Descriptive Complexity in Numeration Systems

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We present some applications of descriptive set theory to numeration systems and dynamics.

Part of the work is joint with D. Airey, D. Kwietniak, and B. Mance. Part is joint with B. Mance and J. Vandehey. We also mention some work with P. Allaart, R. Jones, and D. Lambert

The results we present connect descriptive set theory with numeration systems and dynamics.

In particular we will be concerned with continued fractions, β-expansions, and GLS-expansion.

Descriptive set theory provides a way to calibrate the complexity of sets in Polish spaces. Two motivations for doing this:

- By computing the exact complexity of a set we show there are no further theorems that would result in yet simpler characterizations.
- By computing the exact complexity of the difference of two sets, we establish a notion of the logical independence of the two sets.

Recall that in any uncountable Polish space we have the Borel hierarchy of sets:

$$
\begin{array}{ccccccccccccc}
 & \boldsymbol{\Sigma}^0_1 & & \boldsymbol{\Sigma}^0_2 & & \boldsymbol{\Sigma}^0_3 & & & & \boldsymbol{\Sigma}^0_\alpha & & \\
\rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} \; \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} \; \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\boldsymbol{\Delta}^0_1 & & \boldsymbol{\Delta}^0_2 & & \boldsymbol{\Delta}^0_3 & & \boldsymbol{\Delta}^0_4 \cdots & \boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & \cdots \\
\rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} \; \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} \; \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & \boldsymbol{\Pi}^0_1 & & \boldsymbol{\Pi}^0_2 & & \boldsymbol{\Pi}^0_3 & & & & \boldsymbol{\Pi}^0_\alpha & &
\end{array}
$$

All of these collections are pointclasses.

In any uncountable Polish space there is no collapsing in any of these levels.

$$\begin{array}{ccccc} & & \boldsymbol{\Sigma}^1_1 & & \\ & \subseteq & & \subseteq & \\ \text{Borel}=\boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 \;\cdots \\ & \subseteq & & \subseteq & \\ & & \boldsymbol{\Pi}^1_1 & & \end{array}$$

We have the following classical result of Ki-Linton concerning $\mathcal{N}(b)=$the set of base b normal numbers.

Recall x is b-normal if for every $B = (i_0, \ldots, i_{\ell-1}) \in b^{<\omega}$ we have that $\lim_{N\to\infty} \frac{1}{N}|I(x,B,N)| = \frac{1}{b^\ell}$, where $I(x,B,N) = \{i < N \colon (c_i, c_{i+1}, \ldots, c_{i+\ell-1}) = B\}$.

Theorem

(Ki, Linton) $\mathcal{N}(b)$ is a $\mathbf{\Pi}^0_3$-complete set.

Let

$$\mathcal{N}^{\perp}(b) = \{y \colon \forall x \in \mathcal{N}(b)\ (x+y) \in \mathcal{N}(b)\}$$

be the set of reals x which preserve normality under addition.

- Wall showed that adding or multiplying by a non-zero rational preserves normality in all bases.
- From the definition we have that $\mathcal{N}^{\perp}(b)$ is a $\mathbf{\Pi}^1_1$ set.

However, a theorem of Rauzy reduces the computation of the complexity.

Rauzy introduces a quantity of a base-b expansion called the noise, which is an entropy-like quantity (noise 0 is equivalent to entropy 0).

Theorem (Rauzy)

$y \in \mathcal{N}^{\perp}(b)$ iff y has noise 0.

The set of $y \in b^{\omega}$ of noise 0 is a $\mathbf{\Pi}^0_3$ set, so $\mathcal{N}^{\perp}(d) \in \mathbf{\Pi}^0_3$.

Theorem (Airey, J, Mance)

The set of $y \in b^{\omega}$ of noise 0 is a $\mathbf{\Pi}^0_3$ complete set, so $\mathcal{N}^{\perp}(d)$ is $\mathbf{\Pi}^0_3$-complete.

This says there are no further theorems which would would result in a simpler criterion for checking membership in $\mathcal{N}^{\perp}(b)$.

Some other complexity results along these lines:

Theorem (Becher, Heiber, Slaman)

The set of absolutely normal numbers is $\mathbf{\Pi}^0_3$-complete.

