Uniform Diophantine approximation on the Hecke group $\mathbf{H}_4$

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Uniform approximation and asymptotic approximation

Theorem (Dirichlet)

Let α be a real number and Q be a positive integer. Then there exists a rational number p/q such that $0 < q \leq Q$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qQ}.$$

Theorem (Hurwitz)

For an irrational number α, there exist infinitely many rational number p/q's satisfying that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Uniform and asymptotic approximation rate

Let α be an irrational number.

- Asymptotic approximation rate $C^{\text{asymp}}(\alpha)$ is defined as the infimum of c such that

$$|q\alpha - p| < \frac{c}{q} \qquad \text{for infinitely many } p/q\text{'s.}$$

- Uniform approximation rate $C^{\text{uniform}}(\alpha)$ is defined as the infimum of c satisfying that there exists q with $1 \leq q \leq Q$

$$|q\alpha - p| < \frac{c}{Q} \qquad \text{for sufficiently large } Q.$$

Continued fraction

Let

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad a_0 \in \mathbb{Z}, \quad a_n \in \mathbb{N}, n \geq 1.$$

Principal convergents for α:

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + 1/a_n}}.$$

Theorem (Legendre)

If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$, *then* $\frac{p}{q} = \frac{p_n}{q_n}$ *for some* n.

Best rational approximating numbers

A rational p/q is called a best approximation of α if any rational $a/b \neq p/q$ such that $0 < b \leq q$ satisfies

$$|q\alpha - p| < |b\alpha - a|.$$

Theorem (Lagrange)

For any irrational α, the set of best approximations of α is either

$$\left\{ \frac{p_n}{q_n} : n \geq 1 \right\} \quad \textit{or} \quad \left\{ \frac{p_n}{q_n} : n \geq 0 \right\}$$

depending on $a_1 = 1$ or $a_1 \geq 2$.

Uniform and asymptotic approximation rate

Let α be an irrational and $\frac{p_n}{q_n}$ be its principal convergents.

- Asymptotic approximation rate

$$C^{\mathrm{asymp}}(\alpha) = \liminf_{n\to\infty} q_n|q_n\alpha - p_n|$$

- Uniform approximation rate

$$C^{\mathrm{uniform}}(\alpha) = \limsup_{n\to\infty} q_n|q_{n-1}\alpha - p_{n-1}|$$

[Davenport-Schmidt]

Lagrange number $L(\alpha) = (C^{\mathrm{asymp}}(\alpha))^{-1}$.

Action of the modular group

For

$$g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}), \qquad g^{-1} = \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}.$$

Let α be a real number.

$$\begin{aligned} q^2 \cdot \left| \alpha - \frac{p}{q} \right| &= q^2 \cdot \left| g(g^{-1}(\alpha)) - \frac{p}{q} \right| \\ &= q^2 \cdot \left| \frac{pg^{-1}(\alpha) + r}{qg^{-1}(\alpha) + s} - \frac{p}{q} \right| \\ &= \left| \frac{ps - rq}{g^{-1}(\alpha) + s/q} \right| = \frac{1}{|g^{-1}(\alpha) - g^{-1}(\infty)|}. \end{aligned}$$

We consider rational numbers $\dfrac{p}{q}$ as $g(\infty)$ for $g \in \mathrm{PSL}_2(\mathbb{Z})$.

Let α be an irrational number.

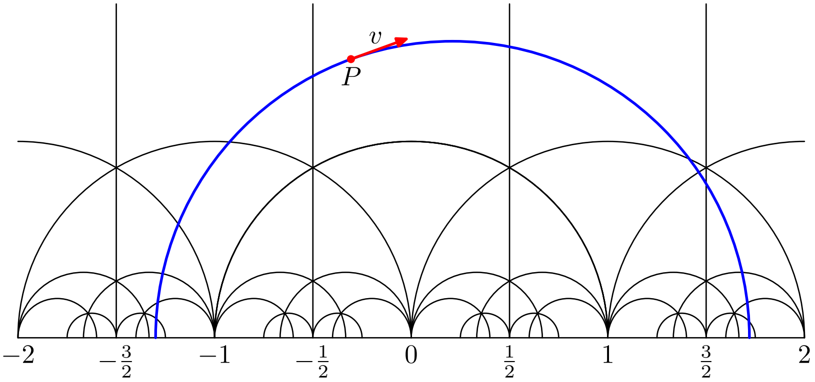
For $g \in \mathrm{PSL}_2(\mathbb{R})$, we write $g = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$.

