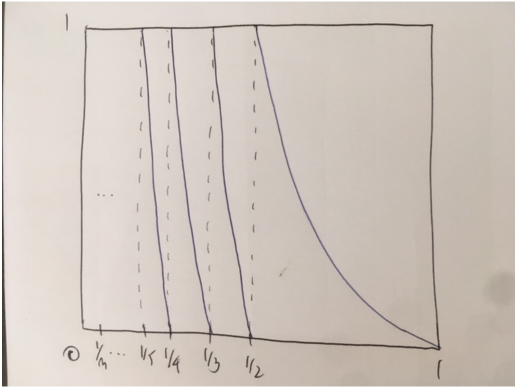

Figure: Gauss map

The Gauss map has infinite entropy and preserves the measure

$$\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx.$$

Arithmetic Mean

It was essentially observed by Khinchine in 1935 that for Lebesgue almost every $x = [a_1, a_2, \dots]$ the arithmetic average of the digits is infinity:

$$\lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = \infty.$$

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For $\alpha \geq 1$ let

$$A(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \alpha \right\}$$

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Theorem (I-Jordan)

The function $\alpha \mapsto \dim_H A(\alpha)$ *in* $(1,\infty)$ *is real analytic, strictly increasing and* $\lim_{\alpha\to\infty} \dim_H A(\alpha) = 1$.

Power Mean

Let $\gamma \in (0, 1]$ and

$$A_\gamma(\alpha) = \left\{ x \in (0,1] : \lim_{n \to \infty} \left(\frac{a_1^\gamma + \cdots + a_n^\gamma}{n} \right)^{1/\gamma} = \alpha \right\}$$

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Theorem (I-Jordan)

If $\gamma \in (0, 1)$ then there exists $K(\gamma) > 1$ such that the function $\alpha \mapsto \dim_H A_\gamma(\alpha)$ is real analytic, strictly increasing in $(1, K(\gamma))$ and $\dim_H A_\gamma(\alpha) = 1$ for $\alpha \geq K(\gamma)$.

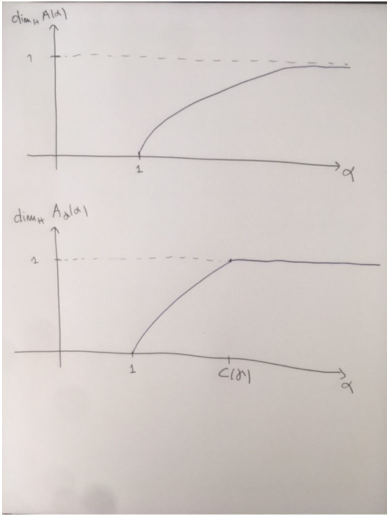

Figure: Arithmetic and Weighted Arithmetic means

Power Mean

If $\varphi_\gamma([a_1, a_2, \ldots]) = a_1^\gamma$ then

$$\lim_{n\to\infty} \frac{a_1^\gamma + \cdots + a_n^\gamma}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi_\gamma(T^i x).$$

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Remark

The shape of $\alpha \mapsto \dim_H(A_\gamma(\alpha))$ *depends on the behaviour of* φ_γ *near zero. More precisely, it depends on whether* $\int \varphi_\gamma d\mu_G = \infty$ *or* $\int \varphi_\gamma d\mu_G < \infty$.

Khinchine observed in 1935 that for Lebesgue almost every $x = [a_1, a_2, \dots]$ the geometric average of the digits is:

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = \prod_{a=1}^{\infty} \left(\frac{(a+1)^2}{a(a+2)} \right)^{\log a / \log 2} := K$$

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For $\alpha \geq 1$ let

$$G(\alpha) = \left\{ x \in (0,1] : \lim_{n \to \infty} \sqrt[n]{a_1 a_2 \cdots a_n} = \alpha \right\}$$

Geometric Mean

Theorem (Fan, Liao, Wang, Wu, 2009)

The function $\alpha \mapsto \dim_H G(\alpha)$ *in* $(1, \infty)$ *is real analytic, it has a unique maximum at* K *and* $\lim_{\alpha \to \infty} \dim_H G(\alpha) = 1/2$.

