Random Lüroth expansions

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Lüroth expansions

In 1883 J. Lüroth introduced number representations that are now called **Lüroth series expansions**:

For each $x \in (0,1]$ there is an infinite sequence $(a_n)_{n\geq 1}$ with $a_n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all n such that

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \frac{1}{a_1(a_1-1)a_2(a_2-1)a_3} + \cdots = \sum_{k\geq 1} \frac{a_k - 1}{\prod_{i=1}^{k} a_i(a_i-1)}.$$

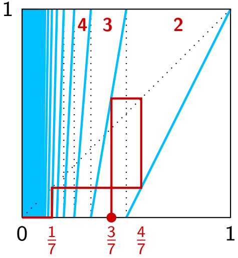
A Lüroth expansion is called **ultimately periodic** if there are $n \geq 0$, $r \geq 1$ such that $a_{n+j} = a_{n+r+j}$ for all $j \geq 1$ and **periodic** if $n = 0$.

Property:
x has an ultimately periodic Lüroth expansion if and only if x is rational.

Lüroth transformation

In 1968 H. Jager and C. de Vroedt proved that Lüroth expansions are generated by iterations of the **Lüroth transformation** $T_L : [0,1] \to [0,1]$ given by $T_L(0) = 0$, $T_L(1) = 1$ and for $x \in (0,1)$,

$$T_L(x) = n(n-1)x - (n-1), \quad \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), \; n \geq 2.$$



$$a_1(3/7) = 3, \; a_2(3/7) = 2, \; a_3(3/7) = 7.$$

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{3 \cdot 2 \cdot 2} + \frac{1}{3 \cdot 2 \cdot 2 \cdot 1 \cdot 7}$$

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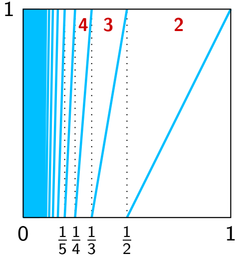

Property:
The digits $a_n(\cdot)$ are i.i.d. random variables with respect to Lebesgue measure with

$$Leb(a_n = k) = \frac{1}{k(k-1)}, \quad k \in \mathbb{N}_{\geq 2}.$$

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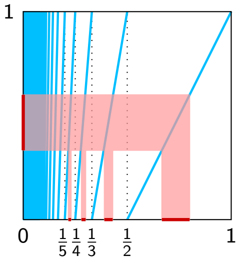


Property:
T_L is measure preserving and ergodic wrt Lebesgue, so for all Borel sets A,

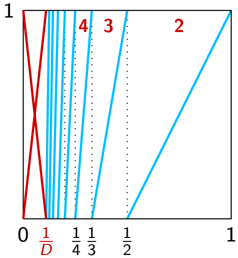
$$Leb(T_L^{-1}A) = Leb(A)$$

and if $T_L^{-1}A = A$, then $Leb(A) \in \{0,1\}$.

Bounds on digits

For each $D \in \mathbb{N}_{\geq 2}$ consider the set

$$E_D = \{x \in [0,1] \,:\, a_n(x) \leq D \text{ for all } n \geq 1\}.$$



From the Birkhoff ergodic theorem it follows that $Leb(E_D) = 0$ for each D.

In 1968 T. Šalát proved that $\dim_H(E_D) < 1$ for each D with

$$\lim_{D\to\infty} \dim_H(E_D) = 1.$$

Convergents and Lyapunov exponent

Approximation of irrationals by Lüroth expansions was considered by various people.

If $x \in [0,1] \setminus \mathbb{Q}$ with

$$x = \sum_{k \geq 1} \frac{a_k - 1}{\prod_{i=1}^{k} a_i(a_i - 1)},$$

then the **convergents** are the rational numbers

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{a_k - 1}{\prod_{i=1}^{k} a_i(a_i - 1)}.$$

Barreira and Iommi, 2009: For *Leb*-a.e. $x \in (0,1)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\sum_{d \geq 2} \frac{\log d(d-1)}{d(d-1)} = -2.04\ldots$$

and the range of possible values is $(-\infty, -\log 2]$.

