On Hermite’s problem, Jacobi-Perron type algorithms, and Dirichlet groups

Oleg Karpenkov, University of Liverpool

7 September 2021
I. Hermite’s problem

II. Euclidean algorithm and its generalisations

III. New algorithms

IV. Application to Dirichlet groups

V. Idea of the proof
I. Hermite’s problem
On periodicity of algebraic numbers

Cubic numbers (roots of cubic integer polynomials):

Hermite's problem (1848):
Find a periodic description of cubic numbers.

Periodic modifications of the J-P algorithm.
— O.K. (2021) based on the geometry of c.f.
Rational numbers: periodic or finite decimal representations.

\[
\frac{4}{3} = 1.3333333\ldots
\]
Rational numbers: periodic or finite decimal representations.

\[ \frac{4}{3} = 1.3333333 \ldots \]

Quadratic numbers (roots of quadratic integer polynomials): periodic continued fractions (J.-L. Lagrange, 1770).

\[ \frac{10 + 3\sqrt{7}}{4} = [4; (2, 15, 2, 3)]. \]
Rational numbers: periodic or finite decimal representations.

\[ \frac{4}{3} = 1.3333333 \ldots \]

Quadratic numbers (roots of quadratic integer polynomials): periodic continued fractions (J.-L. Lagrange, 1770).

\[ \frac{10 + 3\sqrt{7}}{4} = [4; (2, 15, 2, 3)] \]

Cubic numbers (roots of cubic integer polynomials):
Cubic numbers (roots of cubic integer polynomials):
Hermite’s problem (1848): *Find a periodic description of cubic numbers.*
Cubic numbers (roots of cubic integer polynomials):
Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

First steps.
— C. G. J. Jacobi (1868): algorithmic approach
— O. Perron (1907): first realisation of algorithmic approach, which is believed to be non-periodic.
Cubic numbers (roots of cubic integer polynomials): Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

**Development of geometry of numbers.**
— F. Klein, V. Arnold, A. Veselov, E. Korkina, O. German, A.Ustinov, O.K., etc.: geometric approach to continued fractions.

**Remark:** geometry of numbers was limitedly used (overlooked) for multidimensional Euclidaen algorithms.
Cubic numbers (roots of cubic integer polynomials):
Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

Periodic modifications of the J-P algorithm.
— O.K. (2021) based on the geometry of c.f.:
Cubic numbers (roots of cubic integer polynomials):
Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

Periodic modifications of the J-P algorithm.
— O.K. (2021) based on the geometry of c.f.:

▶ **Totally real case:** periodicity is *proved* for sin$^2$-algorithm.
Cubic numbers (roots of cubic integer polynomials):
Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

Periodic modifications of the J-P algorithm.
— O.K. (2021) based on the geometry of c.f.:

- **Totally real case:** periodicity is **proved** for sin²-algorithm.
- **Complex case:** heuristic APD-algorithm that provides periodicity (**no proof**).
Cubic numbers (roots of cubic integer polynomials): Hermite’s problem (1848): *Find a periodic description of cubic numbers.*

Periodic modifications of the J-P algorithm.
— O.K. (2021) based on the geometry of c.f.:

- **Totally real case**: periodicity is *proved* for sin$^2$-algorithm.
- **Complex case**: heuristic APD-algorithm that provides periodicity (*no proof*).
- **Algebraic case of degree $> 3$**: higher-dimensional heuristic APD-algorithm provides periodicity (*no proof*).
II. Euclidean algorithm and its generalisations
Euclid’s algorithm (∼ 300 BC)

Extended Euclid’s algorithm

**Input**: real numbers \((p, q) = (p_0, q_0)\) such that \(q_0 > 0\).
Euclid’s algorithm (∼ 300 BC)

Extended Euclid’s algorithm

**Input:** real numbers \((p, q) = (p_0, q_0)\) such that \(q_0 > 0\).

**Step of the algorithm:** If \(q_i \geq 0\):

\[
(p_i, q_i) \mapsto (p_{i+1}, q_{i+1}) = (q_i, p_i - \lfloor p_i/q_i \rfloor q_i)
\]

Termination of the algorithm: \((p_i, q_i)\) with \(q_i = 0\).

Oleg Karpenkov, University of Liverpool

On Hermite’s problem
Extended Euclid’s algorithm

**Input:** real numbers \((p, q) = (p_0, q_0)\) such that \(q_0 > 0\).

**Step of the algorithm:** If \(q_i \geq 0\):

\[
(p_i, q_i) \mapsto (p_{i+1}, q_{i+1}) = (q_i, p_i - \lfloor p_i/q_i \rfloor q_i)
\]

\[a_i = \lfloor p_i/q_i \rfloor — \text{the } i\text{-th element } \text{of the algorithm.}\]
Euclid’s algorithm (∼ 300 BC)

**Extended Euclid’s algorithm**

**Input:** real numbers \((p, q) = (p_0, q_0)\) such that \(q_0 > 0\).

**Step of the algorithm:** If \(q_i \geq 0\):

\[
(p_i, q_i) \mapsto (p_{i+1}, q_{i+1}) = (q_i, p_i - \lfloor p_i/q_i \rfloor q_i)
\]

\(a_i = \lfloor p_i/q_i \rfloor \) — the \(i\)-th element of the algorithm.

**Termination of the algorithm:** \((p_i, q_i)\) with \(q_i = 0\).
Euclid’s algorithm (≈ 300 BC)

Example
For $(21, 15)$ we have:

$\begin{align*}
(21, 15) &\mapsto (15, 6) \\
(15, 6) &\mapsto (6, 3) \\
(6, 3) &\mapsto (3, 0).
\end{align*}$

Note that $\gcd(21, 15) = 3$ and $21 - 15 = 1 + 1 \cdot 2 + 1 \cdot 2$.
Euclid’s algorithm (∼ 300 BC)

Example
For \((21, 15)\) we have:

\[(21, 15) \rightarrow (15, 6) \rightarrow (6, 3) \rightarrow (3, 0).\]

Output:

\[a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_3 = 2.\]
Euclid’s algorithm (∼ 300 BC)

Example
For (21, 15) we have:

\[(21, 15) \mapsto (15, 6) \mapsto (6, 3) \mapsto (3, 0).\]

Output:

\[a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_3 = 2.\]

Note that

\[\gcd(21, 15) = 3 \quad \text{and} \quad \frac{21}{15} = 1 + \frac{1}{2 + \frac{1}{2}}.\]
Euclid’s algorithm (∼ 300 BC)

