On The Existence of Numbers With Matching Continued Fraction and Decimal Expansions

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Trott's Constant

- Is it possible for a number to have each term in its continued fraction expansion agree with its decimal expansion? That is

$$[0; a_1, a_2, a_3 \dots] = 0.a_1 a_2 a_3 \dots$$

- If we require each $a_i \in \{1, 2, \dots, 9\}$, the answer is clearly no.

- We must have $a_1 = 3$, and $a_2 \in \{1, 2, 3\}$. But $[0; 3, k] \leq 0.3$ for each $k = 1, 2, 3$.

Trott's Constant

The following became known as Trott's constant:

$$\cfrac{1}{1+\cfrac{1}{0+\cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{0+\cfrac{1}{\dots}}}}}}} = 0.1084101\dots$$

Trott Numbers

Definition

- An irrational number $x \in (0,1)$ is a Trott number if

$$[0; a_1, a_2, a_3, \dots] = 0.\hat{a}_1\hat{a}_2\hat{a}_3\dots$$

  where $\hat{a}_i$ is the string corresponding to the base 10 representation of a_i.
- We may of course define Trott numbers in a similar manner for any base $b \in \mathbb{N}_{\geq 2}$, the set of which we call T_b.

Computer Searches

The result of our initial computer search:
[0;3, 29, 5, 7110, 5, 3, 7, 8, 1, 709, 5, 8, 6, 79, 1, 6, 51, 60, 3, 5, 7, 6, 709, 8, 89, 1, 3, 7, 3, 60, 8, 5, 7, 2, 39, 60, 1, 7, 3, 50, 1, 6, 39, 199, 6, 8, 8, 5, 6, 1000, 400, 5, 5, 2, 69, 80, 30, 8, 3, 4, 1, 8, 2, 6, 1, 5, 590, 8, 19, 3, 8, 5, 10, 5, 7, 4, 8, 7, 2, 1, 5, 2, 5, 7, 69, 6, 6, 6, 3, 2, 1, 1, 1, 7, 6, 2, 10, 5, 8, 7, 1, 2, 8, 2, 2, 39, 20, 5, 8, 5, 2, 1, 2, 799, 2, 89, 2, 2, 7, 3, 10, 1, 8, 7, 609, 1, 7, 100, 1, 2, 7, 2, 290, 4, 3, 6, 7, 7, 1, 3, 5, 2, 8, 1, 599, 7, 7, 8, 8, 6, 2, 4, 5, 6, 1, 19, 8, 4, 7, 20, 1, 6, 4, 50, 5, 80, 6, 5, 39, 3, 3, 5, 1, 1, 1, 4, 7, 1, 4, 7, 7, 7, 5, 8, 2, 8, 5, 700, 1, 899, 6, 2, 2, 1, 4, 5, 70, 5, 30, 7, 6, 6, 1, 30, 3, 8, 2, 1, 4, 3, 5, 3, 1, 6, 1, 6, 3, 6, 190, 79, 7, 79, 2, 6, 4, 5, 29, 5, 3, 29, 4, 4, 1, 6, 1, 599, 32, 243, 5340, 10]

Examples in Base 10

On the surface, it appears quite simple to build a Trott number. Start with some initial string

$$[0; 3, 29] = 0.329545454\ldots$$

Then we could continue with any sub-string of $545454\ldots$
That is

$$[0; 3, 29, 545] = 0.329545691471\ldots$$

And so-on and so-forth from there:

$$[0; 3, 29, 545, 6] = 0.3295456913898\ldots$$

Theoretical Issues

It can so happen that this process fails to produce a Trott numbers. For instance

$$[0; 3, 29, 5, 7] = 0.329570398\ldots$$

cannot be continued.

We will call a rational number $x = [0; a_1, \ldots, a_n]$ *pre-Trott* if its decimal expansion begins with $0.\hat{a}_1 \ldots \hat{a}_n$, and the digit immediately following is non-zero.

Existence of Trott Numbers in Base 10

Theorem

Let

$$\Gamma := \{3\} \cup \left[\bigcup_{k \in \mathbb{N}} \bigcup_{j=1}^{k} \{k^2 + j\} \right],$$

then T_b is uncountable for $b \in \Gamma$, and empty for $b \notin \Gamma$.