Approximation coefficients

For $x \in [0,1] \setminus \mathbb{Q}$ the **approximation coefficients** are

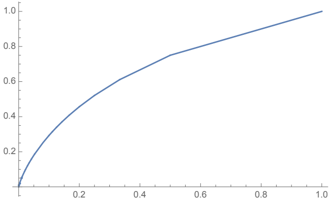
$$\theta_n^L(x) = q_n \left| x - \frac{p_n}{q_n} \right|, \quad \text{where } q_n = a_n \prod_{i=1}^{n-1} a_i(a_i - 1), \quad n \geq 1.$$

Dajani and Kraaikamp, 1996: For *Leb*-a.e. $x \in (0,1)$ and every $z \in (0,1]$ the limit

$$\lim_{N \to \infty} \frac{\#\{1 \leq j \leq N \, : \, \theta_j^L(x) < z\}}{N}$$

exists and equals

$$F_L(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}.$$

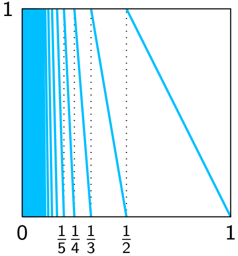


Alternating Lüroth expansions

In 1990 S. Kalpazidou, A. Knopfmacher and J. Knopfmacher introduced **alternating Lüroth series expansions:**

For each $x \in (0,1]$ there is a sequence $(a_n)_{n\geq 1}$ with $a_n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all n such that

$$x = \frac{1}{a_1 - 1} - \frac{1}{a_1(a_1-1)(a_2-1)} + \frac{1}{a_1(a_1-1)a_2(a_2-1)(a_3-1)} - \cdots$$
$$= \sum_{k\geq 1} (-1)^{k-1} \frac{a_k}{\prod_{i=1}^{k} a_i(a_i-1)}.$$



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Property 1:
x has an ultimately periodic Lüroth expansion if and only if x is rational.

Property 2:
The digits $a_n(\cdot)$ are i.i.d. random variables with respect to Lebesgue measure with

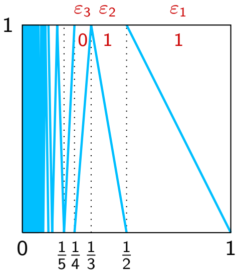
$$Leb(a_n = k) = \frac{1}{k(k-1)}, \quad k \in \mathbb{N}_{\geq 2}.$$

Generalised Lüroth expansions

In 1996 J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp placed both types of expansions in the framework of **generalised Lüroth series expansions:**

Let $\varepsilon = (\varepsilon_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ be a sequence of 0's and 1's.

The map T_ε that maps each interval $(\frac{1}{n}, \frac{1}{n-1})$, $n \geq 2$, linearly onto $(0,1)$ with positive slope if $\varepsilon_{n-1} = 0$ and negative slope if $\varepsilon_{n-1} = 1$.



$\varepsilon = (0)_{n\geq 1}$ gives the Lüroth transformation T_L.

$\varepsilon = (1)_{n\geq 1}$ gives the alternating Lüroth transformation T_A.

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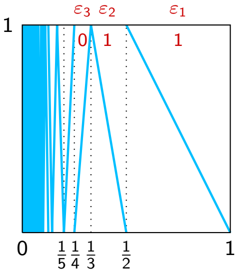


For each $x \in (0,1]$ there is a sequence $(a_n)_{n\geq 1}$ with $a_n \in \mathbb{N}_{\geq 2}$ for all n such that

$$x = \sum_{n\geq 1} (-1)^{\sum_{i=1}^{n-1} \varepsilon_i} \frac{a_n - 1 + \varepsilon_n}{\prod_{i=1}^{n} a_i(a_i - 1)}.$$

Approximations by GSL expansions

For each ε and $n \geq 1$ we can also define

$$\theta_n^\varepsilon(x) = q_n \left| x - \frac{p_n}{q_n} \right|, \quad n \geq 1.$$

J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp, 1996:

The limit

$$F_\varepsilon(z) := \lim_{N \to \infty} \frac{\#\{1 \leq j \leq N \,:\, \theta_n^\varepsilon(x) < z\}}{N}$$

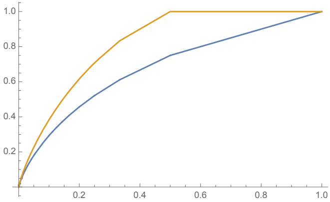
exists for Lebesgue almost all $x \in [0,1]$ and all $z \in (0,1]$.