(This was conjectured by Kechris)

Theorem (Becher, Slaman)

The set of numbers normal in at least one base if $\mathbf{\Sigma}^0_4$-complete.

Theorem (Beros)

For $s > r$, the set $\mathcal{N}_r(b) \setminus \mathcal{N}_s(b)$ is $D_2(\mathbf{\Pi}^0_3)$-complete.

For any pointclass $\mathbf{\Gamma}$, the pointclass $D_2(\mathbf{\Gamma})$ is defined by:

$$D_2(\mathbf{\Gamma}) = \{A \setminus B \colon A, B \in \mathbf{\Gamma}\}.$$

There is also a more general definition of $D_\alpha(\mathbf{\Gamma})$.

So, one can define $D_\alpha(\mathbf{\Pi}^0_\beta)$ for any $\alpha, \beta < \omega_1$. There is no collapse in this hierarchy as well.

Connection with Dynamics

Let (X, T, μ) be a dynamical system: X a Polish space, $T\colon X \to X$ continuous, and μ a T-invariant Borel probability measure on X.

Given a finite or countably infinite partition $X = \bigcup_{k \in \mathcal{D}} X_n$, every $x \in X$ creates an itinerary $i(x) \in \mathcal{D}^\omega$.

$i(x)(n)$ is the $k \in \mathcal{D}$ such that $T^n(x) \in X_k$.

This can be viewed as a general numeration system for the $x \in X$.

Base b expansions: $T\colon [0,1] \to [0,1]$ where $T(x) = bx \mod 1$.
Partition $[0,1]$ into b intervals $[\frac{i}{b}, \frac{i+1}{b})$ for $i \in \{0, \dots, b-1\} = \mathcal{D}$.
$\mu =$Lebesgue measure.

Continued fractions: $T\colon [0,1] \to [0,1]$ where $T(x) = \frac{1}{x} \mod 1$.
$X_i = [\frac{1}{i+1}, \frac{1}{i})$, for $i \in \mathcal{D} = \{1, 2, 3, \dots\}$. Equivalently
$i(x)(k) = \lfloor T^{k+1}(x) \rfloor$. $\mu =$Gauss measure.

β-expansions: $T\colon [0,1] \to [0,1]$ where $T(x) = \beta x \mod 1$.
$\mathcal{D} = \{0, 1, \dots, \lfloor \beta \rfloor\}$. $\mu =$Parry measure.

Generalized GLS expansions: T is piecewise linear on countable partition on $(0,1)$. $\mathcal{D}$ indexed by partition. Includes tent map.
μ=Lebesgue measure.

The notion of normal number is an instance of a more general defiition.

Definition

x is a generic point for (X, T, μ) if the measures $\frac{1}{n}(\delta_x + \delta_{T(x)} + \cdots + \delta_{T^{n-1}(x)})$ converge weakly to μ.

When X is a subshift of a shift space and T the shift map, this is saying that the limiting frequency $\lim_{n\to\infty} \frac{1}{n} N_u(x, n)$ of a word u in $x \restriction n$ is equal to $\mu(C_u)$, where C_u is the cylinder determined by u.

The set of generic points G_μ is a $\mathbf{\Pi}^0_3$ set.

To obtain the complexity results for these numeration systems, we establis a general result for generic points in dynamical systems satisfying a certain property.

We first recall the specification property, which was introduced by R. Bowen

Let $T^{[a,b]}(x) = \langle T^a(x), \dots, T^b(x) \rangle$ denote an orbit segment. Let $\langle x, n \rangle$ abbreviate the orbit segment $T^{[0,n-1]}(x) = \langle x, T(x), \dots, T^{n-1}(x) \rangle$.

A specification of rank k is a k-sequence of orbit segments $\xi = \langle \langle x_1, n_1 \rangle, \dots \langle x_k, n_k \rangle \rangle$.

We say a point y ϵ-traces the specification $\xi = \langle\langle x_1, n_1\rangle, \dots \langle x_k, n_k\rangle\rangle$ provided there are integers ("gaps") $s_1, \dots, s_k$ such that for all $j < k$:

$$\rho(T^{\sum_{i=1}^{j-1}(n_i+s_i)+t}(y), T^t(x_j)) < \epsilon$$

for all $0 \leq t < n_j$.