Example
Now for $(2\sqrt{5}, 1)$:

$$(2\sqrt{5}, 1) \mapsto c_1(1 + \sqrt{5}/2, 1) \mapsto c_2(4 + 2\sqrt{5}, 1) \mapsto c_3(1 + \sqrt{5}/2, 1) \mapsto \ldots$$

where

$$c_1 = 2\sqrt{5} - 4, \quad c_2 = 9 - 4\sqrt{5}, \quad c_3 = 34\sqrt{5} - 76, \quad \ldots$$
Example

Now for \((2\sqrt{5}, 1)\):

\[(2\sqrt{5}, 1) \mapsto c_1(1+\sqrt{5}/2, 1) \mapsto c_2(4+2\sqrt{5}, 1) \mapsto c_3(1+\sqrt{5}/2, 1) \mapsto \ldots \]

where

\[c_1 = 2\sqrt{5} - 4, \quad c_2 = 9 - 4\sqrt{5}, \quad c_3 = 34\sqrt{5} - 76, \quad \ldots \]

The vectors obtained on Step 1 and Step 3 are proportional.
Euclid’s algorithm (∼ 300 BC)

Example
Now for $(2\sqrt{5}, 1)$:

$$(2\sqrt{5}, 1) \mapsto c_1(1+\sqrt{5}/2, 1) \mapsto c_2(4+2\sqrt{5}, 1) \mapsto c_3(1+\sqrt{5}/2, 1) \mapsto \ldots$$

where

$$c_1 = 2\sqrt{5} - 4, \quad c_2 = 9 - 4\sqrt{5}, \quad c_3 = 34\sqrt{5} - 76, \quad \ldots$$

The vectors obtained on Step 1 and Step 3 are proportional. Hence the output is periodic:

$$a_1 = 4, \quad a_{2k} = 2, \quad \text{and} \quad a_{2k+1} = 8$$

So $2\sqrt{5} = [4; 2 : 8 : 2 : 8 : 2 : 8 : \ldots] = [4; (2 : 8)]$. 

Oleg Karpenkov, University of Liverpool

On Hermite’s problem
Jacobi-Perron algorithm (1907)

**Input:** triples of real numbers \((x, y, z)\).
Jacobi-Perron algorithm (1907)

**Input:** triples of real numbers \((x, y, z)\).

**Step of the algorithm:** If \(y_i \neq 0\).

\[
(x_i, y_i, z_i) \mapsto (x_{i+1}, y_{i+1}, z_{i+1}) = \left( y_i, z_i - \left\lfloor \frac{z_i}{y_i} \right\rfloor y_i, x_i - \left\lfloor \frac{x_i}{y_i} \right\rfloor y \right).
\]
Jacobi-Perron algorithm

**Input:** triples of real numbers \((x, y, z)\).

**Step of the algorithm:** If \(y_i \neq 0\).

\[
(x_i, y_i, z_i) \mapsto (x_{i+1}, y_{i+1}, z_{i+1}) = \left( y_i, z_i - \left\lfloor \frac{z_i}{y_i} \right\rfloor y_i, x_i - \left\lfloor \frac{x_i}{y_i} \right\rfloor y \right).
\]

The \textit{i-th element} of the multidimensional continued fraction is

\[
\left( \left\lfloor \frac{z_i}{y_i} \right\rfloor, \left\lfloor \frac{x_i}{y_i} \right\rfloor \right).
\]
Jacobi-Perron algorithm (1907)

Input: triples of real numbers \((x, y, z)\).

Step of the algorithm: If \(y_i \neq 0\).

\((x_i, y_i, z_i) \mapsto (x_{i+1}, y_{i+1}, z_{i+1}) = \left(y_i, z_i - \left\lfloor \frac{z_i}{y_i} \right\rfloor y_i, x_i - \left\lfloor \frac{x_i}{y_i} \right\rfloor y_i \right)\).

The \(i\)-th element of the multidimensional continued fraction is

\[ \left( \left\lfloor \frac{z_i}{y_i} \right\rfloor , \left\lfloor \frac{x_i}{y_i} \right\rfloor \right). \]

Termination of the algorithm: \((x_i, y_i, z_i)\) with \(y_i = 0\).
Let $\xi$ be a real root of the polynomial $x^3 + 2x^2 + x + 4$, namely

$$
\xi = -\frac{(53 + 6\sqrt{78})^{1/3}}{3} - \frac{1}{3(53 + 6\sqrt{78})^{1/3}} - \frac{2}{3}.
$$
Let $\xi$ be a real root of the polynomial $x^3 + 2x^2 + x + 4$, namely

$$
\xi = -\frac{(53 + 6\sqrt{78})^{1/3}}{3} - \frac{1}{3(53 + 6\sqrt{78})^{1/3}} - \frac{2}{3}.
$$

Now consider the vector (not a single number, which is senseless! One can have different periods with the same number)

$$(1, \xi, \xi^2 + \xi).$$
Let $\xi$ be a real root of the polynomial $x^3 + 2x^2 + x + 4$, namely

$$
\xi = -\frac{(53 + 6\sqrt{78})^{1/3}}{3} - \frac{1}{3(53 + 6\sqrt{78})^{1/3}} - \frac{2}{3}.
$$

Now consider the vector (not a single number, which is senseless! One can have different periods with the same number)

$$(1, \xi, \xi^2 + \xi).$$

Jacobi-Perron algorithm periodic output:

<table>
<thead>
<tr>
<th>$[x/y]$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$2k + 1$</th>
<th>$2k + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[z/y]$</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Problem

(Jacobi’s Last Theorem.) Let $K$ be a totally real cubic field. Let $y, z \in K$ satisfy

- $0 < y, z < 1$;
- $1, y, \text{ and } z$ are independent over $\mathbb{Q}$.

Does Jacobi-Perron algorithm generate an eventually periodic continued fraction starting with $\nu = (1, y, z)$?
Problem

(Jacobi’s Last Theorem.) Let $K$ be a totally real cubic field. Let $y, z \in K$ satisfy

- $0 < y, z < 1$;
- $1, y, and z$ are independent over $\mathbb{Q}$.

Does Jacobi-Perron algorithm generate an eventually periodic continued fraction starting with $v = (1, y, z)$?

It is believed that the answer is negative.

