Representations of real numbers on fractal sets

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One World Numeration Seminar

2020.10.13

Integer case:

- (Euclid) Any integer greater than 1 is either a prime number, or can be written as a unique product of prime numbers (ignoring the order);
- (Lagrange) Every positive integer can be represented by the sum of four squares. Generally, we have the Waring problem;
- (Goldbach's conjecture)Every even integer n greater than two is the sum of two primes;
- $\cdots$.

Representations of real numbers

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Representations of real numbers

Continuous case:

- β-expansions;
- continued fractions;
- Lüroth expansions;
- f-expansions;
- $\cdots$.

Given two non-empty sets $A, B \subset \mathbb{R}$. Define

$$A * B = \{x * y : x \in A, y \in B, \}, * = +, -, \cdot, \div.$$

Suppose that f is a continuous function defined on an open set $U \subset \mathbb{R}^2$. Denote the continuous image of f by

$$f_U(A, B) = \{f(x, y) : (x, y) \in (A \times B) \cap U\}.$$

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Some results

(1) (Steinhaus, 1917) $C + C = [0,2]$, $C - C = [-1,1]$, where C is the middle-third Cantor set;

(2) (Athreya, Reznick and Tyson, 2017) $17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9$;

(3)
$$C \div C = \bigcup_{-\infty}^{+\infty} \left[\frac{2}{3}3^m, \frac{3}{2}3^m\right] \cup \{0\}.$$

It is natural to ask what is the topological sturcture of $C \cdot C$.

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Our results

Theorem (Jiang and Xi)

Let C be the middle-third Cantor set. If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that one of the following conditions is satisfied,

$$1 < \left| \frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}} \right| < 3, \text{ or } 1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < 3,$$

then $f_U(C, C)$ has an interior.

Some examples

Example

$$f(x,y)=x^\alpha y^\beta(\alpha\beta\neq 0),\ x^\alpha\pm y^\alpha(\alpha\neq 0),\sin(x)\cos(y),x\sin(xy).$$

then $f_U(C,C)$ contains an interior.

For the function $f(x,y)=xy$, we let $(x_0,y_0)=(8/9,2/3)$ and have

$$1<\left|\frac{\partial_y f|_{(x_0,y_0)}}{\partial_x f|_{(x_0,y_0)}}\right|=\frac{4}{3}<3,$$

Therefore, $C\cdot C$ contains interior.

Main result

Theorem (Jiang and Xi)

$$C \cdot C = \cup_{n=1}^{\infty} A_n \cup B,$$

where A_n is a closed interval, and B has Lebesgue measure zero.

Remark

For some $n \neq m$, we may have $A_n \cap A_m \neq \emptyset$.

Lebesgue measure

With the help of computer program (Python), we are able to calculate the Lebesgue measure of $C \cdot C$.

Theorem (Jiang and Xi)

$$\mathcal{L}(C \cdot C) \approx 0.80955....$$

Similar results can be obtained for uniform λ-Cantor sets.

Multiple respresentations of real numbers

We go back to the middle-third Cantor. Note that

$$-C + C = \{-x + y : x, y \in C\} = [-1, 1].$$

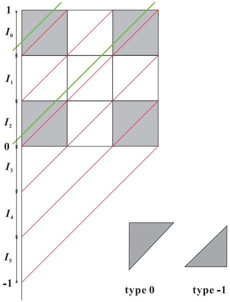
Therefore, for any $t \in [-1, 1]$, there exist some $x, y \in C$ such that

$t = -x + y$(t can be viewed as the intercept of the line $y = x + t$).