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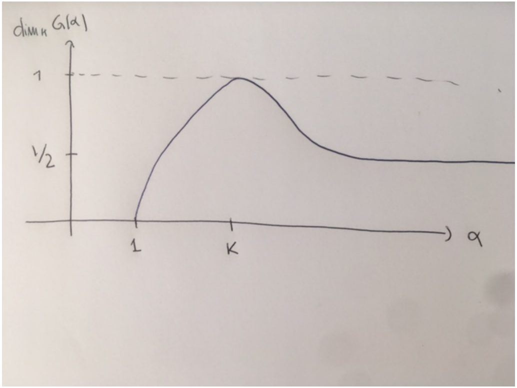

Figure: Geometric mean

Geometric Mean

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Remark

The above result is related to the Hausdorff dimension at infinity,

$$\sup\left\{\limsup_{n\to\infty} \dim_H \mu_n : \mu_n \in \mathcal{M}_T \text{ and } \mu_n \to \delta_0\right\} = \frac{1}{2}.$$

It is a measure theoretic version of Good's theorem (1941).

Speed of approximation

For $x = [a_1 a_2 \ldots]$ let $p_n/q_n = [a_1 a_2 \ldots a_n]$. Let

$$S(\alpha) = \left\{ x \in (0,1) : -\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = \alpha \right\}.$$

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Theorem (Pollicott-Weiss, Kesseböhmer-Stratmann)

The function $\alpha \mapsto \dim_H S(\alpha)$ in $(2\log(1+\sqrt{5})/2, \infty)$ is real analytic. It attains a maximum at $\alpha = \pi^2/(6\log 2)$.

Speed of approximation

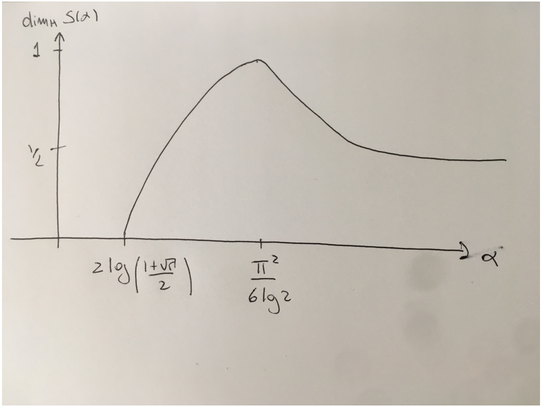

Figure: Speed of approximation by rationals

Sketch of proof Power Mean result

Proof.

Recall that $\varphi_\gamma([a_1, a_2, \dots]) = a_1^\gamma$. To simplify, we remove the power $1/\gamma$.

$$A_\gamma(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{a_1^\gamma + \cdots + a_n^\gamma}{n} = \alpha \right\}$$

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Let μ be an ergodic G-invariant probability measure and

$$\lambda(\mu) = \int \log |T'| d\mu.$$

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Use thermodynamic formalism to find a measure attaining the supremum. □

Proof.

Recall that the pressure of a function ψ is defined by

$$P(\psi) = \sup \left\{ h(\mu) + \int \psi \, d\mu : \mu \in \mathcal{M}_T \text{ and } \int \psi \, d\mu > -\infty \right\}.$$

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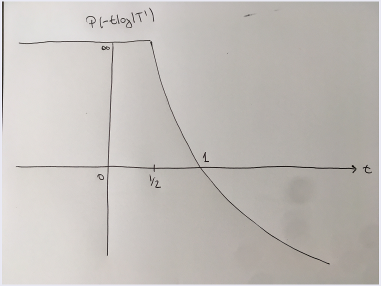

Figure: Map $t \mapsto P(-t \log |T'|)$

Proof.

Study of the function on $\mathbb{R}^2$ defined by

$$(q, \delta) \mapsto P(q(\varphi_\gamma - \alpha) - \delta \log |T'|).$$

Sketch of proof Power Mean result

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Note that, if μ_q is an equilibrium measure for the function $q(\varphi_\gamma - \alpha) - \delta \log |T'|$ then,

$$\frac{\partial}{\partial q} P(q(\varphi_\gamma - \alpha) - \delta \log |T'|) = \int \varphi_\gamma \, d\mu_q - \alpha.$$

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Remark

If $\frac{\partial}{\partial q} P(q(\varphi_\gamma - \alpha) - \delta \log |T'|) = 0$ *then*

$$\mu_q(A_\gamma(\alpha)) = 1.$$

$\square$

Sketch of proof Power Mean result

Proof.

If $\delta = \dim_H A_\gamma(\alpha)$ then

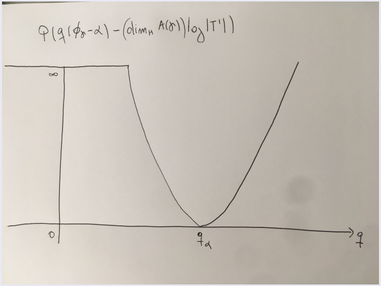

Figure: Map $q \mapsto P(q(\varphi_\gamma - \alpha) - \dim_H A_\gamma(\alpha) \log |T'|)$

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Thus, at the point $(q_\alpha, \dim_H A_\gamma(\alpha))$ the pressure $P(q, \delta)$ is zero and its derivative w/r to q is also zero.

