Lüroth expansions

Alternative Lüroth expansions

Random Lüroth transformations

Approximation
Lüroth expansions

In 1883 J. Lüroth introduced number representations that are now called Lüroth series expansions:

For each $x \in (0, 1]$ there is an infinite sequence $(a_n)_{n \geq 1}$ with $a_n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $n$ such that

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1 - 1)a_2} + \frac{1}{a_1(a_1 - 1)a_2(a_2 - 1)a_3} + \cdots = \sum_{k \geq 1} \frac{a_k - 1}{\prod_{i=1}^{k} a_i(a_i - 1)}.$$

A Lüroth expansion is called **ultimately periodic** if there are $n \geq 0$, $r \geq 1$ such that $a_{n+j} = a_{n+r+j}$ for all $j \geq 1$ and **periodic** if $n = 0$.

**Property:**

$x$ has an ultimately periodic Lüroth expansion if and only if $x$ is rational.

Charlene Kalle, Leiden University
In 1968 H. Jager and C. de Vroedt proved that Lüroth expansions are generated by iterations of the \textbf{Lüroth transformation} $T_L : [0,1] \to [0,1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and for $x \in (0,1)$,

$$T_L(x) = n(n-1)x - (n-1), \quad \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), \; n \geq 2.$$ 

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{3 \cdot 2 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 2 \cdot 1 \cdot 7}$$

$a_1(3/7) = 3$, $a_2(3/7) = 2$, $a_3(3/7) = 7$. 

Lüroth transformation
Lüroth transformation

In 1968 H. Jager and C. de Vroedt proved that Lüroth expansions are generated by iterations of the **Lüroth transformation** $T_L : [0, 1] \rightarrow [0, 1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and for $x \in (0, 1)$,

$$T_L(x) = n(n - 1)x - (n - 1), \quad \text{if } x \in \left[\frac{1}{n}, \frac{1}{n - 1}\right), \; n \geq 2.$$ 

**Property:**
The digits $a_n(\cdot)$ are i.i.d. random variables with respect to Lebesgue measure with

$$\text{Leb}(a_n = k) = \frac{1}{k(k - 1)}, \quad k \in \mathbb{N}_{\geq 2}.$$
In 1968 H. Jager and C. de Vroedt proved that Lüroth expansions are generated by iterations of the **Lüroth transformation** $T_L : [0, 1] \rightarrow [0, 1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and for $x \in (0, 1)$,

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**Property:**

$T_L$ is measure preserving and ergodic wrt Lebesgue, so for all Borel sets $A$,

$$\text{Leb}(T_L^{-1}A) = \text{Leb}(A)$$

and if $T_L^{-1}A = A$, then $\text{Leb}(A) \in \{0, 1\}$. 
For each $D \in \mathbb{N}_{\geq 2}$ consider the set

$$E_D = \{ x \in [0,1] : a_n(x) \leq D \text{ for all } n \geq 1 \}.$$ 

From the Birkhoff ergodic theorem it follows that $\text{Leb}(E_D) = 0$ for each $D$.

In 1968 T. Šalát proved that $\dim_H(E_D) < 1$ for each $D$ with

$$\lim_{D \to \infty} \dim_H(E_D) = 1.$$
Convergents and Lyapunov exponent

Approximation of irrationals by Lüroth expansions was considered by various people.

If \( x \in [0, 1] \setminus \mathbb{Q} \) with
\[
x = \sum_{k \geq 1} \frac{a_k - 1}{\prod_{i=1}^{k} a_i (a_i - 1)},
\]
then the **convergents** are the rational numbers
\[
\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{a_k - 1}{\prod_{i=1}^{k} a_i (a_i - 1)}.
\]

**Barreira and Iommi, 2009:** For \( \text{Leb}-\text{a.e.} \ x \in (0, 1) \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d \geq 2} \frac{\log d (d - 1)}{d (d - 1)} = -2.04 \ldots
\]

and the range of possible values is \((-\infty, -\log 2]\).
Approximation coefficients

For $x \in [0, 1] \setminus \mathbb{Q}$ the approximation coefficients are

$$\theta_n^L(x) = q_n \left| x - \frac{p_n}{q_n} \right|,$$

where $q_n = a_n \prod_{i=1}^{n-1} a_i(a_i - 1)$, $n \geq 1$.