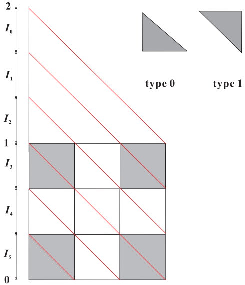
We may ask a natural question:
how many solutions $(x, y) \in C \times C$ can we find such that

$$t = -x + y.$$

Key observation



(a)



(b)

Figure:

Multiple respresentations of real numbers

Let $t \in [-1, 1]$. Define

$$S_t = \{(x, y) : y - x = t, x, y \in C\},$$

and

$$U_r = \{t : \textbf{Card}(S_t) = r\}, r = 1, 2, \cdots ,$$

Multiple representations of real numbers

Theorem (Jiang and Xi)

- $\dim_H(U_{2^k}) = \dfrac{\log 2}{\log 3}, k \in \mathbb{N};$
- $U_{3\cdot 2^k}$ *is countable* , $k \in \mathbb{N}$;
- $U_k = \emptyset$ *for other cases.*

Theorem (Jiang and Xi)

Let $\lambda \in \mathbb{Q} \setminus \{0\}$. *If*

$$(-1, \lambda) \cdot (C \times C) = \{-x + \lambda y : (x, y) \in C \times C\}$$

is an interval, then we can calculate the Hausdorff dimension of U_r, *where*

$$U_r = \{t : \textbf{Card}(S_t) = r\}, r \in \mathbb{Z}^+,$$

$t \in -C + \lambda C$

$$S_t = \{(x, y) : \lambda y - x = t, x, y \in C\}.$$

Representations on self-similar sets with overlaps

Let K be the attractor of the following IFS:

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\}, 1 - \lambda \geq c \geq \lambda,$$

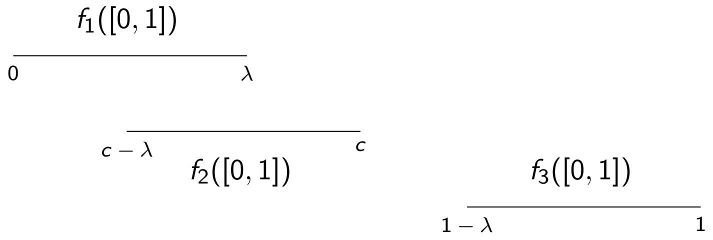


Figure: First iteration

Theorem (Jiang et al.)

$$K \cdot K = [0,1] \textit{ if and only if } c \geq (1-\lambda)^2.$$

$$\sqrt{K} + \sqrt{K} = \{\sqrt{x} + \sqrt{y} : x, y \in K\} = [0,2]$$

if and only if

$$\sqrt{c} + 1 \geq 2\sqrt{1-\lambda}.$$

Theorem (Jiang et al.)

We also prove that the following conditions are equivalent:

(1) *For any* $u \in [0,1]$, *there are some* $x, y \in K$ *such that* $u = x \cdot y$;

(2) *For any* $u \in [0,1]$, *there are some* $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \in K$ *such that*

$$u = x_1 + x_2 = x_3 - x_4 = x_5 \cdot x_6 = x_7 \div x_8 = \sqrt{x_9} + \sqrt{x_{10}};$$

(3) $c \geq (1-\lambda)^2$.

Back to the continuous image of fractal sets

It is natural to consider the when $f(A, B)$ is a closed interval. For this problem, we give a nonlinear version of thickness theorem. This thickness theorem allows us to find many nonlinear equations which can represent real numbers.

Definition of Cantor sets

Let $I_0 = [0, 1]$. In the first level, we delete an open interval from $[0, 1]$, denoted by O. Then there are two closed intervals left, we denote them by B_1 and B_2. Therefore, $[0, 1] = B_1 \cup O \cup B_2$. Let $E_1 = B_1 \cup B_2$.

In the second level, let O_0 and O_1 are open intervals that are deleted from B_1 and B_2 respectively, then we clearly have

$$B_1 = B_{11} \cup O_0 \cup B_{12}, B_2 = B_{21} \cup O_1 \cup B_{22}.$$

Let

$$E_2 = B_{11} \cup B_{12} \cup B_{21} \cup B_{22}.$$

Definition of Cantor sets

Repeating this process, we can generate E_{n+1} from E_n by removing an open interval from each closed interval in the union which consists of E_n. We let

$$K = \cap_{n=1}^{\infty} E_n,$$

and call K a Cantor set.