One has

$$F_A(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor} \frac{z}{k-1} + \frac{1}{\lfloor \frac{1}{z} \rfloor}$$

and

$$F_A \leq F_\varepsilon \leq F_L.$$



First moments

J. Barrionuevo, R. Burton and K. Dajani and C. Kraaikamp, 1996:

For each ε there is a constant M_ε such that for Lebesgue a.e. $x \in [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \theta_i^\varepsilon = M_\varepsilon.$$

For each ε,

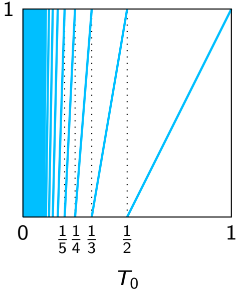
$$M_A = 1 - \frac{1}{2}\zeta(2) \leq M_\varepsilon \leq M_L = \frac{1}{2}(\zeta(2) - 1).$$

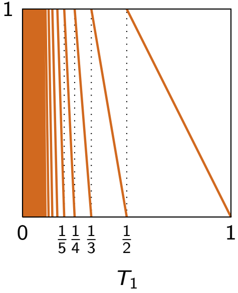
Not every value in $[M_A, M_L]$ can be obtained for M_ε by choosing ε appropriately. It is conjectured that the set of values M_ε can take is a Cantor set.

Random systems

To further investigate the properties of Lüroth expansions we introduce a family of random Lüroth systems.

Set $T_0 = T_L$ and $T_1 = T_A$.





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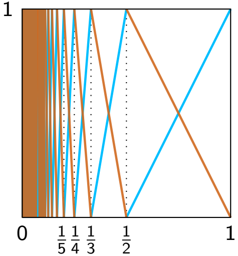

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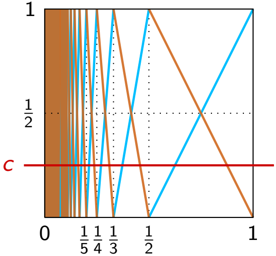

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Parameter: cut-off point $c \in [0, \frac{1}{2}]$.

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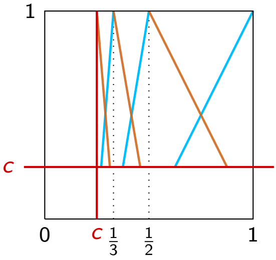

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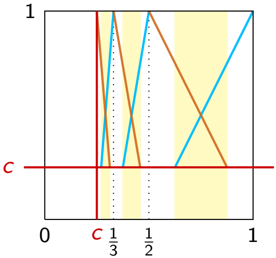

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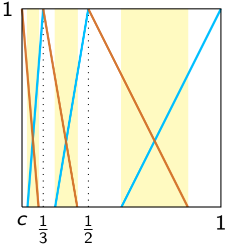

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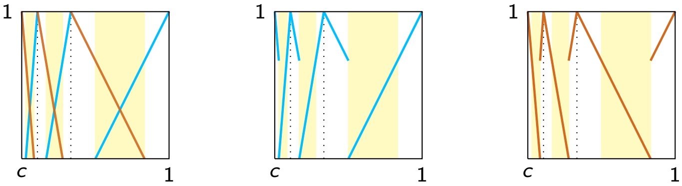


$T_{0,c}$ $T_{1,c}$

Let $\omega = (\omega_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ and $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ the left shift.

The **random Lüroth transformation** is the map
$L_c : \{0,1\}^{\mathbb{N}} \times [c,1] \to \{0,1\}^{\mathbb{N}} \times [c,1]$ given by

$$L_c(\omega, x) = (\sigma(\omega), T_{\omega_1,c}(x)).$$

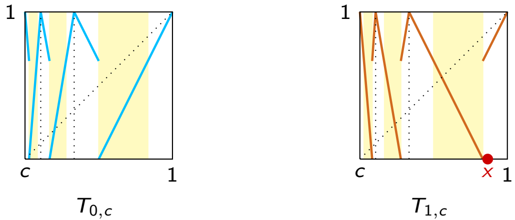
In the second coordinate this yields compositions

$$T^n_{\omega,c}(x) = T_{\omega_n,c} \circ \cdots \circ T_{\omega_2,c} \circ T_{\omega_1,c}(x).$$