Then

$$\begin{aligned} L(\alpha) &= \left(\liminf_{\frac{p}{q} \in \mathbb{Q}} q^2 \left| \alpha - \frac{p}{q} \right| \right)^{-1} \\ &= \limsup_{g \in \mathrm{PSL}_2(\mathbb{Z})} \left(c(g)^2 \left| \alpha - \frac{a(g)}{c(g)} \right| \right)^{-1} \\ &= \limsup_{g \in \mathrm{PSL}_2(\mathbb{Z})} \left(c(g)^2 \left| \alpha - g(\infty) \right| \right)^{-1} \\ &= \limsup_{g \in \mathrm{PSL}_2(\mathbb{Z})} \left| g(\alpha) - g(\infty) \right|. \end{aligned}$$

Therefore, Lagrange number $L(\alpha)$ is the limit superior of the height of the geodesic from ∞ to α.

Lagrange number by the continued fraction expansion

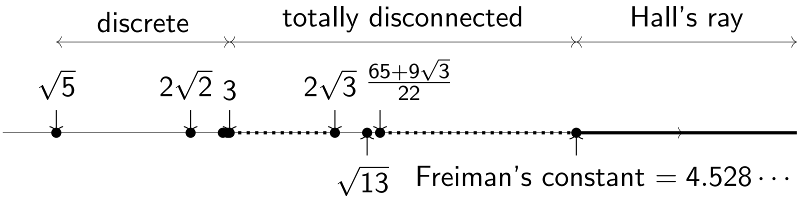


$$L(\alpha) = \limsup \left(q^2 \left| \alpha - \frac{p}{q} \right| \right)^{-1} = \limsup_n \left(q_n^2 \left| \alpha - \frac{p_n}{q_n} \right| \right)^{-1}$$

$$= \limsup_n \left([0; a_{n-1}, \ldots, a_1] + [a_n; a_{n+1}, a_{n+2}, \ldots] \right)^{-1}.$$

Classical Lagrange spectrum

The set of Lagrange numbers is called the Lagrange spectrum $\mathscr{L}$.



Markov (1879,1880) showed

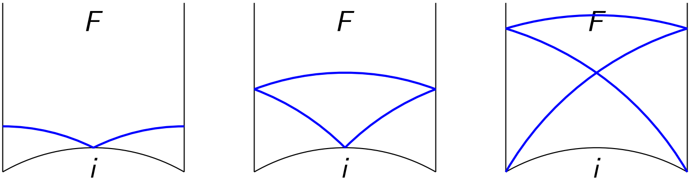
$$\mathscr{L} \cap [0,3) = \left\{ \sqrt{9 - \frac{4}{m^2}} \,\middle|\, m = 1, 2, 5, 13, 29, \ldots \right\}.$$

Here, m is a Markov number, one of solutions to

$$m_1^2 + m_2^2 + m_3^2 = 3m_1 m_2 m_3.$$

Geodesic flow on the fundamental domain

The Lagrange spectrum is the limit superior of the "heights" of the geodesics in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.



Three closed geodesics on the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$.
They have maximal heights $\sqrt{5}$, $2\sqrt{2}$, $2\sqrt{3}$ (from left to right).

Diophantine approximation on group $\mathbf{G}$

Let $\mathbf{G}$ be a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

The group $\mathbf{G}$ acts on the upper half plane $\mathbb{H}$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}, \qquad g(z) = \frac{az+b}{cz+d}.$$

Let $\mathbb{Q}(\mathbf{G})$ be the set of points in $\mathbb{R} \cup \{\infty\}$ that are fixed by parabolic elements of $\mathbf{G}$ and assume that $\infty \in \mathbb{Q}(\mathbf{G})$.

$$\mathbb{Q}(\mathbf{G}) = \{g(\infty) \in \mathbb{R} \cup \{\infty\} : g \in \mathbf{G}\}.$$

As before, for $\alpha \notin \mathbb{Q}(\mathbf{G})$, we define its Lagrange number

$$L_{\mathbf{G}}(\alpha) = \limsup_{g \in \mathbf{G}} \left(c(g)^2 \left|\alpha - g(\infty)\right| \right)^{-1} = \limsup_{g \in \mathbf{G}} \left| g(\alpha) - g(\infty) \right|,$$

the limit superior of the height of the geodesic from ∞ to α.