Definition of Cantor sets

(1) Let $O \cup \cup_{i=0}^{\infty} O_i$ be all the deleted open intervals. We assume that

$$|O| \geq |O_0| \geq |O_1| \geq \cdots \geq |O_n| \geq |O_{n+1}| \geq \cdots .$$

(2) Let B_ω be a closed interval in some level, then we delete an open interval O_ω from B_ω, i.e.

$$B_\omega = B_{\omega 1} \cup O_\omega \cup B_{\omega 2}.$$

We assume that

$$0 < \frac{|O_\omega|}{|B_\omega|} < 1,$$

where $|\cdot|$ means length.

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Definition of thickness

Let B_ω be a closed interval in some level. Then by the construction of K, we have

$$B_\omega = B_{\omega 1} \cup O_\omega \cup B_{\omega 2},$$

where O_ω is an open interval while $B_{\omega 1}$ and $B_{\omega 2}$ are closed intervals. We call $B_{\omega 1}$ and $B_{\omega 2}$ bridges of K, and O_ω gap of K. Let

$$\tau_\omega(B_\omega) = \min\left\{\frac{|B_{\omega 1}|}{|O_\omega|}, \frac{|B_{\omega 2}|}{|O_\omega|}\right\}.$$

We define the thickness of K by

$$\tau(K) = \inf_{B_\omega} \tau_\omega(B_\omega).$$

Here the infimum takes over all bridges in every level.

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Some examples

Example

(1) $\tau(C) = 1$ (middle-third Cantor set);

(2) $\tau(K_\lambda) = \dfrac{\lambda}{1-2\lambda}$, where K_λ is the attractor of

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda, 0 < \lambda < 1/2\}.$$

Some classical results

Theorem (Newhouse)

Let K_1 and K_2 be two Cantor sets with convex hull $[0, 1]$. If

$$\tau(K_1)\tau(K_2) \geq 1,$$

then

$$K_1 + K_2$$

is an interval.

Some classical results

Theorem (Astels)

Let $K_i, 1 \leq i \leq n$ be Cantor sets with convex hull $[0,1]$. If

$$\sum_{i=1}^{n} \frac{\tau(K_i)}{1+\tau(K_i)} \geq 1$$

then

$$\sum_{i=1}^{n} K_i = \left\{ \sum_{i=1}^{n} x_i : x_i \in K_i \right\}$$

is an interval.

Some applications of thickness theorem

Let C be the middle-third Cantor set. Then

Theorem (Wang, Jiang, Li, Zhao)

$$C^2 + C^2 + C^2 + C^2 = [0, 4].$$

Equivalently, for any $u \in [0, 4]$, *there exist some* $x_i, 1 \le i \le 4$ *such that*

$$u = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Theorem (Yu)

For any $k \ge 2$, *there exists some* $n(k) \le 2^k$ *such that*

$$n(k)C^k = \left\{ \sum_{i=1}^{n(k)} x_i^k : x_i \in C \right\} = [0, n(k)].$$

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Some classical results

Theorem (Simon and Taylor)

For any two Cantor sets $A \subset J_1, B \subset J_2$ *with* $\tau(A)\tau(B) > 1$, *if* $f(x, y) \in \mathcal{C}^2$ *with non-vanishing partial derivatives on* $J_1 \times J_2$ *(*J_1 *and* J_2 *are two closed intervals), then* $f(A, B)$ *contains some interiors.*

Remark

This result gives the local property of $f(A, B)$.

Simple observation

Theorem

If $\partial_x f$, $\partial_y f$ are continuous on U, and there is a point $(x_0, y_0) \in (C \times C) \cap U$ such that

$$1 < \left| \frac{\partial_x f|_{(x_0,y_0)}}{\partial_y f|_{(x_0,y_0)}} \right| < 3,$$

then $f_U(C, C)$ has an interior.