(I have learnt this fact from Cor Kraaikamp in 2006 in Leiden.)
Example
Let us consider the vector

\[ \mathbf{v} = (1, 3\sqrt{4}, 3\sqrt{16}). \]

Numerical computations (e.g., by L. Elsner and H. Hasse., 1967) shows

\[ \left\lfloor \frac{x}{y} \right\rfloor \]
\[ \left\lfloor \frac{z}{y} \right\rfloor \]

\[
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots & 94 \\
[0 \ 1 \ 13 \ 1 \ 6 \ 1 \ 1 \ 3 \ 2 \ 3 \ 4 \ 1 \ \ldots & 476 \\
[1 \ 1 \ 9 \ 1 \ 2 \ 0 \ 0 \ 2 \ 0 \ 1 \ 1 \ 1 \ \ldots & 388
\end{array}
\]
Example
Let us consider the vector

\[ \mathbf{v} = (1, 3\sqrt{4}, 3\sqrt{16}). \]

Numerical computations (e.g., by L. Elsner and H. Hasse., 1967) shows

\[
\begin{array}{cccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots & 94 \\
\hline
\lfloor x/y \rfloor & 0 & 1 & 13 & 1 & 6 & 1 & 1 & 3 & 2 & 3 & 4 & 1 & \ldots & 476 \\
\lfloor z/y \rfloor & 1 & 1 & 9 & 1 & 2 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & \ldots & 388 \\
\end{array}
\]

Compare:

\[ \pi = [3 : 7; 15; 1; 292; 1; 1; 2; 1; 3; 1; 14; 2; 1; 1; 2; 2; 2; 2; 1; 84; \ldots ] \]
Gauss-Kuzmin statistics:

$$\frac{1}{\ln(2)} \ln \left(1 + \frac{1}{k(k+1)}\right).$$
Gauss-Kuzmin statistics:

\[ \frac{1}{\ln(2)} \ln \left( 1 + \frac{1}{k(k+1)} \right). \]

Observe:

\[ \frac{1}{\ln(2)} \ln \left( 1 + \frac{1}{k(k+1)} \right) = \frac{\ln[-1, 0, k, k+1]}{\ln[-1, 0, 1, \infty]}. \]

**Hint for us:** this question involves geometry.
The same problem arise with the other Jacobi-Perron type algorithms:

- V. Brun (subtractive algorithm) 1958
- E. S. Selmer (general subtractive algorithm) 1961
- F. Schweiger (fully subtractive algorithm) 1995

etc.
III. New algorithms
Definition
Consider \( u, v, w \in \mathbb{C}^3 \):

\[
u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3), \quad \text{and} \quad w = (w_1, w_2, w_3)
\]
Markov-Davenport Characteristic

Definition
Consider \( u, \nu, w \in \mathbb{C}^3 \):

\[
\begin{align*}
\quad & u = (u_1, u_2, u_3), \quad \nu = (\nu_1, \nu_2, \nu_3), \quad \text{and} \quad w = (w_1, w_2, w_3).
\end{align*}
\]

The Markov-Davenport characteristic is the form \( \chi_{u,\nu,w} \):

\[
\begin{align*}
\det \begin{pmatrix}
x & y & z \\
\nu_1 & \nu_2 & \nu_3 \\
w_1 & w_2 & w_3
\end{pmatrix}
\cdot \det \begin{pmatrix}
u_1 & \nu_2 & \nu_3 \\
u_1 & \nu_2 & \nu_3 \\
\end{pmatrix}
\cdot \det \begin{pmatrix}
x & y & z \\
x & y & z \\
\end{pmatrix}
\end{align*}
\]

in variables \( x, y, \) and \( z \) (with respect to \( u, \nu, w \)).
Markov-Davenport Characteristic

Definition
Consider \( u, v, w \in \mathbb{C}^3 \):

\[
\begin{align*}
u &= (u_1, u_2, u_3), \\
v &= (v_1, v_2, v_3), \\
w &= (w_1, w_2, w_3)
\end{align*}
\]

The Markov-Davenport characteristic is the form \( \chi_{u,v,w} \):

\[
\det \begin{pmatrix} x & y & z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \det \begin{pmatrix} u_1 & u_2 & u_3 \\ x & y & z \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ x & y & z \end{pmatrix}
\]

in variables \( x, y, \) and \( z \) (with respect to \( u, v, w \)).

Remark: MD-characteristic provides a “proper distance” to the cone generated by \( u, v, \) and \( w \).
JP-algorithm searching for the nearest lattice point in a 2-plane $y = 1$. 

Here are the “closest” integer points w.r.t. MD-characteristics:
Markov-Davenport Characteristic

- JP-algorithm searching for the nearest lattice point in a 2-plane \( y = 1 \).
- While a proper point could be different.
Markov-Davenport Characteristic

- JP-algorithm searching for the nearest lattice point in a 2-plane $y = 1$.
- While a proper point could be different.
- Here are the “closest” integer points w.r.t. MD-characteristics:

![Diagram of lattice points in a 2-plane with y = 1]
Heuristic algebraic periodicity detecting algorithm

**Input:** a triple of real vectors \((\xi, \nu, \mu)\) where \(\xi = (x_0, y_0, z_0)\) with \(z_0 > 0\).

**Step 0:** Similar to JP-algorithm:

\[
T_0 : (x, y, z) \mapsto \left( x - \left\lfloor \frac{x}{z} \right\rfloor z, y - \left\lfloor \frac{y}{z} \right\rfloor z, z \right),
\]

namely we consider

\[
(\xi_1, \nu_1, \mu_1) = (T_0(\xi), T_0(\nu), T_0(\mu)).
\]
Heuristic algebraic periodicity detecting algorithm

Heuristic APD-algorithm

**Step i for** \( i \geq 1 \): We have \((\xi_i(x_i, y_i, z_i), \nu_i, \mu_i)\) with \(\xi_i \in \mathbb{R}_+^3\).
Step $i$ for $i \geq 1$: We have $(\xi_i(x_i, y_i, z_i), \nu_i, \mu_i)$ with $\xi_i \in \mathbb{R}_+^3$.

- **Stage 1: Determination of the element of the CF:**
  - we pick $(a_i, b_i)$ satisfying:
    - $0 \leq a_i \leq \lfloor x_i/z_i \rfloor$;
    - $0 \leq b_i \leq \lfloor y_i/z_i \rfloor$;
    - $(a_i, b_i) \neq (0, 0)$;
    - (exception: $(0, 0)$ if $\lfloor x_i/z_i \rfloor < 1$ and $\lfloor y_i/z_i \rfloor < 1$).
Heuristic algebraic periodicity detecting algorithm

Heuristic APD-algorithm

**Step** \( i \) for \( i \geq 1 \): We have \((\xi_i(x_i, y_i, z_i), \nu_i, \mu_i)\) with \( \xi_i \in \mathbb{R}^3_+ \).

