The partial derivative of Okamoto's functions with respect to the parameter

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World Numeration seminar
January 24, 2023

Outline

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Motivation / Background

Lebesgue's singular function: unfair coin tossing

- Imagine flipping a coin.
- Suppose this coin has probability a of landing heads and probability $1 - a$ of landing tails. $(a \neq 1/2)$
- Determine a number $t \in [0, 1]$ by flipping the coin infinitely many times:

$$t = 0.\epsilon_1 \epsilon_2 \cdots = \sum_{k=1}^{\infty} \epsilon_k 2^{-k},$$

  where ϵ_k is 0 if the kth flip is heads, or 1 if it is tails.
- Define Lebesgue's singular function as the probability distribution function (c.d.f):

$$L_a(x) := Prob(t \leq x).$$

Properties:

1. Strictly increasing
2. Derivative zero almost everywhere

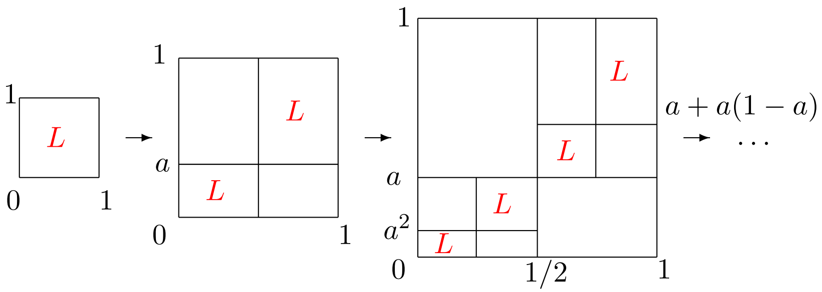
Theorem (De Rham, 1957)

$L_a(x)$ is the unique continuous solution of the functional equation

$$L_a(x) = \begin{cases} aL_a(2x), & 0 \leq x \leq \frac{1}{2}, \\ (1-a)L_a(2x-1)+a, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

where $0 < a < 1$, and $a \neq 1/2$.

Self-affinity: Graphs of $L_a(x)$



Graphs of $L_a(x)$

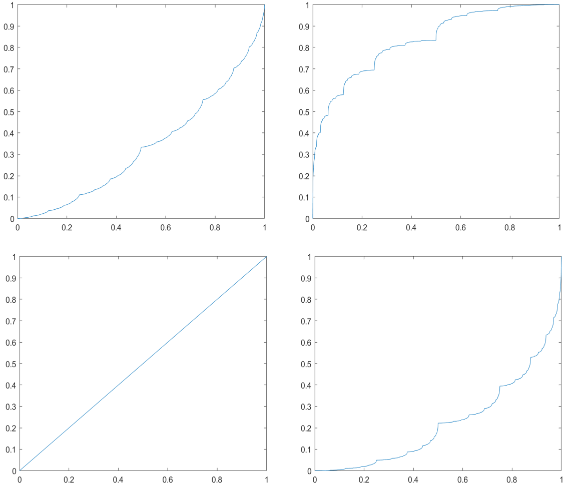


Figure: $L_{\frac{1}{3}}(x)$ (TL) $L_{\frac{5}{6}}(x)$ (TR) $L_{\frac{1}{2}}(x)$ (BL) $L_{\frac{2}{9}}(x)$ (BR)

Hata-Yamaguti's discovery

Theorem (Hata-Yamaguti, 1984)

$$\left.\frac{\partial L_a(x)}{\partial a}\right|_{a=\frac{1}{2}} = 2T(x),$$

where $T(x)$ is Takagi's nowhere differentiable function.

Remark

If $a = 1/2$, $L_a(x) = x$ so that $L_a'(x) = 1$.

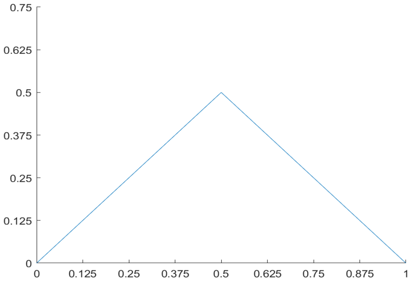
Takagi's nowhere differentiable function

Takagi's function is defined as

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x)$$

where $\phi(x) = dist(x, \mathbb{Z})$.