By the implicit function theorem, we have

$$\frac{dy}{dx} = -\frac{\partial_x f}{\partial_y f} \text{ for the equation } f(x, y) = 0.$$

The slope of the tangent line of the curve is between 1 and 3.

Our results

Theorem (Jiang)

Let K_1 and K_2 be two Cantor sets with convex hull $[0,1]$. Suppose $f(x,y) \in \mathcal{C}^1$. If for any $(x,y) \in [0,1]^2$, we have

$$(\tau(K_1))^{-1} \leq \left|\frac{\partial_x f}{\partial_y f}\right| \leq \tau(K_2),$$

then

$$f(K_1, K_2)$$

is an interval. In particular, if we let

$$f(x,y) = x + y, \text{ and } \tau(K_i) \geq 1, i = 1, 2,$$

then

$$K_1 + K_2$$

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Remark

If K_1 and K_2 do not have the same convex hull, we assume that they are linked, i.e. K_1 is not contained in the gaps of K_2, and vice verse.

Our results

Theorem

Let $\{K_i\}_{i=1}^{d}$ be Cantor sets with convex hull $[0,1]$. Suppose that $f(x_1,\cdots,x_{d-1},z)\in\mathcal{C}^1$. If for any $(x_1,\cdots,x_{d-1},z)\in[0,1]^d$, we have

$$(\tau(K_i))^{-1}\le\left|\frac{\partial_{x_i}f}{\partial_z f}\right|\le\tau(K_d),1\le i\le d-1$$

then $f(K_1,\cdots,K_d)$ is an interval.

For the homogeneous self-similar set, the upper bound can be improved.

Corollary

Let C be the middle-third Cantor set. Suppose $f(x,y) \in \mathcal{C}^1$. If for any $(x,y) \in [0,1]^2$ such that

$$1 \le \left| \frac{\partial_x f}{\partial_y f} \right| \le 3,$$

then $f(C,C)$ has a closed interval.

Applications

We may prove some Waring type results on self-similar sets with overlaps. Let J be the attractor of the IFS

$$f_i(x) = r_i x + a_i, r_i \in (0,1), a_1 = 0 < a_2 \leq \cdots \leq a_n = 1 - r_n, a_i \in \mathbb{R}.$$

Waring type result

Corollary

Let J be the self-similar set defined as above. Given any $d \in \mathbb{N}_{\geq 2}$, and any $\alpha \geq 1$. If

$$\begin{cases} r_1 \geq a_2 \\ \tau(J) \geq a_2^{1-\alpha}, \end{cases}$$

then

$$\sum_{i=1}^{d} J^{\alpha} = \left\{ \sum_{i=1}^{d} x_i^{\alpha} : x_i \in J \right\} = [0, d].$$

Continued fraction with deleted digits

Corollary

$$F^3(7) \pm F(7) = \mathbb{R}, (C+1)^2 + 2F(6) = \mathbb{R}, f(K_1, K_2, K_3) = \mathbb{R},$$

where

$$K_1 = K_2 = C+1, K_3 = F(6), f(x,y,z) = 0.1x + xy + z,$$

and C is the middle-third Cantor set,

$$F(m) = \{[t, a_1, a_2, \cdots] : t \in \mathbb{Z}, 1 \leq a_i \leq m \text{ for } i \geq 1\},$$

A resonant result

Corollary

Let K be the attractor of the following IFS

$$\{f_1(x) = \lambda_1 x, f_2(x) = \lambda_2 x + 1 - \lambda_2, 0 < \lambda_2 \leq \lambda_1 < 1, \lambda_1 + \lambda_2 < 1\}.$$

Then the following conditions are equivalent:

(1)
$$K \cdot K = \{x \cdot y : x, y \in K\} = [0, 1];$$

(2) $\lambda_1 \geq (1 - \lambda_2)^2$;

(3)
$$K \div K = \left\{\frac{x}{y} : x, y \in K, y \neq 0\right\} = \mathbb{R}.$$

Thank you.