An aperiodic monotile

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Basic terminology

A tile is a closed topological disk.

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a finite set of tiles. A tiling from $\mathcal{S}$ is a countable set $\mathcal{T} = \{T_1, T_2, \ldots\}$ such that

1. Every $T_i$ is congruent to a member of $\mathcal{S}$;
2. The interiors of the $T_i$ are pairwise disjoint; and
3. The union of the $T_i$ is the whole plane

We say that $\mathcal{S}$ admits $\mathcal{T}$. If $\mathcal{S}$ consists of a single shape, the tiling is called monohedral.
In the Euclidean plane, a tiling is **periodic** if its symmetry group includes at least two non-parallel translations, and **non-periodic** otherwise.
Non-periodicity is common: many sets of shapes admit both periodic and non-periodic tilings.
A set of tiles is aperiodic if it admits tilings, but none that are periodic.

Aperiodicity is a property of a set of tiles, and not of a tiling! The tiles conspire to prevent periodicity.
Wang tiles

Wang [1961] conjectured that there are no aperiodic sets of (Wang) tiles.

Clearly, a sufficient condition for a set of plates to have a solution is that there exists a cyclic rectangle of the plates.

What appears to be a reasonable conjecture, which has resisted proof or disproof so far, is:

4.1.2 The fundamental conjecture: A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.
Aperiodic Wang tiles

Berger [1966] exhibited an aperiodic set of 20426 Wang tiles (and remarked that smaller sets were possible).

Berger brought the total down to 104; Knuth [1968] managed 92.
Minimal aperiodic Wang tiles


Jeandel and Rao [2021] found a set of size 11 and proved that this was minimal.
Robinson tiles

Robinson [1971] gave an aperiodic set of six tiles.
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The red markings on the tiles must form a pattern of squares of unbounded sizes. Each size of square repeats periodically without overlap. Aperiodicity follows.
Penrose tiles

Penrose [1974] gave two aperiodic sets of size two.

P2, the "kite and dart"
Substitution rules (and the Extension Theorem) show that the kite and dart admit (non-periodic) tilings of the plane.
To show that every tiling by kites and darts is non-periodic, show that tiles can uniquely be composed into "supertiles".
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The quest for an einstein

Since the 1970s, several other small aperiodic sets were discovered by Ammann, Goodman-Strauss, and others.

Does there exist an aperiodic set of size one, AKA an aperiodic monotile, AKA an "einstein"?

Grünbaum and Shephard [1987]: "Though the existence of such a tile may appear unlikely, one must remember that only a few years ago, the existence of aperiodic sets containing just two tiles seemed essentially impossible."
The Socolar-Taylor tile

Socolar and Taylor [2011] presented an aperiodic hexagon!
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Black markings must pass continuously across tile edges
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Flag orientations must agree at an edge's endpoints
The Socolar-Taylor tile
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Variations of the Socolar-Taylor tile can express its matching conditions geometrically.
David Smith
...shape hobbyist
November 2022

The "hat"
November 2022

David emailed me out of the blue: "It has a Heesch number of at least three, if it's a non-tiler (I couldn't get it to tile periodically)."
Measures of disorderliness

Let $T$ be a tile.

If $T$ admits periodic tilings, then the **isohedral number of $T$** is the minimum number of transitivity classes in any of those tilings.

- A rough measure of a tiler's disorderliness.

If $T$ does not admit tilings, then the **Heesch number of $T$** is the maximum number of rings of copies of $T$ that can surround it.

- A rough measure of a non-tiler's disorderliness.
- In my work [2022] I computed Heesch numbers of unmarked polyforms, finding examples up to 4. Current record is 6 [Bašić 2021]
David asked whether my Heesch number software could work with kites (or drafters). Thanks to recent joint work with Ava Pun, it could.
Metatiles

We define four metatiles by observation of recurring patterns in computer-generated patches.
Supertiles

Metatiles beget combinatorially (but not geometrically) equivalent supertiles.
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Iterate this construction to build patches of any size. Thus the hat tiles the plane!

…but is it aperiodic?
Forcing non-periodicity

To complete a proof of aperiodicity, we must show that no tiling by the hat can be periodic.

Past aperiodic sets were generally engineered with matching conditions that facilitate a Berger-style proof of forced non-periodicity. The hat was found "in the wild".

With all the complexity baked into a single shape, a case-based analysis seems daunting.

Happily, the metatiles can help. So can Chaim and Joseph, who joined David and Craig in January!
Forcing non-periodicity

Use the metatiles as an intermediate step to manage complexity.

Step 1: Prove that in any tiling by hats, the tiles are forced to cluster into metatiles.

Step 2: Prove that the matching conditions on the metatiles force them to assemble into larger, combinatorially equivalent copies of themselves.
Surroundable 2-patches

An *n-patch* is a patch of tiles consisting of a tile surrounded by *n* rings of copies.

A *surroundable 2-patch* is a 2-patch that lies in the interior of any 3-patch.
We use software to enumerate the 188 surroundable 2-patches of hats, which loosely simulate neighbourhoods of real tilings.
Validity of clustering

With software we can check that for every surroundable 2-patch,

1. The interior hats can be assigned deterministic identities within metatiles; and
2. Implied metatile boundaries between interior tiles obey prescribed matching conditions.

Thus any hat tiling has a legal decomposition into metatiles.
Assembly of supertiles

Build a big tree of clusters of metatiles, prove that the only legal results are supertiles with combinatorially equivalent matching conditions.
A second einstein!

In December, David had found a second unusual shape (the "turtle").
Joseph: "Tile B is also aperiodic. In fact, we have an infinite family of aperiodic 13-gon tiles, determined by a parameter that can be any positive real except maybe 1…"

5 February

Joseph: "I also now have an outline that *might* work for showing nonexistence of a (strongly) periodic tiling based on the coupling between two polyiamond tilings…"

21 February
The continuum

Hat edges come in two lengths in $\sqrt{3}$ proportion. Every edge has a parallel partner. So we can adjust the two lengths independently to produce a continuum of tile shapes denoted $\text{Tile}(a, b)$ for $a, b \geq 0$.

The tiles $\text{Tile}(0, 1)$ (the "chevron"), $\text{Tile}(1, 1)$, and $\text{Tile}(1, 0)$ (the "comet") admit periodic tilings. All others are aperiodic monotiles with combinatorially equivalent tilings.
La première tuile aperiodique de l'histoire! The Hat - Passe-science #53
Suppose that a tiling by hats were periodic.
There would be corresponding periodic tilings by chevrons and comets.
The affine transformation \( g \) mapping the lattice of translations of the chevron tiling to that of the comet tiling cannot be a similarity: it must scale areas by 2/3, which would scale lengths by an impossible amount.
But by taking into account the distribution of tile orientations in all three tilings, we derive an explicit similarity between the translational lattices, a contradiction.

Therefore, the original tiling by hats could not have been periodic.
Conclusion

The hat is an aperiodic monotile with an unusual origin story.

We provide a "standard" combinatorial proof of aperiodicity, and a new indirect geometric proof.

Related problems for future work:

- Are there "simpler" aperiodic monotiles?
- Is there a chiral aperiodic monotile?
- Is there a 3D aperiodic monotile?
- Are there bounds on isohedral numbers or Heesch numbers?
- Is the tiling problem undecidable for a single tile in the plane?
- Is the periodic tiling problem undecidable for a single tile in the plane?
Thank you!

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