$\phi(x)$

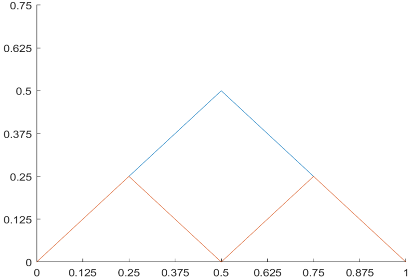


Construction of Takagi's Function

$$C_n := \frac{1}{2^n}\phi(2^n x)$$

where $\phi(x) = dist(x, \mathbb{Z})$

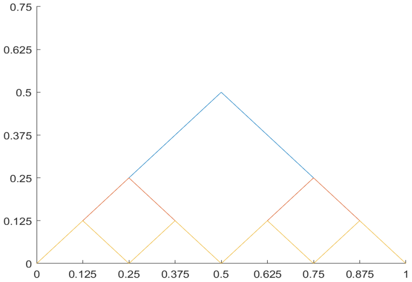
n=0,1



$$C_n := \frac{1}{2^n}\phi(2^n x)$$

where $\phi(x) = dist(x, \mathbb{Z})$

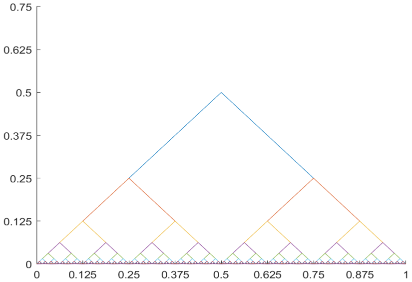
n=0,1,2



$$C_n := \frac{1}{2^n}\phi(2^n x)$$

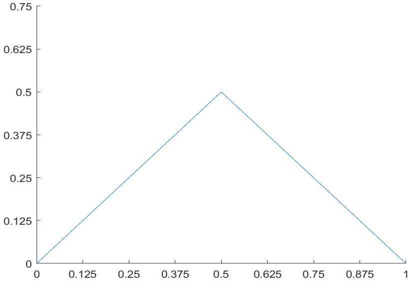
where $\phi(x) = dist(x, \mathbb{Z})$

n=0,1,2,. . . ,10



$$T(x) = \sum_{n=0}^{\infty} C_n$$

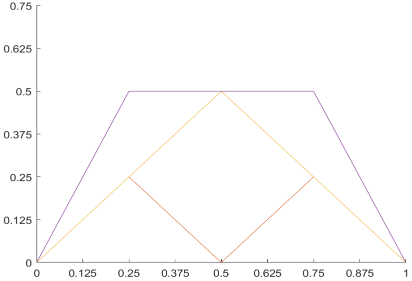
n=0



Construction of Takagi's Function

$$T(x) = \sum_{n=0}^{\infty} C_n$$

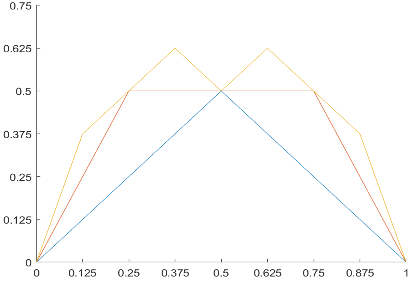
n=0,1



Construction of Takagi's Function

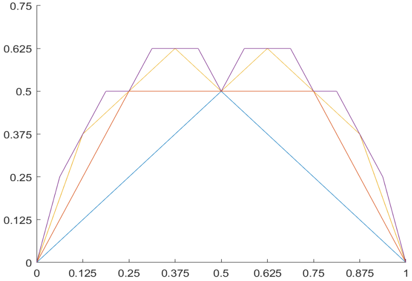
$$T(x) = \sum_{n=0}^{\infty} C_n$$

n=0,1,2



$$T(x) = \sum_{n=0}^{\infty} C_n$$

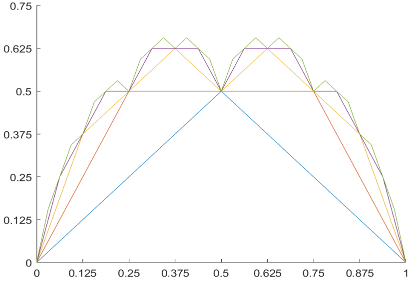
n=0,1,2,3



Construction of Takagi's Function

$$T(x) = \sum_{n=0}^{\infty} C_n$$

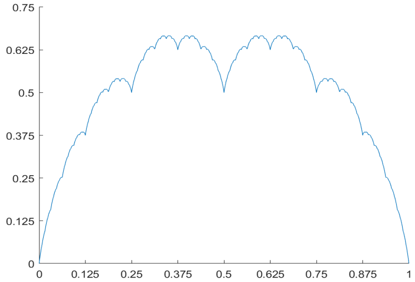
n=0,1,2,3,4



The graph of Takagi's Function

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x)$$

where $\phi(x) = dist(x, \mathbb{Z})$.



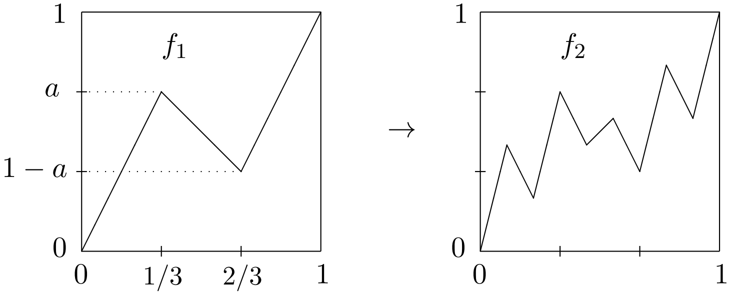
Properties of T:

- T is continuous but nowhere differentiable.
  (Takagi 1903)
