Bernoulli Convolutions and Measures on the Spectra of Algebraic Integers

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The Spectrum of β

- Given a real number $\beta > 1$ and an alphabet $\mathcal{A}$ the spectrum of β is the set

$$X_{\mathcal{A}}(\beta) := \left\{ \sum_{i=0}^{n} a_i \beta^i : n \in \mathbb{N}, a_i \in \mathcal{A} \right\}.$$

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- Today's Questions: What can we say about the measures

$$\mu_n(x) := \mathbb{P}\left(\sum_{i=0}^{n} a_i \beta^i - \sum_{i=0}^{n} b_i \beta^i = x \right)$$

where each a_i, b_i picked independently from $\{0, 1\}$ with probability $(\frac{1}{2}, \frac{1}{2})$.

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- $$\begin{aligned} X_{\mathcal{A}}(\beta) &:= \left\{ \sum_{i=1}^{n} a_i \beta^{(n-i)} : n \in \mathbb{N}, a_i \in \mathcal{A} \right\} \\ &= \{ T_{a_n} \circ \cdots T_{a_1}(0) : n \in \mathbb{N}, a_i \in \mathcal{A} \} \end{aligned}$$

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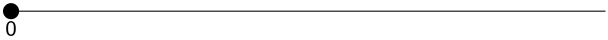

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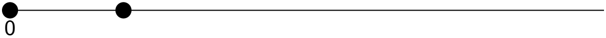

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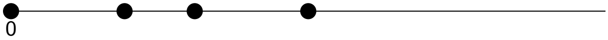

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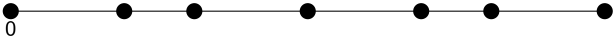

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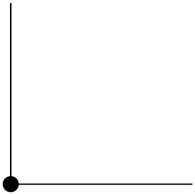
- Gaps of size 1 and $\phi - 1$.
- Gap sequence $ABAABABAABA\cdots$ generated by Fibonacci substitution

$$A \to AB, B \to A$$

Where does the structure come from?

- Using the fact that $\phi^2 = \phi + 1$, we can represent multiplying by ϕ as an action on $\mathbb{Z}^2$. Let $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \sim z_1\phi + z_0$.



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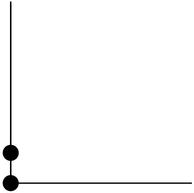

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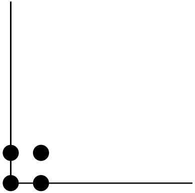

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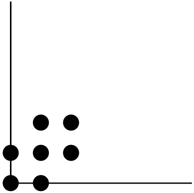

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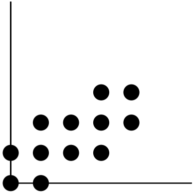

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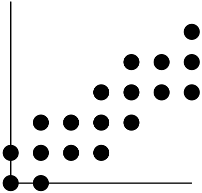

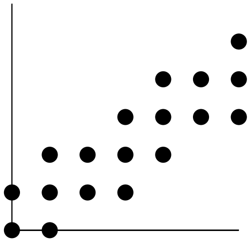

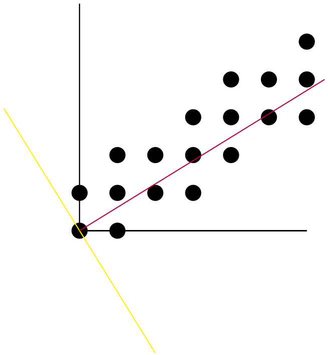

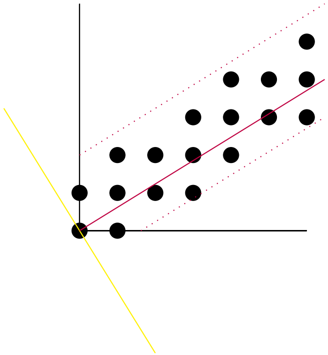

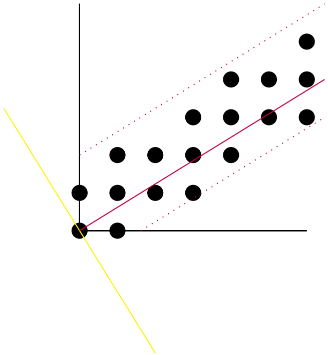

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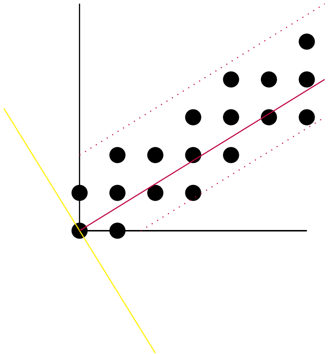

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- Not so hard to see that we can never escape the strip between the two dotted purple lines. Slightly harder - every lattice point (z_1, z_0) between the two dotted lines can be reached:

$$\begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \tilde{T}_{a_1} \circ \cdots \circ \tilde{T}_{a_n} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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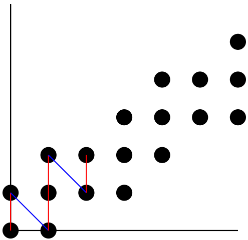
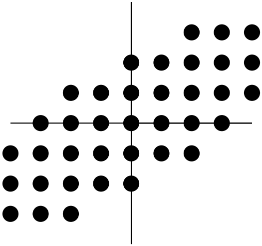


Including digit -1



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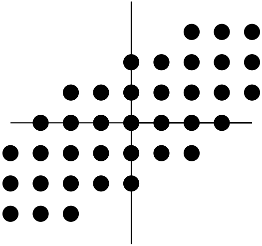