Definition

(X, T) has the specification property if for every $\epsilon > 0$ there is an N_ϵ such that for every k and every specification ξ there is a point y which ϵ-traces ξ with gap sized $\leq N_\epsilon$.

For subshifts X of a shift space $\mathcal{D}^\omega$ on a finite or countable alphabet $\mathcal{D}$, the specification property is equivalent to:

- There is an integer N such that if $w_1, \dots, w_n$ are admissible words, then there are $v_1, \dots, v_{n-1} \in \mathcal{D}^N$ such that $w_1 v_1 \dots w_{n-1} v_{n-1} w_n$ is admissible.

However, some numeration systems of interest such as the β-shift don't satisfy the full specification property.

For the symbolic case (subshifts of $\mathcal{D}^\omega$) we require (where $\mathcal{L}(X)$ is the set of admissible words, and d_H is the Hamming distance):

Definition

We say that a subshift X has the right feeble specification property if there exists a set $\mathcal{G} \subseteq \mathcal{L}(X)$ satisfying:

1. a concatenation of words in $\mathcal{G}$ stays in $\mathcal{G}$, that is, if $u, v \in \mathcal{G}$, then $uv \in \mathcal{G}$;
2. for any $\epsilon > 0$ there is an $N = N(\epsilon)$ such that for every $u \in \mathcal{G}$ and $v \in \mathcal{L}(X)$ with $|v| \geq N$, there are $s, v' \in \mathcal{D}^{<\omega}$ satisfying $|v'| = |v|$, $0 \leq |s| \leq \epsilon|v|$, $d_H(v, v') < \varepsilon$, and $usv' \in \mathcal{G}$.

Theorem (Airey, J, Kwietniak, Mance)

Assume that $\mathcal{D}$ is at most countable and X is a subshift over $\mathcal{D}$ with the right feeble specification property and at least two shift-invariant Borel probability measures. If μ is a shift-invariant Borel probability measure on X, then G_μ is $\mathbf{\Pi}^0_3$-complete.

Remark

The assumption of at least two invariant measures is automatically satisfied when X is compact and $|X| \geq 2$.

All of the above numeration systems satisfy this feeble right specifcation.

Corollary

The set of generic points (normal numbers) with respect to continued fractions, β-shifts, generalized GLS expansions, are all $\mathbf{\Pi}^0_3$-complete.

There is also a version of this theorem for dynamical systems using a weakening of the specification property which we call the strong approximate product structure.

- Pfister and Sullivan introduced the notion of an approximate product structure.

We now discuss the logical independence of the notions of normality for numerations systems.

The logical independence of two notions, which define two sets A and B, can be measured by the descriptive complexity of the set $A \setminus B$.

If A, B are $\mathbf{\Gamma}$-complete sets and $A \setminus B$ is $D_2(\mathbf{\Gamma})$-complete, then there is no logical reduction in the complexity of $A \setminus B$.

That is, $x \in A$ does not simplify $x \in B$, so being in A does not imply any information about being in B.

Theorem (J, Mance, Vandehey)

The set of $x \in (0,1)$ which are continued fraction normal but not base b normal is $D_2(\mathbf{\Pi}^0_3)$-complete.

Theorem (J, Mance, Vandehey)

The set of $x \in (0,1)$ which are continued fraction normal but not base b normal for all $b \geq 2$ is $D_2(\mathbf{\Pi}^0_3)$-hard.

Previously Vandehey had shown that $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_b$ is uncountable using the GRH.

Application: There is no G_δ set $M \supseteq \mathcal{N}_{\mathrm{CF}} \cap \mathcal{N}_b$ such that $M \cap \mathcal{N}_{\mathrm{CF}} \subseteq M \cap \mathcal{N}_b$.

If F is any F_σ set, then either F contain a point of $\mathcal{N}_{\mathrm{CF}} \cap \mathcal{N}_b$ or F^c contains a point of $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_b$.