The Hecke group

The Hecke group $\mathbf{H}_q$ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix},$$

where $\lambda_q = 2\cos\frac{\pi}{q}$ and $q \geq 3$ is an integer.

The Hecke group $\mathbf{H}_q$ has the presentation

$$\mathbf{H}_q \cong \left\langle S, T \mid S^2 = I, (ST)^q = I \right\rangle.$$

If $q = 3$, then $\lambda_3 = 1$ and $\mathbf{H}_3 = \mathrm{PSL}_2(\mathbb{Z})$.
If $q = 4$, then $\lambda_4 = \sqrt{2}$.
Moreover,

$$\lambda_5 = \frac{\sqrt{5}+1}{2}, \qquad \lambda_6 = \sqrt{3}.$$

The Hecke group

Define the Lagrange spectrum on group $\mathbf{G}$ by

$$\mathscr{L}(\mathbf{G}) = \{L_{\mathbf{G}}(\alpha) \,|\, \alpha \in \mathbb{R} \setminus \mathbb{Q}(\mathbf{G})\}\,.$$

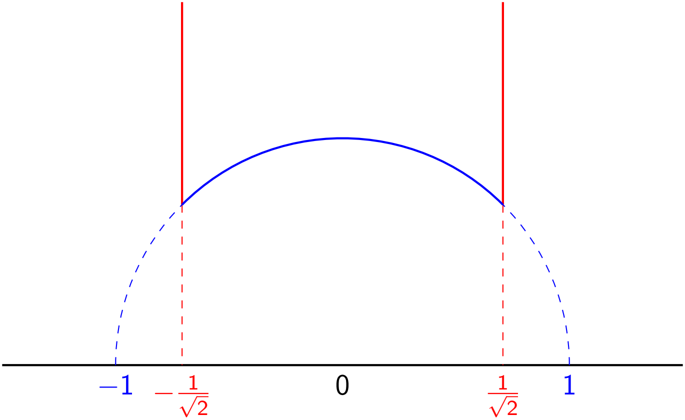
In 1970's, the Lagrange spectrum $\mathscr{L}(\mathbf{H}_4)$ is known as the spectrum of 2-minimal form by A. Schmidt and the spectrum on sublattice of index 2 by Vulakh. The spectrum $\mathscr{L}(\mathbf{H}_6)$ is also studied by A. Schmidt.

Lehner (1985), Haas and Series (1986) studied the minimum of Lagrange spectrum, which is called Hurwitz's constant, for the Hecke group $\mathbf{H}_q$. In particular, if q is even, then the minimum of the Lagrange spectrum $\mathscr{L}(\mathbf{H}_q)$ is equal to 2.

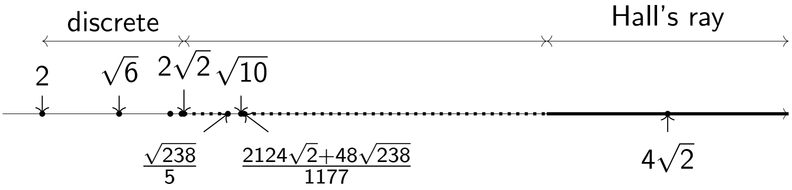
The discrete part of the Lagrange spectrum $\mathscr{L}(\mathbf{H}_q)$ is studied by Series (1988) for $q = 5$ and by Vulakh (1997) for general cases .