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Sketch of proof Power Mean result

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Thus, at the point $(q_\alpha, \dim_H A_\gamma(\alpha))$ the pressure $P(q,\delta)$ is zero and its derivative w/r to q is also zero. Therefore,

1. $\mu_{q_\alpha}(A_\gamma(\alpha)) = 1$
2. $0 = P(q_\alpha, \dim_H A_\gamma(\alpha)) = h(\mu_{q_\alpha}) - \dim_H A_\gamma(\alpha)\lambda(\mu_{q_\alpha})$ that is

$$\dim_H A_\gamma(\alpha) = \frac{h(\mu_{q_\alpha})}{\lambda(\mu_{q_\alpha})}.$$

□

Normal numbers

Definition

A number $x \in (0,1]$ is *continued fraction normal* if the frequency of appearance of every string of digits $(d_1, \ldots, d_m) \in \mathbb{N}^m$ in the expansion $x = [a_1, a_2, \ldots]$ is equal to $\mu_G([d_1, \ldots d_m])$, where

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Remark

Lebesgue almost every point is continued fraction normal.

Normal numbers

Theorem (Moshchevitin and Shkredov 2003, Airy and Mance 2019)

The number $x \in (0,1]$ is continued normal if and only if the sequence $(\nu_n)_n$, with $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}$, is tight in $(0,1]$ and there exists a constant $B > 0$ such that for every interval $[a,c] \subset (0,1]$,

$$\limsup_{n\to\infty} \frac{1}{n} \# \left\{ i \in \{1, \dots, n\} : T^i(x) \in [a,c] \right\} \leq B\mu_G([a,c]).$$

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Example

If $x = [1, 2, 3, \dots, n, n+1, \dots]$ then (ν_n) has no convergent sub-sequence in $(0,1]$ and the frequency of appearance of every string is equal to zero.

Normal numbers

Definition

Let $\alpha \in (0,1)$. A real number $x \in (0,1]$ is $\alpha-$*continued fraction normal* if for every string of digits $(d_1, \dots, d_m) \in \mathbb{N}^m$ we have

$$\lim_{n\to\infty} \frac{1}{n} \# \{ i \in \{1, \dots, n\} : (a_i(x), \dots, a_{i+m}(x)) = (d_1, \dots, d_m) \} \\ = \alpha \mu_G([d_1, \dots d_m])$$

We denote this set by $G(\alpha \mu_G)$

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Construction of such numbers: There exists a sequence of ergodic measures supported on periodic orbits $(\nu_n)_n$ that converges to the measure $\alpha\mu_G + (1-\alpha)\delta_0$.

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Theorem (I-Velozo)

If $\alpha \in (0,1)$ *then* $\dim_H G(\alpha\mu_G) = \frac{1}{2}$.

Normal numbers

Theorem (Good 1941)

$$\dim_H \left(\left\{ x = [a_1, a_2, \dots] \in (0,1] : \lim_{n \to \infty} a_n = \infty \right\} \right) = \frac{1}{2}$$

Normal numbers

Theorem (Good 1941)

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Theorem (I-Jordan 2015)

$$\dim_H \left(\left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(T^i x)| = \infty \right\} \right) = \frac{1}{2}$$

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Theorem (Fan, Jordan, Liao, Rams 2015)

If $(\mu_n)_n \subset \mathcal{M}_G$ *is such that* $\lim_{n\to\infty} \int \log |T'| \, d\mu_n = \infty$ *then*

$$\limsup_{n\to\infty} (\dim_H \mu_n) \leq \frac{1}{2}.$$

Moreover, there exists (ν_n) *s.t.* $\lim_{n\to\infty} \dim_H \nu_n = 1/2$.

Remark

The pressure of the geometric potential satisfies:

$$P(-t \log |T'|) = \begin{cases} \infty & \text{if } t \leq 1/2; \\ \text{finite} & \text{if } t > 1/2. \end{cases}$$

Normal numbers

Remark

The above results are related to the entropy at infinity. If Φ is the suspension flow over T with roof function $\log |T'|$ then Abramov's formula yields,

$$\dim_H \mu = h_\Phi(\mu \times Leb) = \frac{h(\mu)}{\int \log |T'| \, d\mu}.$$