- **Stage 1: Determination of the element of the CF:**
  we pick \((a_i, b_i)\) satisfying:
  - \(0 \leq a_i \leq \lfloor x_i/z_i \rfloor\);
  - \(0 \leq b_i \leq \lfloor y_i/z_i \rfloor\);
  - \((a_i, b_i) \neq (0, 0)\)
  (exception: \((0, 0)\) if \(\lfloor x_i/z_i \rfloor < 1\) and \(\lfloor y_i/z_i \rfloor < 1\));

  such that the triple \((a_i, b_i, 1)\) has minimal possible abs. value of the MD-characteristic \(|\chi_{\xi, \nu, \mu}|\).
Heuristic algebraic periodicity detecting algorithm

Heuristic APD-algorithm

**Step \( i \) for \( i \geq 1 \):** We already have \((\xi_i, \nu_i, \mu_i)\) with \(\xi_i = (x_i, y_i, z_i) \in \mathbb{R}_+^3\) and \((a_i, b_i)\).

▶ **Stage 2: Iteration step:**

\[
T_i : (x, y, z) \mapsto (y - b_i z, z, x - a_i z).
\]

Here we construct

\[
(\xi_{i+1}, \nu_{i+1}, \mu_{i+1}) = (T_i(\xi_i), T_i(\nu_i), T_i(\mu_i))
\]
Step \( i \) for \( i \geq 1 \): We already have \((\xi_i, \nu_i, \mu_i)\) with \(\xi_i = (x_i, y_i, z_i) \in \mathbb{R}^3_+\) and \((a_i, b_i)\).

Stage 2: Iteration step:

\[
T_i : (x, y, z) \mapsto (y - b_i z, z, x - a_i z).
\]

Here we construct

\[
(\xi_{i+1}, \nu_{i+1}, \mu_{i+1}) = (T_i(\xi_i), T_i(\nu_i), T_i(\mu_i))
\]

Termination of the algorithm: when \(z_i = 0\).
Conjecture
Heuristic APD-algorithm is periodic for all cubic triples of vectors.
Conjecture
Heuristic APD-algorithm is periodic for all cubic triples of vectors.

Remark
A triple of cubic vectors
“=”
a triple of linearly independent eigenvectors for an integer matrix (and irreducible characteristic polynomial).
Heuristic algebraic periodicity detecting algorithm

Conjecture
Heuristic APD-algorithm is periodic for all cubic triples of vectors.

Remark
A triple of cubic vectors
“=”
a triple of linearly independent eigenvectors for an integer matrix
(and irreducible characteristic polynomial).

Remark
APD algorithm is defined for both totally real and complex cases.

Remark
APD algorithm has a straightforward generalisation to higher dimensional cases.
Heuristic algebraic periodicity detecting algorithm

Consider:

\[ \xi = (1, \sqrt[3]{4}, \sqrt[3]{16}) = (1, \sqrt[3]{4}, (\sqrt[3]{4})^2). \]

Note that \( \sqrt[3]{4} \) is a root of \( x^3 - 4 \).
Consider:

$$\xi = (1, \sqrt[3]{4}, \sqrt[3]{16}) = (1, \sqrt[3]{4}, (\sqrt[3]{4})^2).$$

Note that $\sqrt[3]{4}$ is a root of $x^3 - 4$.

Let $\beta$ and $\gamma$ be the complex roots of $x^3 - 4$. Consider:

$$\nu = (1, \beta, \beta^2) \quad \text{and} \quad \mu = (1, \gamma, \gamma^2).$$
Heuristic algebraic periodicity detecting algorithm

Consider:

$$\xi = (1, \sqrt[3]{4}, \sqrt[3]{16}) = (1, \sqrt[3]{4}, (\sqrt[3]{4})^2).$$

Note that $\sqrt[3]{4}$ is a root of $x^3 - 4$.

Let $\beta$ and $\gamma$ be the complex roots of $x^3 - 4$. Consider:

$$\nu = (1, \beta, \beta^2) \quad \text{and} \quad \mu = (1, \gamma, \gamma^2).$$

The output of the heuristic APD-algorithm for the triple $(\xi, \nu, \mu)$ is periodic:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>4$k$ + 3</th>
<th>4$k$ + 4</th>
<th>4$k$ + 5</th>
<th>4$k$ + 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
In a few words:
the sin\(^2\)-algorithm repeats heuristic APD-algorithm but it works with sin\(^2\) \(\alpha(u, v, w)\), where \(\alpha(u, v, w)\) is the angle between the planes spanned by \((u, v)\) and by \((u, w)\).
**sin^2-algorithm (skipping technical details)**

**In a few words:**
the sin^2-algorithm repeats heuristic APD-algorithm but it works with sin^2 α(u, v, w), where α(u, v, w) is the angle between the planes spanned by (u, v) and by (u, w).

The sin^2-algorithm is proven to be periodic for algebraic triples.
In a few words:
the sin\(^2\)-algorithm repeats heuristic APD-algorithm but it works with \(\sin^2 \alpha(u, v, w)\), where \(\alpha(u, v, w)\) is the angle between the planes spanned by \((u, v)\) and by \((u, w)\).

The sin\(^2\)-algorithm is proven to be periodic for algebraic triples.

Remark
It is defined for the totally real case (when all \(u, v\) and \(w\) are real vectors).
**Input:** We are given three vectors $\xi, \nu, \mu$ such that
- $\xi(x, y, z)$ satisfy $x > y > z > 0$;,
- all coordinates for $\nu$ and $\mu$ are neither simultaneously positive nor simultaneously negative.