Fundamental domain of the Hecke group $\mathbf{H}_4$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & \lambda_4 \\ 0 & 1 \end{pmatrix}, \quad \lambda_4 = \sqrt{2}.$$



The Lagrange spectrum $\mathscr{L}(\mathbf{H}_4)$

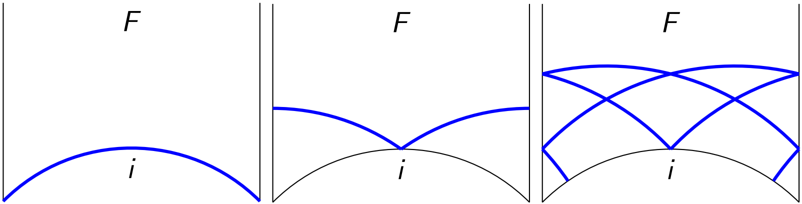


The discrete part of the spectrum on $\mathbf{H}_4$ is known by Schmidt and Vulakh independently

$$\left\{\sqrt{8-\frac{2}{x^2}}\,|x=1,5,29,65,\dots\right\}\cup\left\{\sqrt{8-\frac{4}{y^2}}\,|y=1,3,11,17,\dots\right\}$$

Here, x and y are integral solutions to $2x^2+y_1^2+y_2^2=4xy_1y_2$.

Three closed geodesics in the fundamental domain of Hecke group $\mathbf{H}_4$ on the upper half plane $\mathbb{H}$ with three lowest heights.



Note that the discrete part of the spectrum corresponds to closed geodesics.

Theorem (K- Deokwon Sim)

1. *For any* $\varepsilon > 0$, *we have*

$$\dim_H \left(\mathscr{L}(\mathbf{H}_4) \cap \left[0, 2\sqrt{2} + \epsilon\right)\right) > 0.$$

2. *There are two maximal gaps in* $\mathscr{L}(\mathbf{H}_4)$

$$\left(\frac{\sqrt{238}}{5}, \sqrt{10} \right) \quad \textit{and} \quad \left(\sqrt{10}, \frac{2124\sqrt{2} + 48\sqrt{238}}{1177} \right).$$

3. $\mathscr{L}(\mathbf{H}_4)$ *contains every real number greater than* $4\sqrt{2}$.

Note that $\sqrt{10}$ is an isolated point. Two gaps in look similar to the gaps $(\sqrt{12}, \sqrt{13})$ and $\left(\sqrt{13}, \frac{1}{22}(9\sqrt{3} + 65)\right)$ in the classical Markoff and Lagrange spectra

Rosen continued fraction

The Rosen continued fraction expansion is given by

$$\alpha = a_0\sqrt{2} + \cfrac{\epsilon_1}{a_1\sqrt{2} + \cfrac{\epsilon_2}{a_2\sqrt{2} + \cfrac{\epsilon_3}{a_3\sqrt{2} + \ddots}}},$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$ and $\epsilon_i \in \{-1, +1\}$.

Note that the expansion is not unique.

To make it unique, we introduce one of the following restrictions :

$$\begin{aligned} \text{Rosen CF} &: \epsilon_{i+1} = -1 \text{ implies } a_i \geq 2 \qquad \text{for all } i \geq 1, \\ \text{Dual Rosen CF} &: \epsilon_i = -1 \text{ implies } a_i \geq 2 \qquad \text{for all } i \geq 1. \end{aligned}$$

Let p_n/q_n and p'_n/q'_n be the convergent of α with the Rosen and the dual Rosen continued fraction expansion respectively.

$\mathbf{H}_4$-best approximation numbers

If p/q is a $\mathbf{H}_4$-best approximation of α, then we have

$$|q\alpha - p| < \frac{1}{q}.$$

If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$, then $\frac{p}{q}$ is a $\mathbf{H}_4$-best approximation of α.

Theorem (Ayreena Bakhtawar, K, Seul Bee Lee)

The set of $\mathbf{H}_4$-best approximation of α is

$$\left\{ \frac{p_n}{q_n} \mid n \geq 0 \right\} \cup \left\{ \frac{\tilde{p}_n}{\tilde{q}_n} \mid n \geq 1 \right\},$$

where p_n/q_n and $\tilde{p}_n/\tilde{q}_n$ are the convergents of the Rosen and the dual Rosen continued fraction respectively.