Dajani and Kraaikamp, 1996: For Lebesgue-a.e. $x \in (0, 1)$ and every $z \in (0, 1]$ the limit

$$\lim_{N \to \infty} \frac{\#\{1 \leq j \leq N : \theta_j^L(x) < z\}}{N}$$

exists and equals

$$F_L(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}.$$
Alternating Lüroth expansions

In 1990 S. Kalpazidou, A. Knopfmacher and J. Knopfmacher introduced alternating Lüroth series expansions:

For each \( x \in (0, 1] \) there is a sequence \( (a_n)_{n \geq 1} \) with \( a_n \in \mathbb{N}_{\geq 2} \cup \{\infty\} \) for all \( n \) such that

\[
x = \frac{1}{a_1 - 1} - \frac{1}{a_1(a_1 - 1)(a_2 - 1)} + \frac{1}{a_1(a_1 - 1)a_2(a_2 - 1)(a_3 - 1)} - \cdots
\]

\[
= \sum_{k \geq 1} (-1)^{k-1} \frac{a_k}{\prod_{i=1}^{k} a_i(a_i - 1)}.
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$$= \sum_{k \geq 1} (-1)^{k-1} \frac{a_k}{\prod_{i=1}^{k} a_i(a_i - 1)}.$$

**Property 1:**
$x$ has an ultimately periodic Lüroth expansion if and only if $x$ is rational.

**Property 2:**
The digits $a_n(\cdot)$ are i.i.d. random variables with respect to Lebesgue measure with

$$\text{Leb}(a_n = k) = \frac{1}{k(k - 1)}, \quad k \in \mathbb{N}_{\geq 2}.$$
Generalised Lüroth expansions

In 1996 J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp placed both types of expansions in the framework of **generalised Lüroth series expansions**:

Let \( \varepsilon = (\varepsilon_n)_{n \geq 1} \in \{0, 1\}^\mathbb{N} \) be a sequence of 0’s and 1’s.

The map \( T_\varepsilon \) that maps each interval \( \left( \frac{1}{n}, \frac{1}{n-1} \right) \), \( n \geq 2 \), linearly onto \((0, 1)\) with positive slope if \( \varepsilon_{n-1} = 0 \) and negative slope if \( \varepsilon_{n-1} = 1 \).

\[ \varepsilon = (0)_{n \geq 1} \] gives the Lüroth transformation \( T_L \).

\[ \varepsilon = (1)_{n \geq 1} \] gives the alternating Lüroth transformation \( T_A \).
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The map $T_{\varepsilon}$ that maps each interval $(\frac{1}{n}, \frac{1}{n-1})$, $n \geq 2$, linearly onto $\langle 0, 1 \rangle$ with positive slope if $\varepsilon_{n-1} = 0$ and negative slope if $\varepsilon_{n-1} = 1$.

For each $x \in (0, 1]$ there is a sequence $(a_n)_{n \geq 1}$ with $a_n \in \mathbb{N}_{\geq 2}$ for all $n$ such that

$$x = \sum_{n \geq 1} (-1)^{\sum_{i=1}^{n-1} \varepsilon_i} \frac{a_n - 1 + \varepsilon_n}{\prod_{i=1}^{n} a_i(a_i - 1)}.$$
For each $\varepsilon$ and $n \geq 1$ we can also define

$$\theta_n^\varepsilon(x) = q_n \left| x - \frac{p_n}{q_n} \right|, \quad n \geq 1.$$ 

J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp, 1996:

The limit

$$F_\varepsilon(z) := \lim_{N \to \infty} \frac{\# \{1 \leq j \leq N : \theta_n^\varepsilon(x) < z\}}{N}$$

exists for Lebesgue almost all $x \in [0, 1]$ and all $z \in (0, 1]$.

One has

$$F_A(z) = \sum_{k=2}^{\left\lfloor \frac{1}{z} \right\rfloor} \frac{z}{k-1} + \frac{1}{\left\lfloor \frac{1}{z} \right\rfloor}$$

and

$$F_A \leq F_\varepsilon \leq F_L.$$
First moments

J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp, 1996:

For each \( \varepsilon \) there is a constant \( M_\varepsilon \) such that for Lebesgue a.e. \( x \in [0, 1] \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \theta_i^\varepsilon = M_\varepsilon.
\]

For each \( \varepsilon \),

\[
M_A = 1 - \frac{1}{2}\zeta(2) \leq M_\varepsilon \leq M_L = \frac{1}{2}(\zeta(2) - 1).
\]

Not every value in \([M_A, M_L]\) can be obtained for \( M_\varepsilon \) by choosing \( \varepsilon \) appropriately. It is conjectured that the set of values \( M_\varepsilon \) can take is a Cantor set.
To further investigate the properties of Lüroth expansions we introduce a family of random Lüroth systems.

Set $T_0 = T_L$ and $T_1 = T_A$. 

\begin{figure}
\centering
\includegraphics[width=1\textwidth]{figure.png}
\caption{Graphs of $T_0$ and $T_1$.}
\end{figure}
Random systems

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![Graph showing $T_0$ and $T_1$]
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**Parameter:** cut-off point $c \in [0, \frac{1}{2}]$. 