**Step of the algorithm:** Let us apply the following linear transformation

$$(\xi_i, \nu_i, \mu_i) \rightarrow (\Phi_i(\xi_i), \Phi_i(\nu_i), \Phi_i(\mu_i))$$

with

$$\Phi_i = T_i M_i.$$
\[ \Phi_i = T_i M_i. \]

Here \( M_i \) is taken to be the minimiser of the value of \( \sin^2 \). The minimisation is done among all the transformations

\[ N_{\alpha, \beta, \gamma} : (x, y, z) \mapsto (x - \alpha z - \gamma(y - \beta z), y - \beta z, z) \]

with

\[ 0 \leq \alpha \leq \left\lfloor \frac{x_i}{z_i} \right\rfloor, \quad 0 \leq \beta \leq \left\lfloor \frac{y_i}{z_i} \right\rfloor, \quad \text{and} \quad 0 \leq \gamma \leq \left\lfloor \frac{x_i / z_i - \alpha}{y_i / z_i - \beta} \right\rfloor, \]

and the transformation

\[ N_0 = (x, y, z) \mapsto (x - y, y, z - (x - y)), \]

which is considered only in case \( z_i > x_i - y_i > 0 \).
**sin²-algorithm**

\[ \Phi_i = T_i M_i. \]

Set \( T_i \) as a basis permutation that puts the coordinates of \( M_i(\xi) \) in decreasing order.

At each step the algorithm returns \( \Phi_i \) as an output.

**Termination of the algorithm:** In the case that the last coordinate of \( \xi_i \) is zero (i.e. \( z_i = 0 \)).
IV. Application to Dirichlet groups
Example

Let

\[ A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & 6 & 1 \\ 4 & 7 & 1 \end{pmatrix}. \]

Find an integer matrix with unit determinant commuting with \( A \)?

Let us first peek the answer to this question.

\[ B = \begin{pmatrix} 8 & 87778750433916 & 1881948516620816 \\ -1642359549748757 & -77918418013751 \\ -849278651461089 & 759124773173459 \\ 534000559063825 & -721564227716990 \\ 360094549931638 \end{pmatrix}. \]

Brute force algorithm does not work...

Even if one notices that

\[ B = -147205796095883 A^2 + 1347947957556991 A - 399030223241821. \]

Note: Input is nine 4-bit elements.
Example

Let

\[
A = \begin{pmatrix}
2 & 5 & -1 \\
3 & 6 & 1 \\
4 & 7 & 1
\end{pmatrix}.
\]

Find an integer matrix with unit determinant commuting with \(A\)?

Let us first peek the answer to this question.

\[
B = \begin{pmatrix}
88778750433916 & 1881948516620816 \\
-1642359549748757 & -77918418013751 \\
-849278651461089 & 759124773173459 \\
534000559063825 & -721564227716990 & 360094549931638
\end{pmatrix}
\]

Brute force algorithm does not work...

Even if one notices that

\[
B = -147205796095883 A^2 + 1347947957556991 A - 399030223241821
\]

Note: Input is nine 4-bit elements.
Example

Let

$$A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & 6 & 1 \\ 4 & 7 & 1 \end{pmatrix}.$$ 

Find an integer matrix with unit determinant commuting with $A$?

Let us first peek the answer to this question.

$$B = \begin{pmatrix} 88778750433916 & 1881948516620816 & -1642359549748757 \\ -77918418013751 & -849278651461089 & 759124773173459 \\ 534000559063825 & -721564227716990 & 360094549931638 \end{pmatrix}.$$ 

Note: Input is nine 4-bit elements.
Example

Let

\[
A = \begin{pmatrix}
2 & 5 & -1 \\
3 & 6 & 1 \\
4 & 7 & 1
\end{pmatrix}.
\]

Find an integer matrix with unit determinant commuting with \(A\)?

Let us first peek the answer to this question.

\[
B = \begin{pmatrix}
88778750433916 & 1881948516620816 & -1642359549748757 \\
-77918418013751 & -849278651461089 & 759124773173459 \\
534000559063825 & -721564227716990 & 360094549931638
\end{pmatrix}
\]

Brute force algorithm does not work...
Example

Let

\[ A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & 6 & 1 \\ 4 & 7 & 1 \end{pmatrix}. \]

Find an integer matrix with unit determinant commuting with \( A \)?

Let us first peek the answer to this question.

\[ B = \begin{pmatrix} 88778750433916 & 1881948516620816 & -1642359549748757 \\ -77918418013751 & -849278651461089 & 759124773173459 \\ 534000559063825 & -721564227716990 & 360094549931638 \end{pmatrix}. \]

Brute force algorithm does not work... even if one notices that

\[ B = -147205796095883A^2 + 1347947957556991A - 399030223241821 \]

Note: Input is nine 4-bit elements.
Definition. $A \in \text{SL}(n, \mathbb{R})$. $\Gamma(A)$ — all integer matrices commuting with $A$.

(i) The Dirichlet group $\Xi(A)$ — all invertible matrices in $\Gamma(A)$.

(ii) The positive Dirichlet group $\Xi_+(A) \subset \Xi(A)$: only positive real eigenvalues.
**Definition.** $A \in \text{SL}(n, \mathbb{R})$. $\Gamma(A)$ — all integer matrices commuting with $A$.

(i) The *Dirichlet group* $\Xi(A)$ — all invertible matrices in $\Gamma(A)$.

(ii) The *positive Dirichlet group* $\Xi_+(A) \subset \Xi(A)$: only positive real eigenvalues.

**Dirichlet’s unit theorem.** Let $K$ be a field of algebraic numbers of degree $n = s + 2t$, ($s$ — number of real roots; $2t$ — number of complex roots). Consider an arbitrary order $D$ in $K$. Then $D$ contains units $\varepsilon_1, \ldots, \varepsilon_r$ for $r = s + t - 1$ such that every unit $\varepsilon$ in $D$ has a unique decomposition of the form

$$\varepsilon = \xi \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r},$$

where $a_1, \ldots, a_r$ are integers and $\xi$ is a root of 1 contained in $D$. 
Definition. $A \in \text{SL}(n, \mathbb{R})$. $\Gamma(A)$ — all integer matrices commuting with $A$.

(i) The Dirichlet group $\Xi(A)$ — all invertible matrices in $\Gamma(A)$.

(ii) The positive Dirichlet group $\Xi_+(A) \subset \Xi(A)$: only positive real eigenvalues.

Dirichlet’s unit theorem in the matrix form. Let $A \in \text{SL}(n, \mathbb{Z})$ with irreducible characteristic ($s$ real and $2t$ complex eigenvalues).

Then there exists a finite Abelian group $G$ such that

\[
\Xi(A) \cong G \oplus \mathbb{Z}^{s+t-1}.
\]

\[
\Xi_+(A) \cong \mathbb{Z}^{s+t-1}.
\]

Oleg Karpenkov, University of Liverpool

On Hermite’s problem
Dirichlet groups, Dirichlet’s unit theorem

**Definition.** $A \in \text{SL}(n, \mathbb{R})$. $\Gamma(A)$ — all integer matrices commuting with $A$.

(i) The *Dirichlet group* $\Xi(A)$ — all invertible matrices in $\Gamma(A)$.