Dirichlet theorem for $\mathbf{H}_4$

Theorem (Ayreena Bakhtawar, K, Seul Bee Lee)
Let $\alpha \in \mathbb{R}$. For every Q there exists $p/q \in \mathbb{Q}(\mathbf{H}_4)$ such that

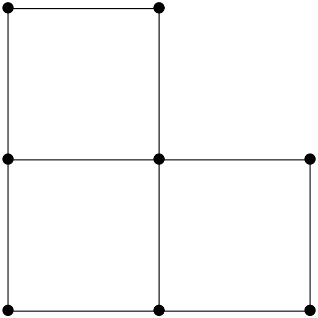
$$|q\alpha - p| < \frac{1+\sqrt{2}}{2}\frac{1}{Q}, \qquad 1 \leq q \leq Q.$$

Note that $\alpha = 1 \notin \mathbb{Q}(\mathbf{H}_4)$ and

$$C^{\mathrm{uniform}}_{\mathbf{H}_4}(1) = \frac{\sqrt{2}+1}{2}.$$

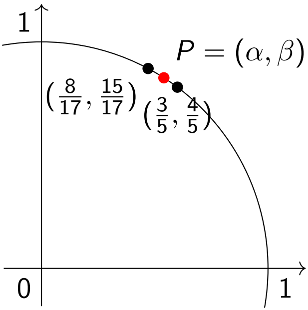
Motivations

1. Diophantine approximation on circles.
2. Diophantine approximation with specific parity rationals.
3. Translation surface - covering time on the translation surface.



Ongoing project with Luca Marchese and Stefano Marmi

Diophantine approximation on the unit circle



$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

A rational point $Z = \left(\frac{a}{c}, \frac{b}{c}\right) \in S^1$ corresponds a primitive Pythagorean triple (a, b, c) since $a^2 + b^2 = c^2$.

The height function $\mathrm{Ht}\left(\frac{a}{c}, \frac{b}{c}\right) = c$.

For an irrational $P \in S^2$, we define the Lagrange number as

$$L(P) = \overline{\lim_{Z}} \frac{1}{\mathrm{Ht}(Z) d(P, Z)} = \overline{\lim} \left(c \sqrt{\left(\alpha - \frac{a}{c}\right)^2 + \left(\beta - \frac{b}{c}\right)^2} \right)^{-1}$$

Height should be modified for the unit circle

Let $\boldsymbol{\sigma} : \mathbb{R} \to S^1$ be the inverse of the stereographic projection.

$$\boldsymbol{\sigma}(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right).$$

Since $\mathrm{d}s = \dfrac{2\,\mathrm{d}t}{t^2+1}$, $\qquad \left| \boldsymbol{\sigma}(\xi) - \boldsymbol{\sigma}\left(\frac{p}{q}\right) \right| \approx \frac{2q^2}{p^2+q^2} \left| \xi - \frac{p}{q} \right|.$

$$\boldsymbol{\sigma}\left(\frac{p}{q}\right) = \left(\frac{2pq}{p^2+q^2}, \frac{p^2-q^2}{p^2+q^2} \right), \quad \mathrm{Ht}(P) = \begin{cases} p^2+q^2, \\ (p^2+q^2)/2. \end{cases}$$

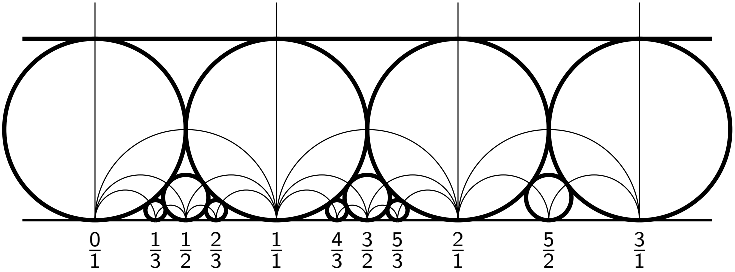
We should consider

$$\mathrm{Ht}_{\mathbb{Q}}^{\text{modified}}\left(\frac{p}{q}\right) := \begin{cases} q^2 & \text{if } p \not\equiv q \bmod 2, \\ q^2/2 & \text{if } p \equiv q \bmod 2, \end{cases}$$

