The Dynamics of The Fibonacci Partition Function.

Tom Kempton

University of Manchester

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The Fibonacci Partition Function

- Let $R(n)$ be the number of ways of writing $n \in \mathbb{N}$ as the sum of **distinct** Fibonacci numbers.
- For example

$$\begin{aligned} 8 &= 8 \\ &= 5 + 3 \\ &= 5 + 2 + 1 \end{aligned}$$

  so $R(8) = 3$.
- $(R(n)) = 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, \cdots$
- Fibonacci numbers $F_1 = F_2 = 1$, $F_3 = 2$, write

$$R(n) = \# \left\{ n = \sum_{i=1}^{k} a_i F_{k+2-i} : \text{k large enough}, a_i \in \{0, 1\} \right\}$$

The Fibonacci Partition Function

- $(R(n))_{n=1}^{\infty} = 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, \cdots$
- Berstel showed how to express $R(n)$ in terms of a product of matrices. Recently, Chow and Slattery gave an explicit (albeit complicated) formula for $R(n)$.
- **Our Goal:** Determine the local, multiplicative structure of R.
- Can't answer: When is $R(n) = 7$?
  Can answer: when is $R(n) = R(n+1)$?
  How often do we have $R(n+2) = 3R(n)$?
  What is the longest

$$R(n) \leq R(n+1) \leq \cdots \leq R(n+k).$$

- Define irrational rotation $T : \left[\frac{-1}{\varphi^2}, \frac{1}{\varphi}\right) \to \left[\frac{-1}{\varphi^2}, \frac{1}{\varphi}\right)$ by

$$T(y) = \begin{cases} y + \frac{1}{\varphi^2} & y \in \left[\frac{-1}{\varphi^2}, \frac{1}{\varphi^3}\right] \\ y + \frac{1}{\varphi^2} - 1 & y \in \left[\frac{1}{\varphi^3}, \frac{1}{\varphi}\right] \end{cases}.$$

- Let $h : \left(\frac{-1}{\varphi^2}, \frac{1}{\varphi^3}\right) \cup \left(\frac{1}{\varphi^3}, \frac{1}{\varphi}\right) \to \mathbb{R}$ be the unique continuous function satisfying

$$h(y) = \begin{cases} 1 + h(-\varphi y) & y \in \left(\frac{-1}{\varphi^2}, \frac{-1}{\varphi^4}\right] \\ 1 & y \in \left[\frac{-1}{\varphi^4}, 0\right] \\ \dfrac{h(-\varphi y + \frac{1}{\varphi})}{1 + h(-\varphi y + \frac{1}{\varphi})} & y \in \left[0, \frac{1}{\varphi^3}\right) \\ h(-\varphi y + \frac{1}{\varphi}) & y \in \left(\frac{1}{\varphi^3}, \frac{1}{\varphi}\right) \end{cases}$$

Dynamics

- T is an irrational rotation by $\frac{1}{\varphi^2}$
- h is a Devil's staircase, $\log(h)$ has graph

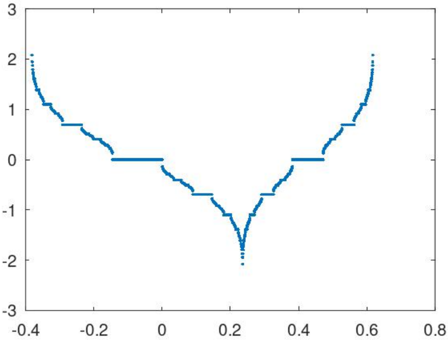
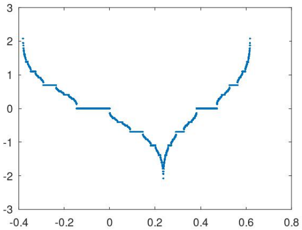


Figure: The function $\log(h)$, drawn by plotting $y_n = T^n(0)$ against $\log\left(\frac{R(n+1)}{R(n)}\right)$ for $0 \leq n \leq 2582$.

