The Dynamics of The Fibonacci Partition Function.

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The Fibonacci Partition Function

- Let $R(n)$ be the number of ways of writing $n \in \mathbb{N}$ as the sum of distinct Fibonacci numbers.
- For example

$$8 = 8 = 5 + 3 = 5 + 2 + 1$$

so $R(8) = 3$.

- $(R(n)) = 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, \cdots$
- Fibonacci numbers $F_1 = F_2 = 1, F_3 = 2$, write

$$R(n) = \#\left\{ n = \sum_{i=1}^{k} a_i F_{k+2-i} : k \text{ large enough, } a_i \in \{0, 1\} \right\}$$
The Fibonacci Partition Function

- \((R(n))_{n=1}^{\infty} = 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, \cdots\)
- Berstel showed how to express \(R(n)\) in terms of a product of matrices. Recently, Chow and Slattery gave an explicit (albeit complicated) formula for \(R(n)\).
- **Our Goal:** Determine the local, multiplicative structure of \(R\).
- Can’t answer: When is \(R(n) = 7\)?
  Can answer: when is \(R(n) = R(n + 1)\)?
  How often do we have \(R(n + 2) = 3R(n)\)?
  What is the longest

\[
R(n) \leq R(n + 1) \leq \cdots \leq R(n + k).
\]
Define irrational rotation $T : \left[ \frac{-1}{\varphi^2}, \frac{1}{\varphi} \right) \rightarrow \left[ \frac{-1}{\varphi^2}, \frac{1}{\varphi} \right)$ by

$$T(y) = \begin{cases} 
  y + \frac{1}{\varphi^2} & y \in \left[ \frac{-1}{\varphi^2}, \frac{1}{\varphi^3} \right] \\
  y + \frac{1}{\varphi^2} - 1 & y \in \left[ \frac{1}{\varphi^3}, \frac{1}{\varphi} \right].
\end{cases}$$

Let $h : \left( \frac{-1}{\varphi^2}, \frac{1}{\varphi^3} \right) \cup \left( \frac{1}{\varphi^3}, \frac{1}{\varphi} \right) \rightarrow \mathbb{R}$ be the unique continuous function satisfying

$$h(y) = \begin{cases} 
  1 + h(-\varphi y) & y \in \left( \frac{-1}{\varphi^2}, \frac{-1}{\varphi^4} \right) \\
  1 & y \in \left[ \frac{-1}{\varphi^4}, 0 \right] \\
  h(-\varphi y + \frac{1}{\varphi}) & y \in \left( 0, \frac{1}{\varphi^3} \right) \\
  \frac{1}{1 + h(-\varphi y + \frac{1}{\varphi})} & y \in \left( \frac{1}{\varphi^3}, \frac{1}{\varphi} \right).
\end{cases}$$
Dynamics

- $T$ is an irrational rotation by $\frac{1}{\varphi^2}$
- $h$ is a Devil’s staircase, $\log(h)$ has graph

**Figure:** The function $\log(h)$, drawn by plotting $y_n = T^n(0)$ against $\log \left( \frac{R(n+1)}{R(n)} \right)$ for $0 \leq n \leq 2582$.

**Theorem:** $\frac{R(n+1)}{R(n)} = h(T^n(0))$
Main Theorems

▶ **Theorem:** $R(n) = \exp\left(\sum_{k=0}^{n-1} \log(h(T^k(0)))\right)$ for all $n \geq 1$.

▶ Given $P = (p_1, \ldots, p_k) \in \mathbb{Q}^k$, say $R$ contains patch $P$ at time $n$ if

$$R(n + i) = p_i R(n) \quad \forall \ i \in \{1, \ldots, k\}.$$  

▶ For any patch $P$, the set

$$\{ n \in \mathbb{N} : \ R \text{ contains patch } P \text{ at time } n \}$$

is a cut and project set.
Trying to do dynamics

We could try to do dynamics on $\mathbb{N} \cup \{0\}$ to study Fibonacci codings.