(ii) The *positive Dirichlet group* $\Xi_+(A) \subset \Xi(A)$: only positive real eigenvalues.

**Example**

In the three-dimensional case:

- **Complex case:** $s = 1, t = 1$ then

  \[ \Xi(A) \cong \Xi_+(A) \cong \mathbb{Z}. \]

- **Totally real case:** $s = 3, t = 0$. Then

  \[ \Xi(A) \cong G \oplus \mathbb{Z}^2 \quad \text{and} \quad \Xi_+(A) \cong \mathbb{Z}^2. \]
Question 1. Given $\xi$ — a cubic vector. Find $A \in \text{SL}(3, \mathbb{Z})$ such that

$$A\xi = \lambda\xi.$$
Questions that JP-type algorithms answer

**Question 1.** Given $\xi$ — a cubic vector.
Find $A \in \text{SL}(3, \mathbb{Z})$ such that

$$A\xi = \lambda\xi.$$ 

**Question 2.** Given $M \in \text{Mat}(3, \mathbb{Z})$ with irreducible characteristic polynomial over $\mathbb{Q}$.
Find an $A \in \text{SL}(3, \mathbb{Z})$ such that

$$AM = MA.$$ 

**Question 3.** (in totally real case).
Let $M \in \text{SL}(3, \mathbb{Z})$-matrix.
Find an $\text{SL}(3, \mathbb{Z})$-matrix commuting with $M$ that is not a power of $M$. 

Oleg Karpenkov, University of Liverpool
On Hermite’s problem
Questions that JP-type algorithms answer

**Question 1.** Given $\xi$ — a cubic vector. 
Find $A \in \text{SL}(3, \mathbb{Z})$ such that

$$A\xi = \lambda \xi.$$ 

**Question 2.** Given $M \in \text{Mat}(3, \mathbb{Z})$ with irreducible characteristic polynomial over $\mathbb{Q}$. 
Find an $A \in \text{SL}(3, \mathbb{Z})$ such that

$$AM = MA.$$ 

**Question 3. (in totally real case).** Let $M \in \text{SL}(3, \mathbb{Z})$-matrix. 
Find an $\text{SL}(3, \mathbb{Z})$-matrix commuting with $M$ that is not a power of $M$. 

Oleg Karpenkov, University of Liverpool
On Hermite’s problem
Answer to Question 1: Example 1

Consider \((1, \xi, \xi^2 + \xi)\) with \(\xi\) satisfying

\[x^3 + 2x^2 + x + 4 = 0.\]

The JP-algorithm generates a periodic sequence:
Consider \((1, \xi, \xi^2 + \xi)\) with \(\xi\) satisfying

\[x^3 + 2x^2 + x + 4 = 0.\]

The JP-algorithm generates a periodic sequence:

\[
\begin{array}{cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 2k + 1 & 2k + 2 \\
\lfloor x/y \rfloor & -1 & 1 & 1 & 1 & 2 & 6 & 3 & 7 \\
\lfloor z/y \rfloor & -2 & 0 & 0 & 0 & 2 & 4 & 1 & 1 \\
\end{array}
\]
Consider \((1, \xi, \xi^2 + \xi)\) with \(\xi\) satisfying

\[x^3 + 2x^2 + x + 4 = 0.\]

The JP-algorithm generates a periodic sequence:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2k + 1</th>
<th>2k + 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x/y)</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>(z/y)</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Matrix for the pre-period:

\[
M_1 = \begin{pmatrix}
-1 & 1 & 0 \\
1 & 0 & 0 \\
-2 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
2 & 1 & 0 \\
1 & 0 & 0 \\
2 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
6 & 1 & 0 \\
1 & 0 & 0 \\
4 & 0 & 1
\end{pmatrix}
\]

\[= \begin{pmatrix}
-22 & -1 & -3 \\
51 & 2 & 7 \\
-67 & -3 & -9
\end{pmatrix};\]
Answer to Question 1: Example 1

Consider \((1, \xi, \xi^2 + \xi)\) with \(\xi\) satisfying

\[ x^3 + 2x^2 + x + 4 = 0. \]

The JP-algorithm generates a periodic sequence:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2k + 1</th>
<th>2k + 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>([x/y])</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>([z/y])</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Matrix for the period:

\[
M_2 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 22 & 1 & 3 \\ 7 & 0 & 1 \\ 8 & 0 & 1 \end{pmatrix}.
\]
Consider \((1, \xi, \xi^2 + \xi)\) with \(\xi\) satisfying
\[
x^3 + 2x^2 + x + 4 = 0.
\]

The JP-algorithm generates a periodic sequence:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2k + 1</th>
<th>2k + 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>([x/y])</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>([z/y])</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Finally we get
\[
M = M_1 M_2 (M_1)^{-1} = \begin{pmatrix} 5 & -4 & 3 \\ -12 & 9 & -7 \\ 16 & -12 & 9 \end{pmatrix}.
\]

This concludes the computation of \(M\).
Answer to Question 1: Example 2

Consider $v = (1, 3\sqrt{4}, 3\sqrt{16})$.

Here Jacobi-Perron algorithm does not work. Heuristic APD-algorithm produces

$$
\begin{pmatrix}
5 & 8 & 12 \\
3 & 5 & 8 \\
2 & 3 & 5
\end{pmatrix}
$$

Oleg Karpenkov, University of Liverpool
Consider $v = (1, \sqrt[3]{4}, \sqrt[3]{16})$.
Here Jacobi-Perron algorithm does not work.
Answer to Question 1: Example 2

Consider \( v = (1, \sqrt[3]{4}, \sqrt[3]{16}) \).
Here Jacobi-Perron algorithm does not work.
Heuristic APD-algorithm produces

\[
M = \begin{pmatrix}
5 & 8 & 12 \\
3 & 5 & 8 \\
2 & 3 & 5 
\end{pmatrix}.
\]
Consider an irreducible cubic polynomial

\[ p(x) = 2x^3 - 4x^2 - 7x - 2 \]

with roots \( \alpha, \beta, \gamma \in \mathbb{R} \).
Consider an irreducible cubic polynomial

\[ p(x) = 2x^3 - 4x^2 - 7x - 2 \]

with roots \( \alpha, \beta, \gamma \in \mathbb{R} \).