- T has a (two-sided) infinite derivative at many points.
  (Cater 1984)
- The graph of T has self-affinity.
  (de Rham 1957)
- $\max T = 2/3$, attained at a continuum of points.
  (Kahane 1959)
- The graph of T has Hausdorff dimension 1.
  (Mauldin & Williams 1986)
- The complete characterization of infinite derivatives of T.
  (Allaart & Kawamura 2010, Kruppel 2010)

First Question

Okamoto's self-affine functions (2005)

In 2005, Okamoto introduced a family of self-affine functions, $\{F_a : 0 < a < 1\}$ on the interval $[0, 1]$.
Construction: Fix a parameter $0 < a < 1$.



- Continuing this way, obtain a sequence of continuous functions $f_n : [0, 1] \to [0, 1]$.
- Let $F_a(x) := \lim_{n\to\infty} f_n(x)$.

Functional equation

Okamoto's function $F_a(x)$ satisfies the following functional equation:

Okamoto's Function

$$F_a(x) = \begin{cases} aF_a(3x), & 0 \leq x \leq \frac{1}{3} \\ (1-2a)F_a(3x-1)+a, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ aF_a(3x-2)+(1-a), & \frac{2}{3} \leq x \leq 1 \end{cases}$$

where $0 < a < 1$.

Examples of the graph of $F_a(x)$

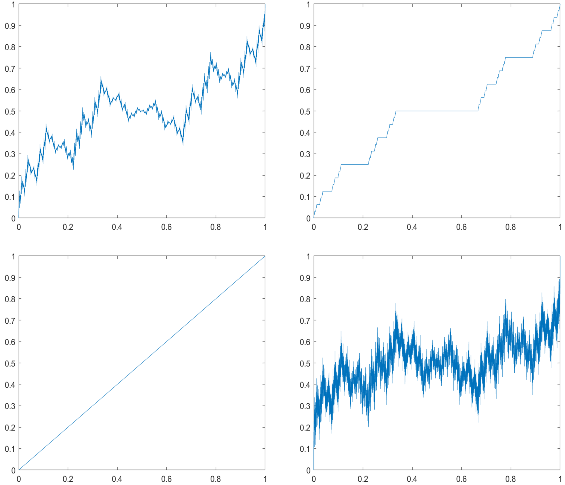


Figure: $F_{\frac{2}{3}}(x)$ (Bourbaki's Function), $F_{\frac{1}{2}}(x)$ (Cantor's Devil's staircase), $F_{\frac{1}{3}}(x)$ (Identity Function), $F_{\frac{5}{6}}(x)$ (Perkins's Function)