For example, $11 = 1 \times 8 + 0 \times 5 + 1 \times 3 + 0 \times 2 + 0 \times 1$. 
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Trying to do dynamics

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For example, $11 = 1 \times 8 + 0 \times 5 + 1 \times 3 + 0 \times 2 + 0 \times 1$. $11 \sim 10100$. Could think of sequence of codes

$$
\begin{align*}
0 & \rightarrow 1 \rightarrow 10 \rightarrow 101 \rightarrow 1010 \rightarrow 10100 \\
0 & \rightarrow 1 \rightarrow 2 \rightarrow 3 + 1 = 4 \rightarrow 5 + 2 = 7 \rightarrow 8 + 3 = 11
\end{align*}
$$

This dynamics doesn’t work very well. Complicating factor is that the map $F_n \rightarrow F_{n+1}$ is not quite linear.
Making things linear

We can write

\[ F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \]

where \( \varphi = \frac{1 + \sqrt{5}}{2} \), \( \psi = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\varphi} \).

**Idea:** Replace \( F_n \) with the pair \((\varphi^n, \psi^n)\).

**Lemma:** Let \( n \in \mathbb{N} \) For any \( a_1 \cdots a_k \in \{0, 1\}^k \) with \( n = \sum_{i=1}^{k} a_i F_{k+2-i} \) define

\[
x_n = \sum_{i=1}^{k} a_i \varphi^{k+2-i}
\]

\[
y_n = \sum_{i=1}^{k} a_i \psi^{k+2-i}.
\]

The map \( n \to (x_n, y_n) \) is well defined and invertible.

\[ R(n) = \# \{ a_1 \cdots a_k \in \{0, 1\}^k : \sum_{i=1}^{k} a_i \varphi^{k+2-i} = x_n \} \text{ for large } k. \]
\[ n = \sum_{i=1}^{k} a_i F_{k+2-i} \rightarrow (x_n, y_n) = \left( \sum_{i=1}^{k} a_i \varphi^{k+2-i}, \sum_{i=1}^{k} a_i \psi^{k+2-i} \right). \]

The set \( \overline{X} = \{(x_n, y_n) : n \in \mathbb{N}\} \) is a strip through a lattice.

Define \( T_i, S_i : \mathbb{R} \rightarrow \mathbb{R}, T_i(x) = \varphi x + i \varphi^2, S_i(y) = \psi y + i \psi^2. \)

\[ \overline{X} = \{( T_{a_k} \circ \ldots \circ T_{a_1}(0), S_{a_k} \circ \ldots \circ S_{a_1}(0)) : k \in \mathbb{N}, a_i \in \{0, 1\}\}. \]

\[ R(n) = \#\{ a_1 \cdots a_k \in \{0, 1\}^k : T_{a_k} \circ \ldots \circ T_{a_1}(0) = x_n \} \]
Dynamics on \( \overline{X} \)

**Proposition:**

\[
\overline{X} = \left\{ (n + m\varphi, n + m\psi) : n + m\psi \in \left[ \frac{-1}{\varphi^2}, \frac{1}{\varphi} \right] \right\}
\]

**Proof:** Since \( y_n \) are generated from 0 using the contractive maps \( S_0, S_1 \), they must lie in the attractor of the iterated function system \( \{ S_0, S_1 \} \). Reverse inclusion slightly less clean (still short).

**Proposition:**

\[
(x_{n+1}, y_{n+1}) = \begin{cases} 
(x_n + 1 + \varphi, y_n + 1 + \psi) & y_n \in \left[ \frac{-1}{\varphi^2}, \frac{1}{\varphi^3} \right] \\
(x_n + \varphi, y_n + \psi) & y \in \left[ \frac{1}{\varphi^3}, \frac{1}{\varphi} \right]
\end{cases}
\]

**Proof:** \( (x_{n+1}, y_{n+1}) - (x_n, y_n) \in \overline{X} - \overline{X} \), a discrete set. Use \( n = (x_n + y_n)/\sqrt{5} \) and \( y_n \) in bounded set.
The Zeckendorf expansion of $n$ writes $n$ as the sum of non-consecutive Fibonacci numbers. It is the greedy expansion of $n$ by Fibonacci numbers.