Random systems

For each (ω, x) we define two sequences:

1. A sequence of signs $(s_n)_{n\geq 1}$, where $s_n = 0$ if the slope of $T_{\omega_n,c}$ at $T^{n-1}_{\omega,c}(x)$ is positive and 1 otherwise.
2. A sequence of digits $(d_n)_{n\geq 1}$, where $d_n = k$ if $T^{n-1}_{\omega,c}(x) \in [\frac{1}{k}, \frac{1}{k-1})$.



$$\omega_1 = 1,\ \omega_2 = 0,\ \omega_3 = 1,\ \ldots$$

$s_1 = 0$
$d_1 = 2$

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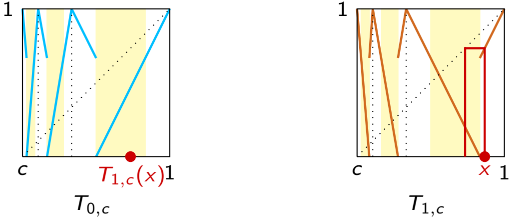

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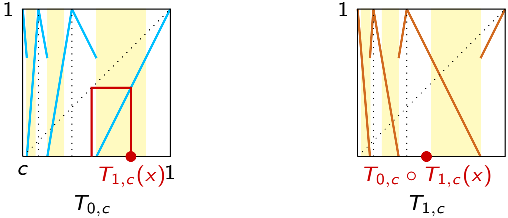

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Random Lüroth expansions

The sequences $(s_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ give a *c*-**Lüroth expansion** of x:

$$x = \sum_{k\geq 1} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i(d_i - 1)}.$$

Or: x has digit sequence $(s_n, d_n)_{n\geq 1}$.

First observations:

1. For each $D \geq 2$,

$$\tilde{E}_D = \left\{ x \in \left[\frac{1}{D}, 1\right] \, : \, \exists \, \omega \;\; \text{s.t.} \;\; d_n \leq D \;\; \text{for all} \;\; n \right\} = \left[\frac{1}{D}, 1\right].$$

Random Lüroth expansions

The sequences $(s_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ give a c-**Lüroth expansion** of x:

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2. A c-Lüroth expansion is called ultimately periodic if there are $n \geq 0$ and $r \geq 1$ such that

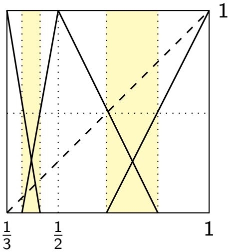
$$T_\omega^{n+j}(x) = T_\omega^{n+r+j}(x) \quad \text{for all } j \geq 1.$$

If $x \in [c, 1] \setminus \mathbb{Q}$, then the c-Lüroth expansion cannot be ultimately periodic.

Periodicity

Let $c \in [0, \frac{1}{2}]$ and $x \in [c, 1] \cap \mathbb{Q}$. One of the following cases occurs.

- x has a unique and ultimately periodic c-Lüroth expansion.
- All c-Lüroth expansions are ultimately periodic (so there are at most countably many).
- x has uncountably many c-Lüroth expansions that are not ultimately periodic and countably many c-Lüroth expansions that are ultimately periodic.



$\frac{1}{3}$ has unique digit sequence $(1,3)\overline{(0,2)}$.

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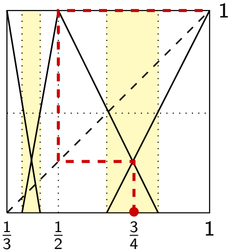

$\frac{3}{4}$ has the two digit sequences:

$(0,2)(1,2)\overline{(0,2)}$,

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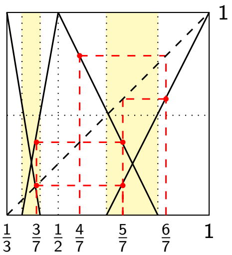

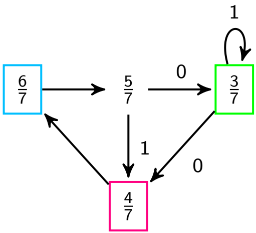

Universal expansions

Results on numbers of different expansions:

1. Let $c \in [0, \frac{2}{5}]$. Then any $x \in [c, 1]$ has uncountably many different c-Lüroth expansions.

2. Let $c = \frac{1}{D}$ for some $D \in \mathbb{N}_{\geq 3}$ and consider the alphabet

$$\mathcal{A}_c = \{(s, d) : s \in \{0, 1\}, d \in \{2, 3, \ldots, D\}\}.$$

A c-Lüroth expansion

$$x = \sum_{k \geq 1} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i(d_i - 1)}$$

is called **universal** if all blocks $(t_1, b_1), \ldots, (t_j, b_j) \in \mathcal{A}_c^j$ occur in the expansion, so if there is a $k \geq 1$ such that $s_{k+i} = t_i$ and $d_{k+i} = b_i$ for all $1 \leq i \leq j$.

For any $c = \frac{1}{D}$ Lebesgue almost every $x \in [c, 1]$ has uncountably many different universal c-Lüroth expansions.

The same holds for $c = 0$.

Convergents

Take $c = 0$.

For any $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ set, like before,

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} (-1)^{\sum_{i=1}^{k-1} s_i} \frac{d_k - 1 + s_k}{\prod_{i=1}^{k} d_i(d_i - 1)}.$$

Fix some $0 < p < 1$, let m_p be the $(p, 1-p)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ and consider the measure $m_p \times Leb$ on $\{0,1\}^{\mathbb{N}} \times [0,1]$.

For $m_p \times Leb$-a.e. (ω, x),

$$\lim_{n \to \infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = - \sum_{d \geq 2} \frac{\log d(d-1)}{d(d-1)}$$

and the range of possible values is $(-\infty, -\log 2]$.

Approximation coefficients

Take $c = 0$, fix some $0 < p < 1$.

For any $(\omega, x) \in \{0,1\}^{\mathbb{N}} \times [0,1]$ set, like before,

$$\theta_n(\omega, x) = q_n \left| x - \frac{p_n}{q_n} \right|, \quad \text{with } \; q_n = (d_n - s_n) \prod_{i=1}^{n-1} d_i(d_i - 1).$$

For $m_p \times Leb$-a.e. (ω, x) the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \theta_i(\omega, x)$$

exists and equals

$$M_p = p \frac{2\zeta(2) - 3}{2} + \frac{2 - \zeta(2)}{2}.$$

The function $p \mapsto M_p$ maps the interval $[0,1]$ to the interval $[M_A, M_L]$.

Invariant measures

Both results use the fact that the measure $m_p \times Leb$ is invariant and ergodic for the random map L_c, in particular Birkhoff's ergodic theorem.

If $c > 0$, then $m_p \times Leb$ is no longer invariant, but there exists a unique invariant and ergodic measure of the form $m_p \times \mu_{c,p}$ with $\mu_{c,p} \ll Leb$ a probability measure.

To obtain the speed of convergence, one needs a good expression for the density of $\mu_{c,p}$. In specific cases this can be computed.

Example: For $c = \frac{1}{8}$ and $0 < p < 1$ the density of $\mu_{p,\frac{1}{8}}$ is

$$f_{p,\frac{1}{8}}(x) = \frac{1}{2p^2 + 3p + 5} \begin{cases} 8, & \text{if } x \in [1/8, 1/4), \\ 4p + 4, & \text{if } x \in [1/4, 1/2), \\ 4p^2 + 2p + 4, & \text{if } x \in [1/2, 3/4), \\ 4p^2 + 6p + 4, & \text{if } x \in [3/4, 7/8), \\ 4p^2 + 6p + 12, & \text{if } x \in [7/8, 1). \end{cases}$$

Summary

	deterministic	random
number of expansions	one unique	uncountably many different, $c < \frac{2}{5}$ uncountably many universal, $c = \frac{1}{D}$ or $c = 0$
periodic expansions	periodic iff rational	many possibilities, all c
bound on digits	Hausdorff dimension smaller than 1	full interval
speed of convergence	typical value $-\sum_{d \geq 2} \frac{\log d(d-1)}{d(d-1)}$ range $(-\infty, -\log 2]$	typical value $-\sum_{d \geq 2} \frac{\log d(d-1)}{d(d-1)}$ range $(-\infty, -\log 2]$, $c = 0$
approximation coefficients	$M_A \leq M_\varepsilon \leq M_L$ not all values	$M_A \leq M_p \leq M_L$ all values, $c = 0$