- **Theorem:** $\frac{R(n+1)}{R(n)} = h(T^n(0))$

Main Theorems



- **Theorem:** $R(n) = \exp(\sum_{k=0}^{n-1} \log(h(T^k(0)))$ for all $n \geq 1$.
- Given $P = (p_1, .., p_k) \in \mathbb{Q}^k$, say R contains patch P at time n if

$$R(n+i) = p_i R(n) \; \forall i \in \{1, \cdots, k\}.$$

- For any patch P, the set

$$\{n \in \mathbb{N} : \text{ R contains patch P at time n}\}$$

is a cut and project set.

Trying to do dynamics

We could try to do dynamics on $\mathbb{N} \cup \{0\}$ to study Fibonacci codings.

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For example, $11 = 1x8 + 0x5 + 1x3 + 0x2 + 0x1$. $11 \sim 10100$. Could think of sequence of codes

$$\begin{array}{cccccc} 0 \rightarrow & 1 \rightarrow & 10 \rightarrow & 101 \rightarrow & 1010 \rightarrow & 10100 \\ 0 \rightarrow & 1 \rightarrow & 2 \rightarrow & 3+1=4 \rightarrow & 5+2=7 \rightarrow & 8+3=11 \end{array}$$

This dynamics doesn't work very well. Complicating factor is that the map $F_n \rightarrow F_{n+1}$ is not quite linear.

Making things linear

We can write

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi}$.

Idea: Replace F_n with the pair (φ^n, ψ^n).

Lemma: Let $n \in \mathbb{N}$ For any $a_1 \cdots a_k \in \{0,1\}^k$ with $n = \sum_{i=1}^k a_i F_{k+2-i}$ define

$$x_n = \sum_{i=1}^k a_i \varphi^{k+2-i}$$

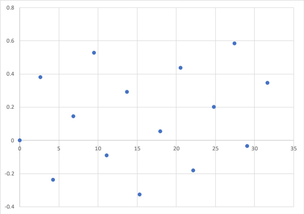
$$y_n = \sum_{i=1}^k a_i \psi^{k+2-i}.$$

The map $n \to (x_n, y_n)$ is well defined and invertible.

$R(n) = \#\{a_1 \cdots a_k \in \{0,1\}^k : \sum_{i=1}^k a_i \varphi^{k+2-i} = x_n\}$ for large k.

$$n = \sum_{i=1}^{k} a_i F_{k+2-i} \to (x_n, y_n) = \left(\sum_{i=1}^{k} a_i \varphi^{k+2-i}, \sum_{i=1}^{k} a_i \psi^{k+2-i} \right).$$

The set $\overline{X} = \{(x_n, y_n) : n \in \mathbb{N}\}$ is a strip through a lattice.



Define $T_i, S_i : \mathbb{R} \to \mathbb{R}$, $T_i(x) = \varphi x + i\varphi^2$, $S_i(y) = \psi y + i\psi^2$.

$$\overline{X} = \{(T_{a_k} \circ \cdots T_{a_1}(0), S_{a_k} \circ \cdots S_{a_1}(0)) : k \in \mathbb{N}, a_i \in \{0, 1\}\}.$$

$$R(n) = \#\{a_1 \cdots a_k \in \{0, 1\}^k : T_{a_k} \circ \cdots T_{a_1}(0) = x_n\}$$

Dynamics on $\overline{X}$

Proposition:

$$\overline{X} = \left\{(n + m\varphi, n + m\psi) : n + m\psi \in \left[\frac{-1}{\varphi^2}, \frac{1}{\varphi}\right]\right\}$$

Proof: Since y_n are generated from 0 using the contractive maps S_0, S_1, they must lie in the attractor of the iterated function system $\{S_0, S_1\}$. Reverse inclusion slightly less clean (still short).