Some remarks

- F_a is continuous, mapping $[0,1]$ onto itself.
- F_a is self-affine, and

$$\dim_B \mathrm{Graph}(F_a) = \begin{cases} 1 & \text{if } a \le 1/2, \\ 1 + \log_3(4a-1) & \text{if } a > 1/2. \end{cases}$$

  (Hausdorff dimension unknown; seems very difficult!)
- F_a is singular for $a \le 1/2, a \ne 1/3$.
- $F_{1/3}(x) = x$.
- F_a is of unbounded variation when $a > 1/2$.

Theorem (Okamoto 2006)

Let $a_0 \approx .5592$ be the unique root in $(0,1)$ of

$$54a^3 - 27a^2 = 1.$$

1. *If $a \geq 2/3$, then F_a is nowhere differentiable.*
2. *If $a_0 < a < 2/3$, then F_a is non-differentiable a.e., but differentiable at uncountably many points.*
3. *If $a < a_0$, then $F_a' = 0$ a.e., but F_a is non-differentiable at uncountably many points.*

Theorem (Kobayashi 2009)

If $a = a_0$, F_a satisfies case (ii) above.

Remark

- $F_a(x)$ is an analytic function w.r.t. $a \in (0,1)$.
- If $a = 1/3$, $F_a(x) = x$ so that $F_a'(x) = 1$.

Motivated by Hata and Yamaguti's result, we define

$$K(x) := \left. \frac{\partial F_a(x)}{\partial a} \right|_{a=1/3}.$$

First Question

1. What does the graph of $K(x)$ look like?
2. Is there a simple expression for $K(x)$?
3. For what values of x is $K(x)$ differentiable?

Simple expression for $K(x)$

Lemma (Dalaklis-K.-Mathis-Paizanis, 2021)

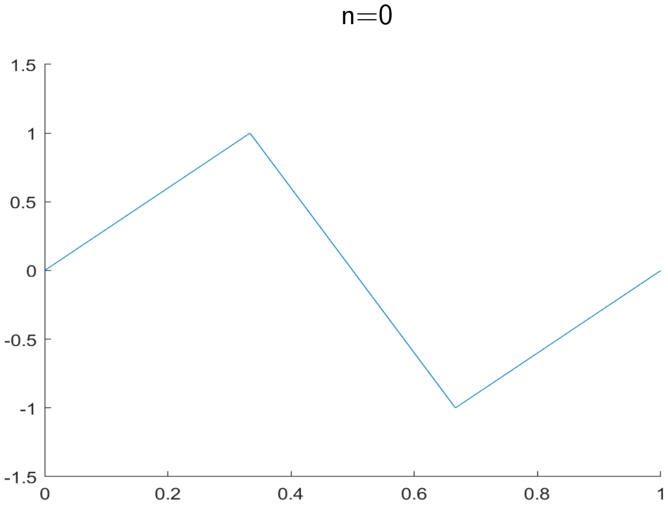
$$K(x) := \left.\frac{\partial F_a(x)}{\partial a}\right|_{a=1/3} = \sum_{n=0}^{\infty} \frac{1}{3^n} \Phi(3^n x)$$

where

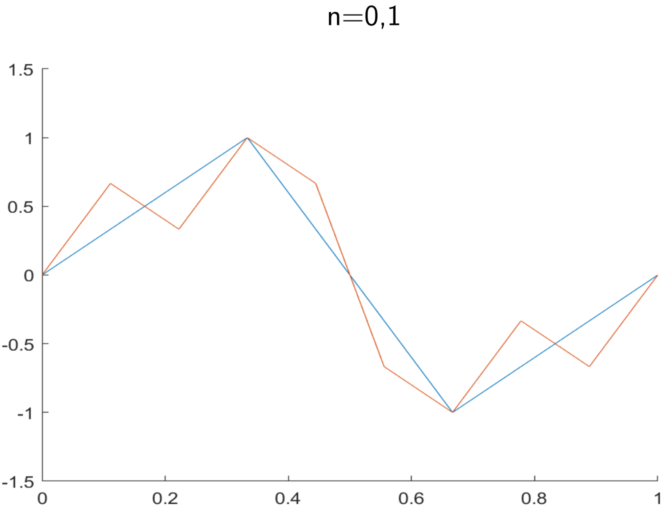
$$\Phi(x) := \begin{cases} 3x, & 0 \le x \le \frac{1}{3} \\ 3(1-2x), & \frac{1}{3} \le x \le \frac{2}{3} \\ 3(x-1), & \frac{2}{3} \le x \le 1 \end{cases}$$