In the above, G_δ could be replaced by $\boldsymbol{\Sigma}^0_3$.

Proof.

If $M \supseteq \mathcal{N}_{\mathrm{CF}} \cap \mathcal{N}_b$ were $\boldsymbol{\Sigma}^0_3$ and $M \cap \mathcal{N}_{\mathrm{CF}} \subseteq \mathcal{N}_b$, then $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_b = \mathcal{N}_{\mathrm{CF}} \setminus M \in \boldsymbol{\Pi}^0_3$, a contradiction. □

We also the following result concerning normality in different bases.

Theorem

If $b, c \geq 2$ are relatively prime then $\mathcal{N}_b \setminus \mathcal{N}_c$ is $D_2(\mathbf{\Pi}^0_3)$-complete.

The same application above for $\mathcal{N}_b$, $\mathcal{N}_c$ normality when $(b, c) = 1$.

The following is open.

Question

Is $\mathcal{N}_2 \setminus \mathcal{N}_{\text{CF}}$ a $D_2(\mathbf{\Pi}^0_3)$-complete set?

Comparing the sets of normal numbers for base b, base c, and continued fractions can be viewed a form of comparing the itineraries of different dynamical systems.

One could ask to what extent the itineraries of a point $x \in X$ under two different systems on X could be similar.

Definition

An $x \in (0, 1)$ is a Trott number base b if x has an infinite CF expansion $x = [a_1, a_2, \dots]$ and a base b expansion $x = 0.\hat{a}_1 \hat{a}_2 \cdots$, where $\hat{a}_i$ is the base b expansion of a_i.

We can characterize for which bases Trott numbers exist.

Theorem (Allaart, J , Jones, Lambert)

There is a Trott number in base b iff $b \in \{3\} \cup \bigcup_{k=1}^{\infty} [k^2+1, k^2+k]$. *For b in this set, the set of Trott numbers is a* G_δ *complete set.*

So, we have Trott numbers in bases $b = 2, 3, 5, 6, 10, 11, 12, 17, 18, 19, 20, \dots$.

Theorem (Allaart, J , Jones, Lambert)

The set of Trott numbers has Hausdorff dimension < 1*.*

Question

Does T_b have positive Hausdorff dimension?

Sketch of Proof

We outine the proof that $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_2 \in D_2(\mathbf{\Pi}^0_3)$-complete.

We use the method of Wadge reduction. Let
$C = \{z \in \omega^\omega \colon z(2n+1) \to \infty\}$, $D = \{z \in \omega^\omega \colon z(2n) \to \infty\}$.

We easily have that C, D are $\mathbf{\Pi}^0_3$-complete and $C \setminus D$ is $D_2(\mathbf{\Pi}^0_3)$-complete.

We reduce $C \setminus D$ to $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_2$. That is, we find a continuous function $\varphi \colon \omega^\omega \to \mathbb{R}$ such that $C \setminus D = \varphi^{-1}(\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_2)$.

This suffices to show that $\mathcal{N}_{\mathrm{CF}} \setminus \mathcal{N}_2$ is $\mathbf{\Pi}^0_3$ hard, as $D_2(\mathbf{\Pi}^0_3)$ is a pointclass.

Given a $z \in \omega^\omega$, we construct a series of CF blocks B_i, where B_i depend only on $z(2i)$, $z(2i+1)$. We take $\varphi(z)$ to have CF expansion $B_1 B_2 \cdots$.

Let $\bar{B}_i$ denote the concatenation of the blocks $B_1, \dots, B_{i-1}$.

We wish to show that in any interval, most 2-adic rationals $\frac{a}{2^d}$ for large enough d will have good CF behavior. To do this, we need to relate fractions with a fixed denominator with two fractions of a variable denominator.

This uses an idea of Avdeeva and Bykovski.

Given $\bar{B}_i$, and $\frac{a}{d}$ with CF expansion $\bar{B}_i B$ then $\bar{B}_i B$ can, up to at most 9 digits in the middle, be written as the concatenation of blocks B' and $(B'')^*$ have denominators at most $\sqrt{d}$.

Here $(B'')^*$ is the reversal of B''.