Proposition:

$$(x_{n+1}, y_{n+1}) = \begin{cases} (x_n + 1 + \varphi, y_n + 1 + \psi) & y_n \in \left[\frac{-1}{\varphi^2}, \frac{1}{\varphi^3}\right) \\ (x_n + \varphi, y_n + \psi) & y \in \left[\frac{1}{\varphi^3}, \frac{1}{\varphi}\right] \end{cases}$$

Proof: $(x_{n+1}, y_{n+1}) - (x_n, y_n) \in \overline{X} - \overline{X}$, a discrete set. Use $n = (x_n + y_n)/\sqrt{5}$ and y_n in bounded set.

Counting:

The Zeckendorf expansion of n writes n as the sum of non-consecutive Fibonacci numbers. It is the greedy expansion of n by Fibonacci numbers.

If $x_n = T_{b_k} \circ \cdots T_{b_1}(0)$ (Zeckendorf expansion),

$R(n) = \#\{a_1 \cdots a_k \in \{0,1\}^k : T_{b_k} \circ \cdots T_{b_1}(0) = T_{a_k} \circ \cdots T_{a_1}(0)\}$

i.e. want to count $a_1 \cdots a_k$ such that

$$T_{b_k - a_k} \circ \cdots \circ T_{b_1 - a_1}(0) = 0$$

$T_i(x) = \varphi x + i\varphi^2$, $b_i - a_i \in \{1, 0, -1\}$.

Lemma: If $T_{b_k - a_k} \circ \cdots \circ T_{b_1 - a_1}(0) = 0$ and $b_1 \cdots b_k$ greedy, then for $i \leq k$

$$T_{b_i - a_i} \circ \cdots \circ T_{b_1 - a_1}(0) \in \{0, \varphi, 1 + \varphi\}.$$

Matrices:

Let

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prop: If n has Zeckendorf expansion $b_1 \cdots b_k$ and if

$$(v_1 \ v_2 \ v_3) := (1 \ 0 \ 0) A_{b_1} \cdots A_{b_k}$$

then $R(n) = v_1$, $R(n-1) = v_2 + v_3$. One of v_2, v_3=0.

Ratio $\frac{R(n)}{R(n-1)} = \frac{v_1}{v_2+v_3}$.

Certain tails $b_{k-m} \cdots b_k$ determine $R(n)/R(n-1)$ exactly, e.g. since

$$A_1 A_0 A_0 A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$R(n) = R(n-1)$ whenever $b_{k-3}..b_k = 1001$.

Two Iterated Function Systems

If n has Zeckendorf coding $b_1 \cdots b_k$ then

$$y_n = S_{b_k} \circ \cdots \circ S_{b_1}(0)$$

A word $b_k \cdots b_{k-m}$ corresponds to an interval in $[\frac{-1}{\varphi^2}, \frac{1}{\varphi}]$. n has Zeckendorf coding ending $b_{k-m} \cdots b_m$ if and only if

$$T^n(0) = y_n \in S_{b_k} \circ \cdots S_{b_{k-m}} \left(\left[\frac{-1}{\varphi^2}, \frac{1}{\varphi} \right] \right).$$

But these tails also tell me the ratio $R(n)/R(n-1)$

Putting it all together

We turned $n \in \mathbb{N}$ into a pair (x_n, y_n) with good dynamics. $y_n = T^n(0)$ where T is an irrational rotation.

We could see $R(n)$ and $R(n-1)$ by instead counting $R(x_n)$, $R(x_n - \varphi), R(x_n - \varphi - 1)$ and using the good linear dynamics, turning this into a matrix product indexed by the Zeckendorf coding.

Certain tails in the Zeckendorf coding correspond to knowing the ratio $R(n)/R(n-1)$ exactly. They also correspond to an interval in which y_n must lie. Defining $h(T^n(0)) = R(n)/R(n-1)$ on these intervals gives a Devil's staircase structure.

We can now understand the local multiplicative structure of $R(n)$ by seeing where $T^n(0)$ lies. For any given local pattern in the multiplicative structure, it occurs whenever $T^n(0)$ lies in a region corresponding to the pattern.