and $\Phi(x+1) = \Phi(x)$

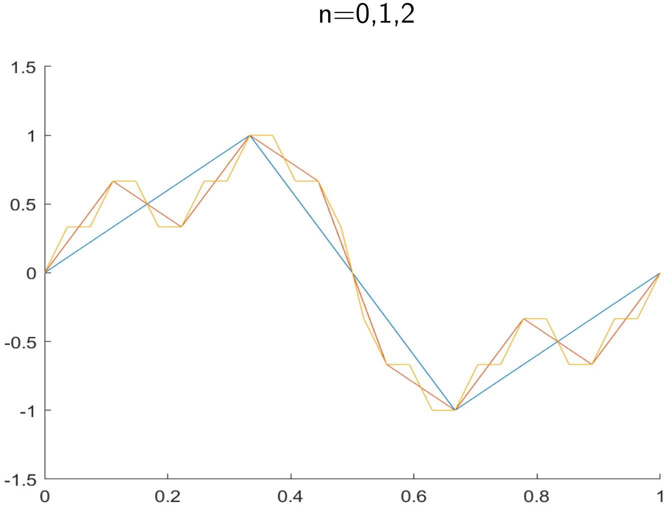
Construction of K(x)



Construction of K(x)

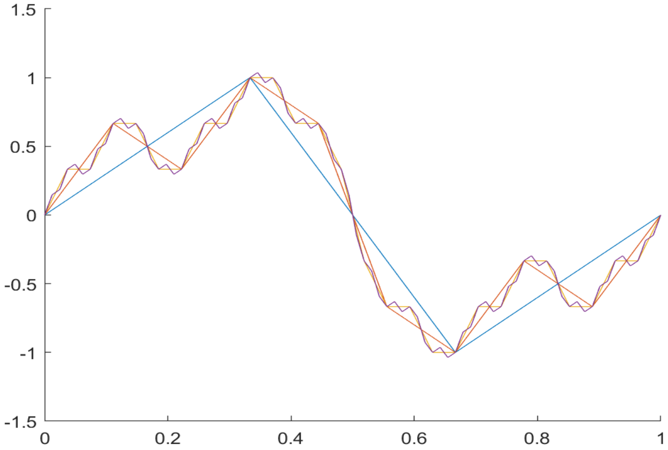


Construction of K(x)

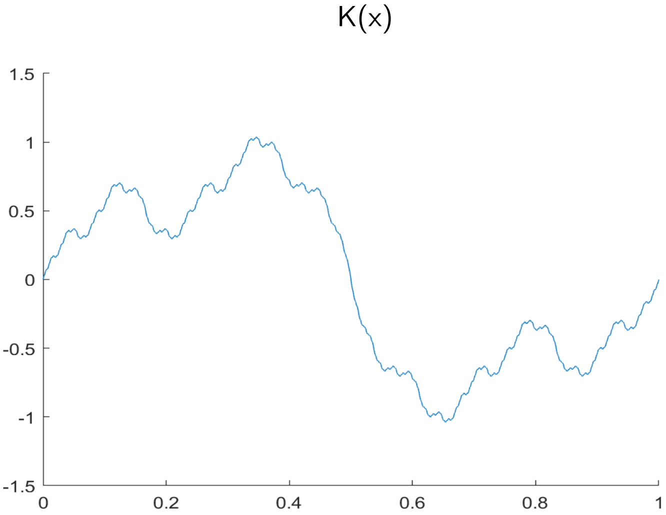


Construction of K(x)

n=0,1,2,3



Construction of K(x)