Ford's circles and the Diophantine approximation on $\mathbb{R}$

For an irrational ξ, there are infinitely many p/q's satisfying

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

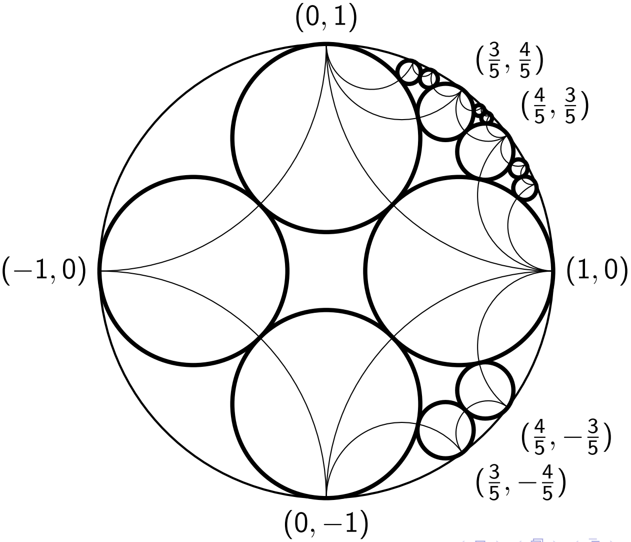


The radius of the horocycle based at $\dfrac{p}{q}$ is $\dfrac{1}{2q^2}$.

Two circles based at $\dfrac{p}{q}$ and $\dfrac{p'}{q'}$ are tangent $\Longleftrightarrow |pq' - qp'| = 1$.

Ford's circles for the unit circle

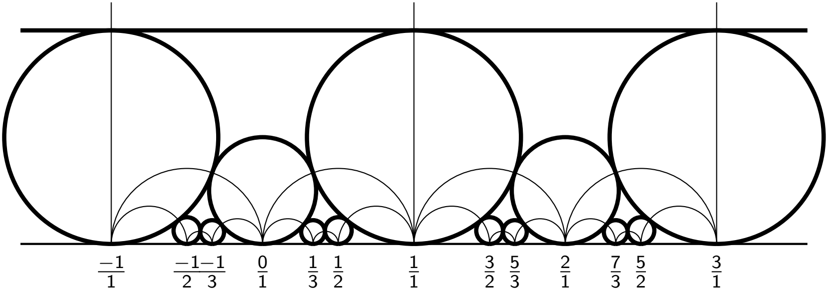
The radius of the horocycle at $(\frac{a}{c}, \frac{b}{c})$ is $1/(1+\sqrt{2}c)$. Horocycles at $(\frac{a}{c}, \frac{b}{c})$ and $(\frac{a'}{c'}, \frac{b'}{c'})$ are tangent $\Leftrightarrow aa' + bb' - cc' = -1$.



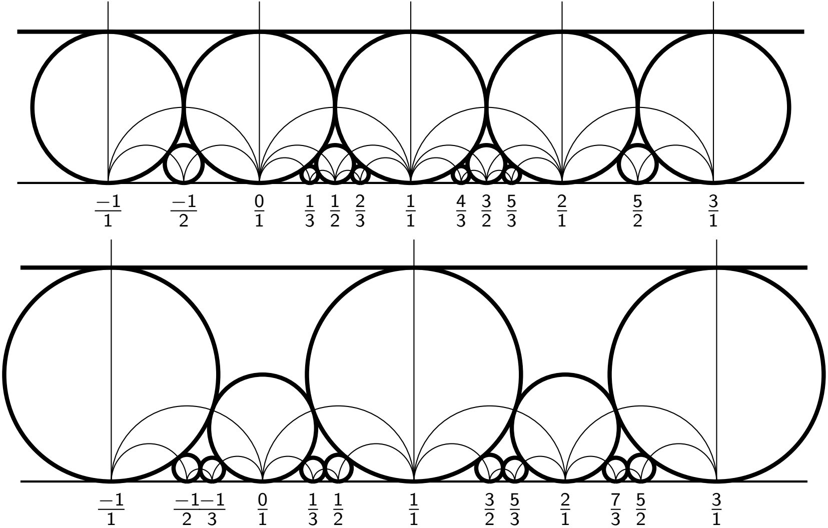
Stereographic projection $(\alpha, \beta) \mapsto \frac{\alpha}{1-\beta}$

The horocycle based at $(a/c, b/c)$ with radius $1/(1+\sqrt{2}c)$ on $\mathbf{S}^1_{\mathrm{I}}$ is mapped to the horocycle based at $a/(c-b)$ with radius $1/(\sqrt{2}(c-b))$ in $\mathbb{H}^2$.