So the first few good bases are 2,3,5,6,10,11,12,17,18,19,20, ...

Inductive Hypothesis

For $n \geq 2$, suppose that there exists positive integers $m \geq 2$, and $a_1, a_2, \ldots, a_n$ satisfying:

1. The number $x_n := \frac{p_n}{q_n} := [0; a_1, a_2, \ldots, a_n]$ is pre-Trott.
2. There exists a length $m+1$ sub-word, ω, of $\hat{a}_1\hat{a}_2\ldots\hat{a}_n$ such that:
   - ω begins with 0, followed by a non-zero.
   - $\omega \neq 01\underbrace{000\ldots0}_{m-1 \text{ times}}$
   - $\omega \neq 0\underbrace{999\ldots9}_{m \text{ times}}$
3. $q_n^2 > 10^{S_n+m+1}$, where $S_n = \text{length}(\hat{a}_1\hat{a}_2\ldots\hat{a}_n)$.
4. $(q_{n-1}, 10) = (q_n, 10) = 1$.

Base Case

The base case must be done by tedious trial and error.
A valid starting point for base 10 is

$$[0; 3, 329, 9595, 15237720207] =$$

$$0.3329959515237720207 5599542\ldots$$

In this example, $\omega = 0207$, $m = 3$, and $n = 4$. The denominators $q_4 \equiv 7 \pmod{10}$, and $q_3 \equiv 3 \pmod{10}$ are co-prime to 10, and q_4 is 18 digits long (so $q_4^2 > 10^{19+4}$)

Inductive Step

Suppose the previous statement holds for $n > 2$. Property $\boxed{4}$ tells us that x_n is purely periodic. That is

$$x_n = 0.\overline{c_1 c_2 \dots c_p}$$

for some $p \in \mathbb{N}$, (which we may choose as large as we want).
Therefore, $\hat{a}_1 \hat{a}_2 \dots \hat{a}_n$, and in particular ω, appears infinitely often in the decimal expansion of x_n by $\boxed{1}$ and $\boxed{2}$.
Thus, there are infinitely many j, for which we have

$$\omega = c_{S_n+j} \dots c_{S_n+j+m}.$$

Inductive Step

We take $a_{n+1} = c_{S_n+1} \ldots c_{S_n+j}$. By $\boxed{1}$, $c_{S_n+1} \neq 0$ so a_{n+1} is a valid continuation of x_n.
Next, since $10|a_{n+1}$, $q_{n+1} = a_{n+1}q_n + q_{n-1}$, and $(q_{n-1}, 10) = 1$, we have that $(q_{n+1}, 10) = 1$.
Furthermore,

$$q_{n+1}^2 > a_{n+1}^2 q_n^2 > 10^{2(j-1)} 10^{S_n+m+1} > 10^{S_{n+1}+m+1}.$$

Then

$$|x_n - x_{n+1}| = \frac{1}{q_{n+1}q_n} < \frac{1}{a_{n+1}q_n^2} < \frac{1}{10^{(S_{n+1}+m)}}$$

By the properties in $\boxed{2}$, we have that x_{n+1} is pre-Trott.

Known Results So Far

- T_b is a complete G_δ set for all $b \in \Gamma$.
- $\mathrm{Dim}_H(T_b) < 1$ for all b.
- If $b \neq c$, and $b > 1.185 \times 10^{29}$, then $T_b \cap T_c$ is empty.

Questions for Further Research

- Is it true that $\lim_{b\to\infty} \mathrm{Dim}_H(T_b) = 0$?
- Are there Trott numbers for β-expansions? What would be the natural definition for such numbers?
- Is $\mathrm{Dim}_H(T_b) > 0$ for $b \in \Gamma$?
- Is $T_b \cap T_c$ empty for all $b \neq c$?
- Are there any badly approximable Trott numbers? Normal Trott numbers?

References

1. M. Trott, Finding Trott Constants. *Mathematica J.* **10** (2006), 303–322
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