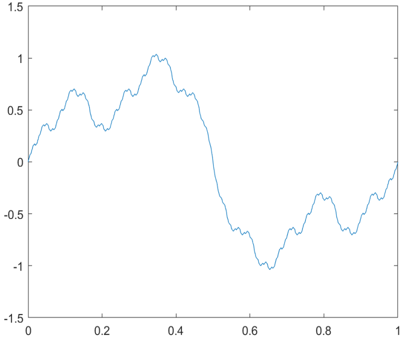


Dalaklis-K.-Mathis-Paizanis's result

Using this simple expression for K(x), we proved

Theorem (Dalaklis-K.-Mathis-Paizanis, 2021)

$K(x)$ is continuous, but it does not possess a finite derivative at any point on $[0, 1]$.



Second Question

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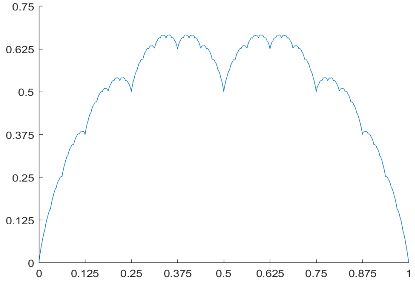
Second Question

$K(x)$ does not have a finite derivative anywhere, but at which points does it have an infinite derivative?

Is it a good project for undergraduate?

For Takagi's function, it is well-known that at dyadic rational points $(x = k/2^n)$,

$$T'_+(x) = \infty, \qquad T'_-(x) = -\infty$$



Theorem (Allaart & K., 2010)

Assume $x \in [0,1]$ *is not dyadic rational, and write*

$$x = \sum_{n=1}^{\infty} 2^{-a_n} \qquad (a_n \in \mathbb{N}, \quad a_{n+1} > a_n).$$

Then $T'(x) = \infty$ *iff*

$$a_{n+1} - 2a_n + 2n - \log_2(a_{n+1} - a_n) \to -\infty.$$

Be brave! Let's go!

Ternary representation of $x \in [0, 1]$

$$x = \sum_{k=1}^{\infty} \varepsilon_k / 3^k \text{ where } \varepsilon_k \in \{0, 1, 2\}$$

For numbers with multiple representations, we take the representation which is eventually all zeroes.

Example

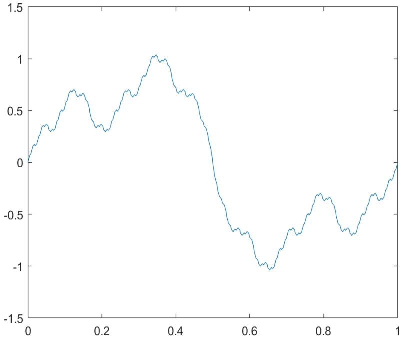
$$x = \frac{1}{3} = 0.1\overline{0}_3 = 0.0\overline{2}_3$$

Result

Theorem (Dalaklis-K.-Mathis-Paizanis, 2021)

Let $I_1(n)$ be the number of 1's in the first n ternary digits of $x \in (0,1)$.

$$K'(x) = \pm\infty \iff n - 3I_1(n) \to \pm\infty \qquad \text{as } n \to \infty.$$



Define

$$p_1(x) := \lim_{n\to\infty} \frac{I_1(n)}{n},$$

assuming the limit exists. Note that $p_1(x)$ is the frequency of the digit 1 in the ternary expansion of x.

Corollary 1

Assume $p_1(x)$ exists. Then

$$\begin{cases} K'(x) = +\infty, & \text{if} \quad p_1(x) < 1/3, \\ K'(x) = -\infty, & \text{if} \quad p_1(x) > 1/3. \end{cases}$$

Corollary 2

Let $F := \{x \in [0,1] : K'(x) = \pm\infty\}$. The Lebesgue measure of F is 0 while the Hausdorff dimension of F is 1.