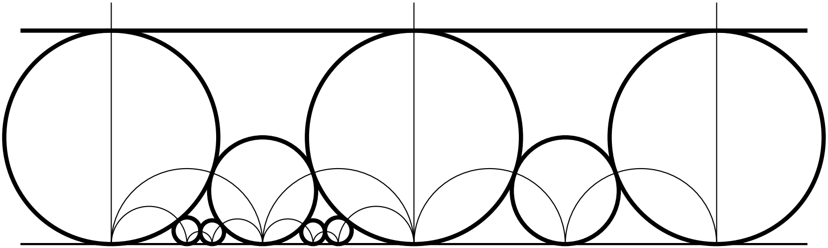
$$(a, b, c) \leftrightarrow \left(\frac{a}{c}, \frac{b}{c}\right) \in S^1 \mapsto \frac{a/c}{1 - b/c} = \frac{a}{c-b}.$$



Comparision of Ford circles



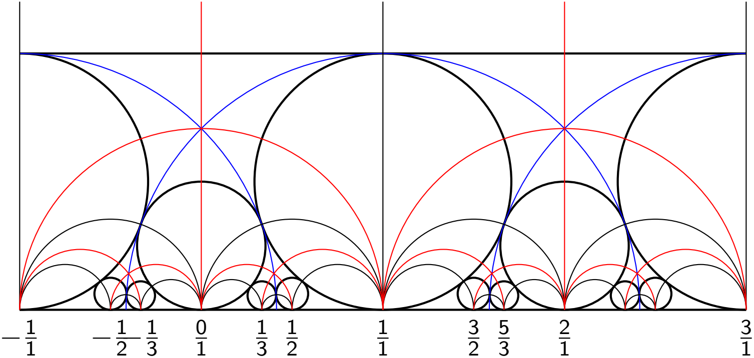
Approximation by elements of $\sqrt{2}\mathbb{Q}$ instead of $\mathbb{Q}$.



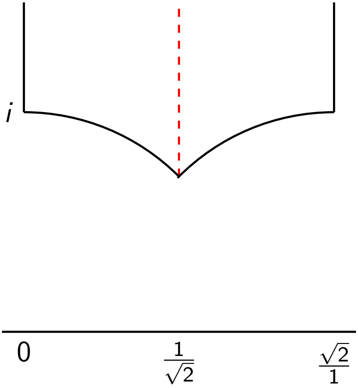
$$\frac{p}{q} \mapsto \frac{\sqrt{2}p}{q} = \begin{cases} \frac{\sqrt{2}p}{q} & \text{if } q \text{ is odd,} \\ \frac{p}{\sqrt{2}q'} & \text{if } q = 2q'. \end{cases}$$

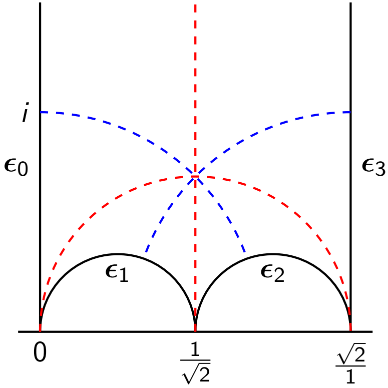
If q is even, then the height become $2q'^2 = q^2/2$.

The spectrum on the unit circle is a constant multiple of and the spectrum on the Hecke group $\mathbf{H}_4$.



Fundamental domain of $\mathbf{H}_4$





Let $T = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $R = ST^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{pmatrix}$.

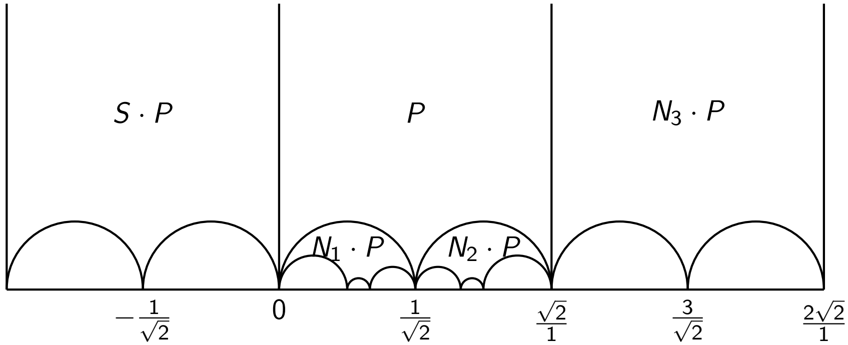
We have fundamental domain F of $\mathbf{H}_4$ (left) and the Ideal quadrilateral $Q = F \cup R(f) \cup R^2(F) \cup R^3(F)$ (right).