---

\[ T_0 \text{ and } T_1 \]
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Set $T_0 = T_L$ and $T_1 = T_A$.

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\[ 0 \quad c \quad \frac{1}{3} \quad \frac{1}{2} \quad 1 \]

$T_0$ and $T_1$
Random systems

To further investigate the properties of Lüroth expansions we introduce a family of random Lüroth systems.

Set $T_0 = T_L$ and $T_1 = T_A$.

Parameter: cut-off point $c \in [0, \frac{1}{2}]$. 
Let \( \omega = (\omega_n)_{n \geq 1} \in \{0, 1\}^\mathbb{N} \) and \( \sigma : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N} \) the left shift.

The random Lüroth transformation is the map
\[
L_c : \{0, 1\}^\mathbb{N} \times [c, 1] \to \{0, 1\}^\mathbb{N} \times [c, 1]
\]
given by
\[
L_c(\omega, x) = (\sigma(\omega), T_{\omega_1,c}(x)).
\]

In the second coordinate this yields compositions
\[
T^n_{\omega,c}(x) = T_{\omega_n,c} \circ \cdots \circ T_{\omega_2,c} \circ T_{\omega_1,c}(x).
\]
Random systems

For each \((\omega, x)\) we define two sequences:

1. A sequence of signs \((s_n)_{n \geq 1}\), where \(s_n = 0\) if the slope of \(T_{\omega, c}^n\) at \(T_{\omega, c}^{n-1}(x)\) is positive and 1 otherwise.
2. A sequence of digits \((d_n)_{n \geq 1}\), where \(d_n = k\) if \(T_{\omega, c}^{n-1}(x) \in \left[\frac{1}{k}, \frac{1}{k-1}\right)\).

\[
T_{0,c} \quad T_{1,c}
\]

\[
\omega_1 = 1, \; \omega_2 = 0, \; \omega_3 = 1, \; \ldots
\]

\[
s_1 = 0, \quad d_1 = 2
\]
For each \((\omega, x)\) we define two sequences:

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\[
\begin{align*}
&s_1 = 0, \quad s_2 = 0 \\
&d_1 = 2, \quad d_2 = 2
\end{align*}
\]
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\[
\omega_1 = 1, \omega_2 = 0, \omega_3 = 1, \ldots
\]

\(s_1 = 0, s_2 = 0, s_3 = 1\)

\(d_1 = 2, d_2 = 2, d_3 = 2\)
The sequences \((s_n)_{n \geq 1}\) and \((d_n)_{n \geq 1}\) give a \(c\)-Lüroth expansion of \(x\):

\[
x = \sum_{k \geq 1} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i(d_i - 1)}.
\]

Or: \(x\) has digit sequence \((s_n, d_n)_{n \geq 1}\).

**First observations:**

1. For each \(D \geq 2\),

\[
\tilde{E}_D = \left\{ x \in \left[ \frac{1}{D}, 1 \right] : \exists \omega \text{ s.t. } d_n \leq D \text{ for all } n \right\} = \left[ \frac{1}{D}, 1 \right].
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Random Lüroth expansions

The sequences \((s_n)_{n \geq 1}\) and \((d_n)_{n \geq 1}\) give a \(c\)-Lüroth expansion of \(x\):

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First observations:

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\tilde{E}_D = \left\{ x \in \left[ \frac{1}{D}, 1 \right] : \exists \omega \text{ s.t. } d_n \leq D \text{ for all } n \right\} = \left[ \frac{1}{D}, 1 \right].
\]

2. A \(c\)-Lüroth expansion is called ultimately periodic if there are \(n \geq 0\) and \(r \geq 1\) such that

\[
T_{\omega}^{n+j}(x) = T_{\omega}^{n+r+j}(x) \quad \text{for all } j \geq 1.
\]

If \(x \in [c, 1] \setminus \mathbb{Q}\), then the \(c\)-Lüroth expansion cannot be ultimately periodic.
Let $c \in [0, \frac{1}{2}]$ and $x \in [c, 1] \cap \mathbb{Q}$. One of the following cases occurs.

- $x$ has a unique and ultimately periodic $c$-Lüroth expansion.
- All $c$-Lüroth expansions are ultimately periodic (so there are at most countably many).
- $x$ has uncountably many $c$-Lüroth expansions that are not ultimately periodic and countably many $c$-Lüroth expansions that are ultimately periodic.

\[ \frac{1}{3} \text{ has unique digit sequence } (1, 3)(0, 2). \]
Let $c \in [0, \frac{1}{2}]$ and $x \in [c, 1] \cap \mathbb{Q}$. One of the following cases occurs.