We take

$$\frac{q(\bar{B}B)}{2} \leq m < q(\bar{B}B),$$

where the CF block B is the rational $r(B) = \frac{p(B)}{q(B)}$.

Thus, the dyadic rational $\frac{a}{d}$, which has CF expansion $\bar{B}_i B$, gives two Farey fractions in $\mathcal{F}_m$.

There is a tight relationship between the number of elements is a set $U \subseteq \mathcal{F}_m$ and the measure of the union of the corresponding intervals $I_{\frac{p}{q}}$:

$$\lambda\left(\bigcup_{\frac{p}{q}\in U} I_{\frac{p}{q}}\right) \geq \frac{\#U}{m^2}$$

$$\lambda\left(\bigcup_{\frac{p}{q}\in U} I_{\frac{p}{q}}\right) \leq \frac{1}{m}\sqrt{\#U} \qquad \text{[Avdeeva, Bykovsky]}$$

We need results which control the lengths, the behavior of the denominators, and the normality properties of most CF expansions.

1.) (controlling lengths) As a corollary to a result of Hensley we have:

Let N, $\epsilon > 0$ be given, and let $m = e^{(\lambda_{\mathrm{KL}}+\epsilon)N}$. Then the proportion of elements in $\mathcal{F}_m$ having fewer than N digits in their CF expansion is at most $O(\frac{1}{\sqrt{N}})$.

2.) (controlling denominators) From a result of Vandehey we have: Let N, ϵ, m be as above. Let $\mathcal{G}_m \subseteq \mathcal{F}_m$ be those $\frac{p}{q} \in \mathcal{F}_m$ with corresponding block B of CF digits such that for some n which is a multiple of $\sqrt{N}$ we have

$$\left| \frac{q_n(B)}{n} - \lambda_{\mathrm{KL}} \right| > \epsilon.$$

Then the proportion of elements in $\mathcal{F}_m$ which are in $\mathcal{G}_m$ is $O(\frac{\log(N)}{\sqrt{N}})$.

A slight refinement of a result of Adrian-Maria Scheerer gives:

3.) (controlling CF normality) Let $\mathcal{A}$ be a finite collection of blocks, and $\epsilon > 0$. Then there are $\eta, \xi > 0$ such that for large enough n

$$\mu\left(\bigcup_{k=1}^{\infty} E^c_{\mathrm{CF}}(\epsilon, \mathcal{A}, kn)\right) \leq \xi e^{-\eta \frac{n}{\log(n)}},$$

where $E_{\mathrm{CF}}(\epsilon, \mathcal{A}, n)$ is the set of $x \in (0,1)$ whose first n digits are (ϵ, A) normal.

We get a similar but slightly worse estimate for the reverse of the blocks:

$$\mu\left(\bigcup_{k=1}^{\infty} E^c_{\mathrm{CF}}(\epsilon, \mathcal{A}, kn, K)\right) \leq K\xi e^{-\eta \frac{n}{\log(n)}},$$

where $E_{\mathrm{CF}}(\epsilon, \mathcal{A}, n, K)$ is the union of the CF cylinders C_B with $|B| \leq K$ and B^* is not (ϵ, A, n) normal.

Using (1), (2) and (3), we can choose a block B_i with the above *CF* properties and also base 2 normal:

Choose first N, then let $m = e^{(\lambda_{\mathrm{KL}}+\epsilon)N}$, and let $\frac{d}{2} \leq m^2 < d$. Let $\frac{a}{d}$ be such that $\frac{a}{d} = r_{\bar{B}B}$.

Note that we can't use all of B as this wouldn't give base 2 normality. We take $\log_d(d)\left(1 + \frac{1}{z(2i)+1}\right)$ many base 2 digits are determined by $\bar{B}B$.

We add a string of 1's in the CF expansion after $\bar{B}B$ of length $\frac{1}{z(2i+1)}|\bar{B}B|$.

If $z(2i+1)$ does not go to ∞, then $\varphi(z)$ is not CF normal.

If $z(2i+1) \to \infty$ then $\varphi(z) \in \mathcal{N}_{\mathrm{CF}}$. Then $z(2i) \to \infty$ iff $\varphi(z) \in \mathcal{N}_2$.