Proof Outline of Main theorem

Key of the proof (Right-hand derivative of $K(x)$)

Fix $0 \le x < x + h < 1$. Define

$$p := p(h) \in \mathbb{N} \text{ such that } 3^{-p} \le h < 3^{-p+1}.$$

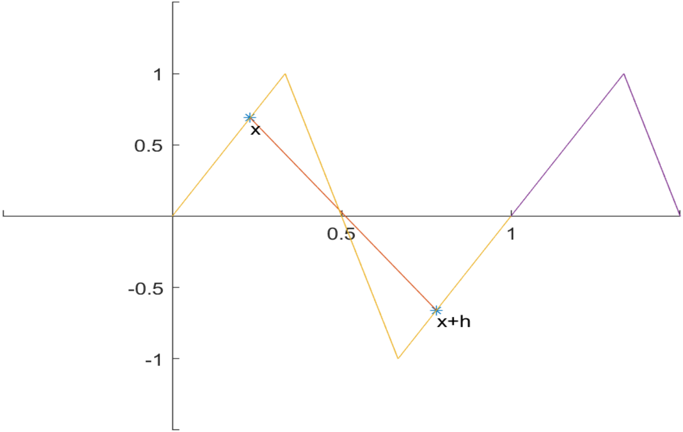
Define the right-hand derivative of $K(x)$ by

$$K'_+(x) := \lim_{h \to 0+} \frac{K(x+h) - K(x)}{h}.$$

$$\frac{K(x+h) - K(x)}{h} = \sum_{n=0}^{\infty} \frac{\Phi(3^n(x+h)) - \Phi(3^n x)}{3^n h} =: \sum_{n=0}^{\infty} D_n(x,h).$$

Notice that $D_n(x,h)$ represents the slope of the secant line connecting the points $(3^n x, \Phi(3^n x))$ and $(3^n(x+h), \Phi(3^n(x+h)))$.

Graphical Representation of $D_n(x, h)$



Define

$$I_i(n) := \#\{j : 1 \leq j \leq n, \varepsilon_j = i\},$$

and

$$f(n) := f_x(n) = 3I_0(n) - 6I_1(n) + 3I_2(n).$$

After the detailed analysis, we have

$$f(p-1) - 18 \leq \frac{K(x+h) - K(x)}{h} \leq f(p-1) + 18.$$

Next notice that as $h \to 0+$, $p \to \infty$. Therefore, it follows that

$$K'_+(x) = \pm\infty \text{ iff } f(n) \to \pm\infty \text{ as } n \to \infty.$$

Since the graph of $K(x)$ is symmetric about the point $(1/2, 0)$, $K(x) = -K(1-x)$ for every $x \in \mathbb{R}$. Thus, we have

$$K'_{-}(x) = K'_{+}(1-x).$$

Notice that $f_x(n) \to \pm\infty \iff f_{1-x}(n) \to \pm\infty$ as $n \to \infty$. Therefore, the left-side derivative follows

$$K'_{-}(x) = \pm\infty \text{ iff } f(n) \to \pm\infty \text{ as } n \to \infty.$$

This completes the proof.

Future/ongoing projects

Future/ongoing Projects

Let

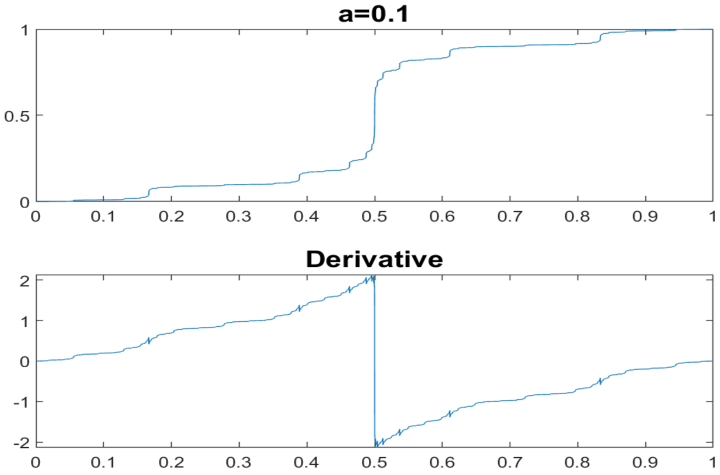
$$K_r(x) := \left.\frac{\partial F_a(x)}{\partial a}\right|_{a=r}.$$

1. For a fixed $r = 1/10$, is $K_r(x)$ continuous everywhere?
2. For a fixed $r = 1/2$, is $K_r(x)$ a modification of Cantor's Devil's staircase function?
3. What is the box-counting dimension of the graph of $K_r(x)$?
4. For a fixed $a = r$, define the n-th partical derivative of $F_a(x)$.