- $x$ has a unique and ultimately periodic $c$-Lüroth expansion.
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- $x$ has uncountably many $c$-Lüroth expansions that are not ultimately periodic and countably many $c$-Lüroth expansions that are ultimately periodic.

$\frac{3}{4}$ has the two digit sequences:

$(0, 2)(1, 2)(0, 2)$,

$(1, 2)(1, 2)(0, 2)$. 
Let $c \in [0, \frac{1}{2}]$ and $x \in [c, 1] \cap \mathbb{Q}$. One of the following cases occurs.

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- All $c$-Lüroth expansions are ultimately periodic (so there are at most countably many).
- $x$ has uncountably many $c$-Lüroth expansions that are not ultimately periodic and countably many $c$-Lüroth expansions that are ultimately periodic.
Universal expansions

Results on numbers of different expansions:

1. Let \( c \in [0, \frac{2}{5}] \). Then any \( x \in [c, 1] \) has uncountably many different \( c \)-Lüroth expansions.

2. Let \( c = \frac{1}{D} \) for some \( D \in \mathbb{N}_{\geq 3} \) and consider the alphabet

\[
\mathcal{A}_c = \{(s, d) : s \in \{0, 1\}, \, d \in \{2, 3, \ldots, D\}\}.
\]

A \( c \)-Lüroth expansion

\[
x = \sum_{k \geq 1} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i(d_i - 1)}
\]

is called universal if all blocks \((t_1, b_1), \ldots, (t_j, b_j) \in \mathcal{A}_c^j \) occur in the expansion, so if there is a \( k \geq 1 \) such that \( s_{k+i} = t_i \) and \( d_{k+i} = b_i \) for all \( 1 \leq i \leq j \).

For any \( c = \frac{1}{D} \) Lebesgue almost every \( x \in [c, 1] \) has uncountably many different universal \( c \)-Lüroth expansions.

The same holds for \( c = 0 \).
Convergents

Take $c = 0$.

For any $(\omega, x) \in \{0, 1\}^\mathbb{N} \times [0, 1]$ set, like before,

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i (d_i - 1)}.$$

Fix some $0 < p < 1$, let $m_p$ be the $(p, 1 - p)$-Bernoulli measure on $\{0, 1\}^\mathbb{N}$ and consider the measure $m_p \times \text{Leb}$ on $\{0, 1\}^\mathbb{N} \times [0, 1]$.

For $m_p \times \text{Leb}$-a.e. $(\omega, x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = - \sum_{d \geq 2} \frac{\log d (d - 1)}{d(d - 1)}$$

and the range of possible values is $(-\infty, -\log 2]$. 
Approximation coefficients

Take $c = 0$, fix some $0 < p < 1$.

For any $(\omega, x) \in \{0, 1\}^\mathbb{N} \times [0, 1]$ set, like before,

$$\theta_n(\omega, x) = q_n \left| x - \frac{p_n}{q_n} \right|, \quad \text{with} \quad q_n = (d_n - s_n) \prod_{i=1}^{n-1} d_i (d_i - 1).$$

For $m_p \times \text{Leb}$-a.e. $(\omega, x)$ the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \theta_i(\omega, x)$$

exists and equals

$$M_p = p \frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2}.$$ 

The function $p \mapsto M_p$ maps the interval $[0, 1]$ to the interval $[M_A, M_L]$. 
Invariant measures

Both results use the fact that the measure $m_p \times \text{Leb}$ is invariant and ergodic for the random map $L_c$, in particular Birkhoff’s ergodic theorem.

If $c > 0$, then $m_p \times \text{Leb}$ is no longer invariant, but there exists a unique invariant and ergodic measure of the form $m_p \times \mu_{c,p}$ with $\mu_{c,p} \ll \text{Leb}$ a probability measure.

To obtain the speed of convergence, one needs a good expression for the density of $\mu_{c,p}$. In specific cases this can be computed.

**Example:** For $c = \frac{1}{8}$ and $0 < p < 1$ the density of $\mu_{p,\frac{1}{8}}$ is

$$f_{p,\frac{1}{8}}(x) = \frac{1}{2p^2 + 3p + 5} \begin{cases} 8, & \text{if } x \in [1/8, 1/4), \\ 4p + 4, & \text{if } x \in [1/4, 1/2), \\ 4p^2 + 2p + 4, & \text{if } x \in [1/2, 3/4), \\ 4p^2 + 6p + 4, & \text{if } x \in [3/4, 7/8), \\ 4p^2 + 6p + 12, & \text{if } x \in [7/8, 1). \end{cases}$$
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