If $x_n = T_{b_k} \circ \cdots \circ T_{b_1}(0)$ (Zeckendorf expansion),

$$R(n) = \#\{a_1 \cdots a_k \in \{0, 1\}^k : T_{b_k} \circ \cdots \circ T_{b_1}(0) = T_{a_k} \circ \cdots \circ T_{a_1}(0)\}$$

i.e. want to count $a_1 \cdots a_k$ such that

$$T_{b_k-a_k} \circ \cdots \circ T_{b_1-a_1}(0) = 0$$

$$T_i(x) = \varphi x + i \varphi^2, \ b_i - a_i \in \{1, 0, -1\}.$$ 

**Lemma**: If $T_{b_k-a_k} \circ \cdots \circ T_{b_1-a_1}(0) = 0$ and $b_1 \cdots b_k$ greedy, then for $i \leq k$

$$T_{b_i-a_i} \circ \cdots \circ T_{b_1-a_1}(0) \in \{0, \varphi, 1 + \varphi\}.$$
Matrices:

Let

\[ A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \]

**Prop:** If \( n \) has Zeckendorf expansion \( b_1 \cdots b_k \) and if

\[(v_1 \ v_2 \ v_3) := (1 \ 0 \ 0)A_{b_1} \cdots A_{b_k}\]

then \( R(n) = v_1, \ R(n-1) = v_2 + v_3. \) One of \( v_2, v_3=0. \)

Ratio \( \frac{R(n)}{R(n-1)} = \frac{v_1}{v_2 + v_3}. \)

Certain tails \( b_{k-m} \cdots b_k \) determine \( R(n)/R(n-1) \) exactly, e.g. since

\[ A_1 A_0 A_0 A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\( R(n) = R(n-1) \) whenever \( b_{k-3} \cdots b_k = 1001. \)
Two Iterated Function Systems

If $n$ has Zeckendorf coding $b_1 \cdots b_k$ then

$$y_n = S_{b_k} \circ \cdots \circ S_{b_1}(0)$$

A word $b_k \cdots b_{k-m}$ corresponds to an interval in $\left[\frac{-1}{\varphi^2}, \frac{1}{\varphi}\right]$. $n$ has Zeckendorf coding ending $b_{k-m} \cdots b_m$ if and only if

$$T^n(0) = y_n \in S_{b_k} \circ \cdots S_{b_{k-m}} \left(\left[\frac{-1}{\varphi^2}, \frac{1}{\varphi}\right]\right).$$

But these tails also tell me the ratio $R(n)/R(n-1)$
Putting it all together

We turned $n \in \mathbb{N}$ into a pair $(x_n, y_n)$ with good dynamics.
$y_n = T^n(0)$ where $T$ is an irrational rotation.

We could see $R(n)$ and $R(n - 1)$ by instead counting $R(x_n)$, $R(x_n - \varphi)$, $R(x_n - \varphi - 1)$ and using the good linear dynamics, turning this into a matrix product indexed by the Zeckendorf coding.

Certain tails in the Zeckendorf coding correspond to knowing the ratio $R(n)/R(n - 1)$ exactly. They also correspond to an interval in which $y_n$ must lie. Defining $h(T^n(0)) = R(n)/R(n - 1)$ on these intervals gives a Devil’s staircase structure.

We can now understand the local multiplicative structure of $R(n)$ by seeing where $T^n(0)$ lies. For any given local pattern in the multiplicative structure, it occurs whenever $T^n(0)$ lies in a region corresponding to the pattern.