We can move through the spectrum

Lemma

Let the map G on $X_{\{-1,0,1\}}(\phi) \times (-\phi^2, \phi^2)$ be given by

$$G(x,y) = \begin{cases} (x + 2\phi - 3, y - \frac{2}{\phi} - 3) & y \in [\phi, \phi^2) \\ (x + \phi - 1, y - \frac{1}{\phi} - 1) & y \in (0, \phi) \\ (x + 2 - \phi, y + 2 + \frac{1}{\phi}) & y \in (-\phi^2, 0] \end{cases}$$

Then if x is the nth element to the right of 0 in $X_{\{-1,0,1\}}(\phi)$ and x_c is the corresponding point in the contracting direction we have that

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  where $c_i \in \{-1, 0, 1\}$ w.p. $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$
- $\mu_n(x)$ supported on $X_{\{-1,0,1\}}(\beta)$. Maps $T_i(x) = \beta x + i$ expand, so we have a finite recurrent bit near 0 and a dissipative bit.

- $\phi = \frac{1+\sqrt{5}}{2}$, $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ i.i.d. with probability $(\frac{1}{2}, \frac{1}{2})$.

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Theorem (Batsis, K.)

There exist $\theta > 1$ and a function $f : X_{\{-1,0,1\}}(\phi) \to \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \theta^n \mathbb{P}\left\{\sum_{i=0}^{n} a_i\phi^i - \sum_{i=0}^{n} b_i\phi^i = x\right\} = f(x).$$

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- ϕ can be replaced by any algebraic integer $\beta \in (1, 2)$ for which all of the other Galois conjugates β_i have $|\beta_i| \neq 1$.

Let $T_i(x) = \phi x + i$.
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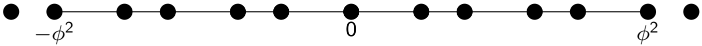

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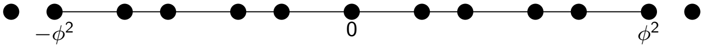

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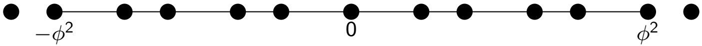

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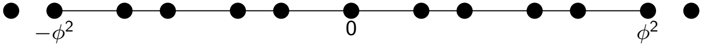

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  decays like θ^n where θ is the max eigenvalue of the matrix encoding the dynamics on the recurrent piece.

Let μ be the infinite, locally finite measure on $X_{\{-1,0,1\}}(\beta)$ given by

$$\mu = \sum_{x \in X_{\{-1,0,1\}}(\beta)} f(x)\delta_x.$$

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Goal for next section: We can move through $X_{\{-1,0,1\}}(\beta)$ using an odometer map. Can we do something similar for the measure?

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If $x = \sum_{i=1}^{n} c_i \beta^{n-i}$ *and* $x = \sum_{i=1}^{n} d_i \beta^{n-i}$, $c_i, d_i \in \{-1, 0, 1\}$ *then for all* $m < n$

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Strictly positive matrices contract projective space. Our matrices aren't strictly positive, but we mess around a bit. For $x \in \mathcal{X}_{\{-1,0,1\}}(\phi)$ let $x_c \in [0, 1]$ be the corresponding point in the contracting window and x' be the nearest neighbour of x.

Theorem [Batsis, K.] There exists a function $\psi : [0, 1] \to \mathbb{R}$, continuous except on a set of Hausdorff dimension < 1, such that $\frac{\mu(x')}{\mu(x)} = \psi(x_c)$.

A Concrete Example

Theorem (Batsis, K.)

Let the map F on $X_{\{-1,0,1\}}(\phi) \times (-\phi^2, \phi^2) \times \mathbb{R}$ be given by

$$F(x,y,z) = \begin{cases} (x + 2\phi - 3, y - \frac{2}{\phi} - 3, z + \psi(y)) & y \in [\phi, \phi^2) \\ (x + \phi - 1, y - \frac{1}{\phi} - 1, z + \psi(y)) & y \in (0, \phi) \\ (x + 2 - \phi, y + 2 + \frac{1}{\phi}, z + \psi(y)) & y \in (-\phi^2, 0] \end{cases}$$

Then if x is the nth element to the right of 0 in $X_{\{-1,0,1\}}(\phi)$ we have that

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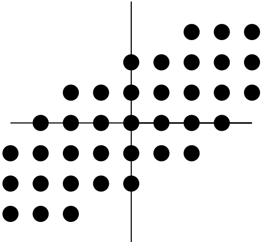

Bernoulli Convolutions:

- Given $\beta \in (1,2)$ the Bernoulli convolution ν_β is the weak* limit of the measures $\nu_{\beta,n}$ given by

$$\nu_{\beta_n} = \frac{1}{2^n} \sum_{a_1 \cdots a_n \in \{0,1\}^n} \delta_{\sum_{i=1}^n a_i \beta^{-i}}$$

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- Very few specific examples of absolutely continuous Bernoulli convolutions, due to Garsia (1950s) and Varju (2020).

If ν_β is absolutely continuous then the density h_β also satisfies a self-similarity relation.

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Theorem (Batsis, K.)

Suppose there exists a constant C that the total number $\mathcal{N}_n$ of words $a_1 \cdots a_n, b_1 \cdots b_n \in \{0,1\}^n$ satisfying

$$|\sum_{i=1}^{n}(a_i - b_i)\beta^{n-i}| < \frac{1}{\beta - 1}$$

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Goal: Let $\beta \in (1,2)$ be an algebraic unit, non-Pisot, no Galois conjugates of absolute value one. Use the cut and project structure of the sets $X_{\mathcal{A}}(\beta)$ to study pairs of words $a_1, \cdots a_n, b_1, \cdots b_n$ as above. Use the measures μ_n to count them.