Our goal is to compute two independent \( SL(3, \mathbb{Z}) \)-matrices with eigenvectors

\[ \xi = (1, \alpha, \alpha^2), \quad \nu = (1, \beta, \beta^2), \quad \text{and} \quad \mu = (1, \gamma, \gamma^2). \]
Consider an irreducible cubic polynomial

\[ p(x) = 2x^3 - 4x^2 - 7x - 2 \]

with roots \( \alpha, \beta, \gamma \in \mathbb{R} \).

Direct computations using the heuristic APD-algorithm applied to triples

\( (\xi, \nu, \mu), \ (\nu, \mu, \xi), \ \text{and} \ (\mu, \xi, \nu) \)

result in the following three matrices:

\[
A = \begin{pmatrix}
55 & 210 & 176 \\
176 & 671 & 562 \\
562 & 2143 & 1795
\end{pmatrix}; \quad
B = \begin{pmatrix}
-497 & -1122 & 400 \\
400 & 903 & -322 \\
-322 & -727 & 259
\end{pmatrix};
\]

\[
C = \begin{pmatrix}
185 & 172 & -72 \\
-72 & -67 & 28 \\
28 & 26 & -11
\end{pmatrix}
\]
Answer to Question 3: Example

Consider an irreducible cubic polynomial

\[ p(x) = 2x^3 - 4x^2 - 7x - 2 \]

with roots \( \alpha, \beta, \gamma \in \mathbb{R} \).

\[
A = \begin{pmatrix}
55 & 210 & 176 \\
176 & 671 & 562 \\
562 & 2143 & 1795
\end{pmatrix};
B = \begin{pmatrix}
-497 & -1122 & 400 \\
400 & 903 & -322 \\
-322 & -727 & 259
\end{pmatrix};
\]

\[
C = \begin{pmatrix}
185 & 172 & -72 \\
-72 & -67 & 28 \\
28 & 26 & -11
\end{pmatrix}.
\]

Brute force search of the powers of matrices show that

\[ A^3 B^5 C^7 = \text{Id}. \]
Answer to Question 3: Example

Consider an irreducible cubic polynomial

\[ p(x) = 2x^3 - 4x^2 - 7x - 2 \]

with roots \( \alpha, \beta, \gamma \in \mathbb{R} \).

Brute force search of the powers of matrices show that

\[ A^3 B^5 C^7 = \text{Id} \, . \]

All these matrices represent different eigenvectors with maximal eigenvalue, hence they there are two of them that are not powers of each other.
V. Idea of the proof
Given a basis $s_0(\xi, \nu_1, \nu_2)$. Then

$$s = ((x_0, y_0, 1), (x_1, y_1, 1), (x_2, y_2, 1))$$

is a $\xi$-state if in some $\mathbb{Z}^3$-basis the coordinates the vectors of $s$ are proportional to the vectors of $s_0$. 

Oleg Karpenkov, University of Liverpool

On Hermite's problem
Separating $\xi$-states

- Given a basis $s_0(\xi, \nu_1, \nu_2)$. Then

$$s = ((x_0, y_0, 1), (x_1, y_1, 1), (x_2, y_2, 1))$$

is a $\xi$-state if in some $\mathbb{Z}^3$-basis the coordinates the vectors of $s$ are proportional to the vectors of $s_0$.

- A $\xi$-state $(\xi(x_0, y_0, 1), \nu_1, \nu_2)$ is *separating* if
  - $\xi \in \mathbb{R}^3_+$ and $x_0 > y_0 > 1$
  - $\nu_1, \nu_2 \notin \mathbb{R}^3_+$.
Separating $\xi$-states

Given a basis $s_0(\xi, \nu_1, \nu_2)$. Then

$$s = ((x_0, y_0, 1), (x_1, y_1, 1), (x_2, y_2, 1))$$

is a $\xi$-state if in some $\mathbb{Z}^3$-basis the coordinates the vectors of $s$ are proportional to the vectors of $s_0$.

A $\xi$-state $(\xi(x_0, y_0, 1), \nu_1, \nu_2)$ is separating if

- $\xi \in \mathbb{R}_+^3$ and $x_0 > y_0 > 1$
- $\nu_1, \nu_2 \notin \mathbb{R}_+^3$.

Proposition (Nose sharpening.)

*If $s$ is a separating $\xi$-state, then $\Phi(s)$ is a separating $\xi$-state.*

(Here $\Phi(s)$ is the iterative step of sin$^2$-algorithm).
Main theorem

Let $s_0$ be the input separating state. Denote

$$\Omega_{\varepsilon}(s_0)$$

the set of all $\xi$-states $s$ where $\sin\alpha(s) > \varepsilon$.

$$\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$$

the set of all $\xi$-states $s$ satisfying $\sin^2\alpha(s) > \sin^2\alpha(\Phi(s))$.

Theorem

Let $s = (\xi, \nu_1, \nu_2)$ be three conjugate cubic vectors. Assume that $s$ is separating. Then:

(i) $\Omega_{\varepsilon}(\xi; \nu_1, \nu_2)$ is finite ($\forall \varepsilon > 0$).

(ii) $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ is finite.

Corollary

The dynamical system with a map $\Phi$ (on separating states) is periodic for triples of cubic conjugate vectors.

Remark:

There is a simple way to find the basis for $s$ in which it is separating.
Main theorem

Let $s_0$ be the input separating state. Denote

- $\Omega_\varepsilon(s_0)$ — the set of all $\xi$-states $s$ where $\sin \alpha(s) > \varepsilon$. 

Theorem

Let $s = (\xi, \nu_1, \nu_2)$ be three conjugate cubic vectors. Assume that $s$ is separating. Then

(i) $\Omega_\varepsilon(s) > \varepsilon$.  

(ii) $\Omega_{\text{max}}((\xi; \nu_1, \nu_2))$ is finite.

Corollary

The dynamical system with a map $\Phi$ on separating states is periodic for triples of cubic conjugate vectors.

Remark: There is a simple way to find the basis for $s$ in which it is separating.

Oleg Karpenkov, University of Liverpool

On Hermite's problem
Main theorem

Let $s_0$ be the input separating state. Denote

- $\Omega_\varepsilon(s_0)$ — the set of all $\xi$-states $s$ where $\sin \alpha(s) > \varepsilon$.
- $\Omega_{\max}(\xi; \nu_1, \nu_2)$ — the set of all $\xi$-states $s$ satisfying
  \[
  \sin^2 \alpha(s) > \sin^2 \alpha(\Phi(s)).
  \]
Main theorem

Let $s_0$ be the input separating state. Denote

- $\Omega_\varepsilon(s_0)$ — the set of all $\xi$-states $s$ where $\sin\alpha(s) > \varepsilon$.
- $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ — the set of all $\xi$-states $s$ satisfying $\sin^2\alpha(s) > \sin^2\alpha(\Phi(s))$.