Expansion of a real number by $\mathbf{H}_4$

Let

$$N_1 = RS = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}, \qquad N_2 = R^2 S = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix},$$

$$N_3 = R^3 S = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}.$$



Expansion of the real number

Since

$$N_1 \cdot [0, \infty] = \left[0, \frac{1}{\sqrt{2}}\right], \qquad N_2 \cdot [0, \infty] = \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right],$$
$$N_3 \cdot [0, \infty] = [\sqrt{2}, \infty],$$

we have

$$\begin{aligned} [0, \infty] &= N_1 \cdot [0, \infty] \cup N_2 \cdot [0, \infty] \cup N_3 \cdot [0, \infty] \\ &= \bigcup_{(d_1, \ldots, d_k) \in \{1,2,3\}^n} N_{d_1} N_{d_2} \cdots N_{d_n} \cdot [0, \infty]. \end{aligned}$$

We write $\alpha = [d_1, d_2, \ldots] \geq 0$ if

$$\alpha \in N_{d_1} N_{d_2} \cdots N_{d_n} \cdot [0, \infty] \quad \text{for all } n \geq 1.$$

Let

$$M_n = N_{d_1} N_{d_2} \cdots N_{d_n} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad \alpha = M_n \cdot \alpha_n,$$

where $\alpha_n \in [0, \infty]$. Then for all $n \geq 1$,

$$M_n \cdot 0 = \frac{b_n}{d_n} < \alpha < \frac{a_n}{c_n} = M_n \cdot \infty.$$

Theorem

- *If $d_n < c_n$, then $\frac{b_n}{d_n}$ is a $\mathbf{H}_4$-best approximation.*
- *If $c_n < d_n$, then $\frac{a_n}{c_n}$ is a $\mathbf{H}_4$-best approximation.*

- *If $\alpha_n < 1$, then $\frac{b_n}{d_n}$ is a $\mathbf{H}_4$-best approximation.*
- *If $\alpha_n > 1$, then $\frac{a_n}{c_n}$ is a $\mathbf{H}_4$-best approximation.*

There are no more $\mathbf{H}_4$-best approximations.

Let

$$A = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Then

$$N_1 = JAJ, \quad N_2 = B, \quad N_3 = A, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using the fact that $BJ = JB$, we can expand a real number with

$$A \cdot \alpha = \sqrt{2} + \alpha, \quad B \cdot \alpha = \sqrt{2} - \frac{1}{\sqrt{2} + \alpha}, \quad JA \cdot \alpha = \frac{1}{\sqrt{2} + \alpha}$$

or

$$A \cdot \alpha = \sqrt{2} + \alpha, \quad B \cdot \alpha = \sqrt{2} - \frac{1}{\sqrt{2} + \alpha}, \quad AJ \cdot \alpha = \sqrt{2} + \frac{1}{\alpha}.$$

Both expansions gives convergent matrix of M_n or $M_n J$.

Remind that $\alpha = M_n \cdot \alpha_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \cdot \alpha_n$.

The expansion by A, B, JA gives the Rosen continued fraction matrix. Moreover, the product matrix is

$$L_n = \begin{cases} M_n & \text{if } c_n < d_n, \\ M_n J & \text{if } c_n > d_n. \end{cases}$$

The expansion by A, B, AJ gives the dual Rosen continued fraction matrix. Moreover, the product matrix is

$$N_n = \begin{cases} M_n & \text{if } \alpha_n \geq 1, \\ M_n J & \text{if } \alpha_n < 1. \end{cases}$$

Note $L_n \cdot \infty$ is the convergent of the Rosen continued fraction and $N_n \cdot \infty$ is the convergent of the dual Rosen continued fraction.

Thank you very much for attention!