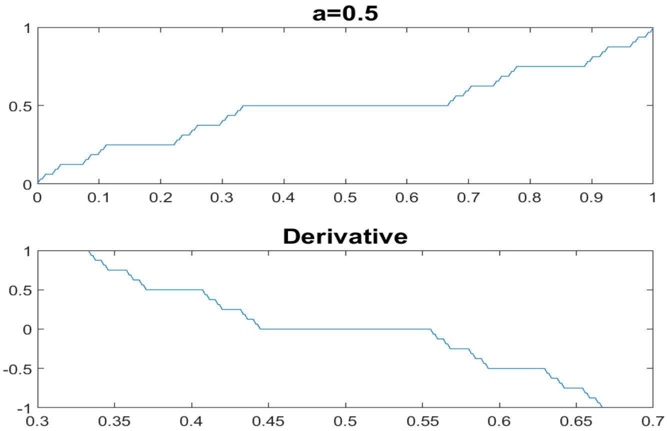
$$K_{r,n}(x) := \left.\frac{\partial^n F_a(x)}{\partial a^n}\right|_{a=r}, \qquad n = 1, 2, 3, \ldots.$$

What does the graph of $K_{r,n}(x)$ look like?

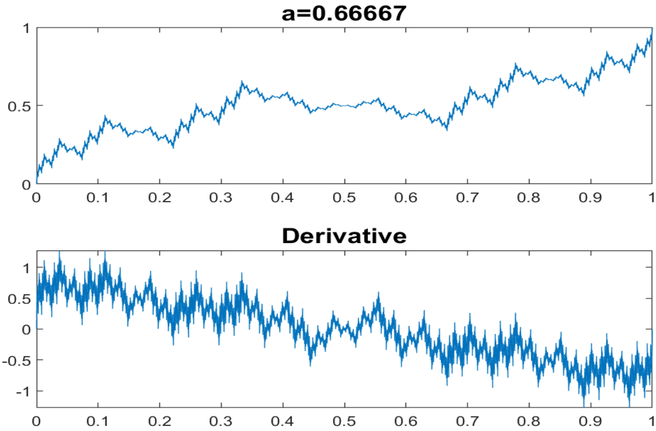
Graphs of $F_a(x)$ and $\partial F_a(x)/\partial a$ where $a = 1/10$



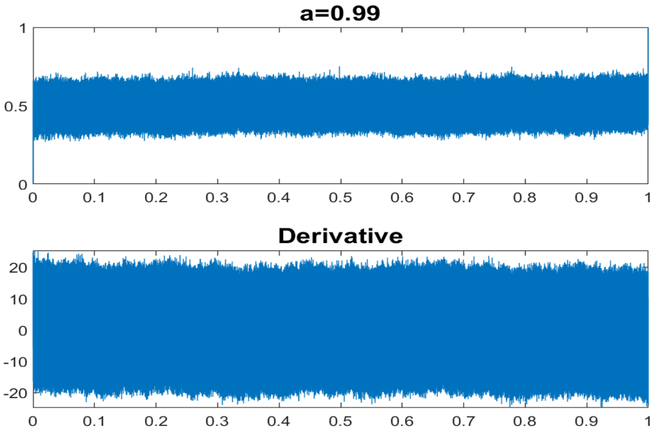
Graphs of $F_a(x)$ and $\partial F_a(x)/\partial a$ where $a = 1/2$



Graphs of $F_a(x)$ and $\partial F_a(x)/\partial a$ where $a = 2/3$



Graphs of $F_a(x)$ and $\partial F_a(x)/\partial a$ where $a = 99/100$



Special Thanks

The authors of this paper would like to thank

Dr. Pieter Allaart
Dr. Joseph Iaia
and
Taylor Jones

for their help and insight on this project.

Thank you, everyone! Happy Lunar New Year!