Theorem

Let $s = (\xi, \nu_1, \nu_2)$ be three conjugate cubic vectors. Assume that $s$ is separating. Then

(i) $\Omega_\varepsilon(\xi; \nu_1, \nu_2)$ is finite ($\forall \varepsilon > 0$).
(ii) $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ is finite.

Corollary

The dynamical system with a map $\Phi$ on separating states is periodic for triples of cubic conjugate vectors.

Remark:

There is a simple way to find the basis for $s$ in which it is separating.
Main theorem

Let $s_0$ be the input separating state. Denote

- $\Omega_\varepsilon(s_0)$ — the set of all $\xi$-states $s$ where $\sin \alpha(s) > \varepsilon$.
- $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ — the set of all $\xi$-states $s$ satisfying

$$
\sin^2 \alpha(s) > \sin^2 \alpha(\Phi(s)).
$$

**Theorem**

Let $s = (\xi, \nu_1, \nu_2)$ be three conjugate cubic vectors. Assume that $s$ is separating. Then

(i) $\Omega_\varepsilon(\xi; \nu_1, \nu_2)$ is finite ($\forall \varepsilon > 0$).

(ii) $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ is finite.

**Corollary**

The dynamical system with a map $\Phi$ (on separating states) is periodic for triples of cubic conjugate vectors.

Remark:

There is a simple way to find the basis for $s$ in which it is separating.
Main theorem

Let $s_0$ be the input separating state. Denote

- $\Omega_\varepsilon(s_0)$ — the set of all $\xi$-states $s$ where $\sin \alpha(s) > \varepsilon$.
- $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ — the set of all $\xi$-states $s$ satisfying

\[
\sin^2 \alpha(s) > \sin^2 \alpha(\Phi(s)).
\]

**Theorem**

Let $s = (\xi, \nu_1, \nu_2)$ be three conjugate cubic vectors. Assume that $s$ is separating. Then

(i) $\Omega_\varepsilon(\xi; \nu_1, \nu_2)$ is finite ($\forall \varepsilon > 0$).

(ii) $\Omega_{\text{max}}(\xi; \nu_1, \nu_2)$ is finite.

**Corollary**

The dynamical system with a map $\Phi$ (on separating states) is periodic for triples of cubic conjugate vectors.

**Remark:** There is a simple way to find the basis for $s$ in which it is separating.
Proof of Item (i): finiteness statements

*Integer distance* $= \text{number of integer planes plus } 1 \text{ between the point and our plane:}$

![Diagram showing integer distance concept]
Proof of Item (i): finiteness statements

Proposition

Let $P$ be a triangular pyramid with vertex at $O(0,0,0)$. Let $C$ be $\text{Cone}(P)$ and $d \in \mathbb{Z}_+$. **Only finitely many:** integer planes $\pi$ satisfying
— integer distance$(O, \pi) \leq d$;
— $\pi$ divides $C$ into two parts, one of which is bounded and contains $P$.

Remark. This is due to finiteness of affine types of faces for two-dimensional Klein's continued fraction.
Proof of Item (i): finiteness statements

**Definition**

Consider an integer plane $\pi$ and a convex polygon $P$ in it. Let $S$ be the basis square of the integer lattice in it. Set *integer area*

$$\text{Area}_\pi(P) = \frac{\text{Area}(P)}{\text{Area}(S)}.$$
Proof of Item (i): finiteness statements

Definition

Consider an integer plane $\pi$ and a convex polygon $P$ in it. Let $S$ be the basis square of the integer lattice in it. Set \textit{integer area}

$$\text{Area}_\pi(P) = \frac{\text{Area}(P)}{\text{Area}(S)}.$$

Proposition

Consider a totally-real cubic cone $C$ in $\mathbb{R}^3$ centered at the origin. Then there exists $M$ s.t.

$$\text{Area}_\pi(C \cap \pi) < M$$

uniformly for all planes at distance 1 to the origin and cutting $C$. 
Proof of Item (i): finiteness statements

Remark. This is due to finiteness of affine types of faces for two-dimensional Klein's continued fraction.
Proof of Item (i): finiteness statements

Proposition

Let $T$ be an arbitrary triangle on an integer plane $\pi$, and $M \in \mathbb{R}_+$. Only finitely many: integer affine bases $(O, e_1, e_2)$ in which all absolute values of the coordinates of vertices of $T$ are bounded by $M$ from above.
Proof of Item (i): finiteness statements

Proposition

Consider a totally-real cubic cone in $\mathbb{R}^3$ centered at $O(0, 0, 0)$. Let $\varepsilon \in \mathbb{R}_+$. Only finitely many (up to the action of $\Xi_+(C)$): integer base planes $\pi$ on the unit integer distance to the origin satisfying

$$\text{Vol}(\text{Pyr}(C, \pi)) > \varepsilon.$$
Proof of Item (i): finiteness statements

Proposition

Consider a totally-real cubic cone in $\mathbb{R}^3$ centered at $O(0, 0, 0)$. Let $\varepsilon \in \mathbb{R}_+$. Only finitely many (up to the action of $\Xi_+(C)$): integer base planes $\pi$ on the unit integer distance to the origin satisfying

$$\text{Vol}(\text{Pyr}(C, \pi)) > \varepsilon.$$ 

Remark. This is due to finiteness of affine types of faces for two-dimensional Klein’s continued fraction.
Proof of Item (ii): case study

Ideas of the proof:
— Compactify the configuration space of all possible $\xi$-states (use projectivisation and certain asymptotics).
— Split the configuration space to several cases and compute finiteness for each of them separately.
Thank you.
Klein polyhedra

F. Klein (1895) – geometric generalization.

H. Tsuchihashi (1973) – relation to cusp singularities.

V. Arnold (1990) – formulated many problems.


...
From sail to torus decomposition

A sail for an algebraic operator $A$.
Let $\Xi(A)$ is generated by $X$ and $Y$. 
$X$ acts on the sail as a shift.
$Y$ acts on the sail as a shift.
The orbits under the action of $\Xi(A)$. 
The fundamental domain of the action of $\Xi(A)$. 
So the factor of the sail under the action of $\Xi(A)$ is a compact torus (finitely many faces).