Linear repetition in polytopal cut and project sets

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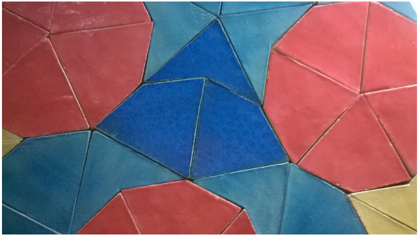
Periodic and aperiodic tilings

Periodic tiling

Aperiodic tiling

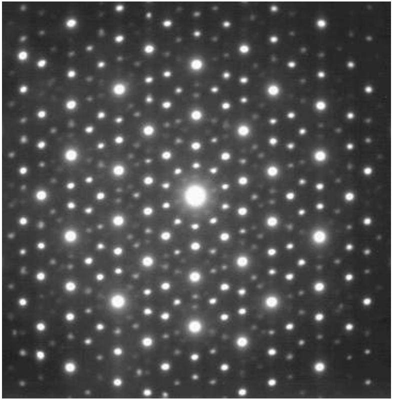
Aperiodic order

- Wang tiles 1961, Berger 1966
- Penrose tiling 1970s

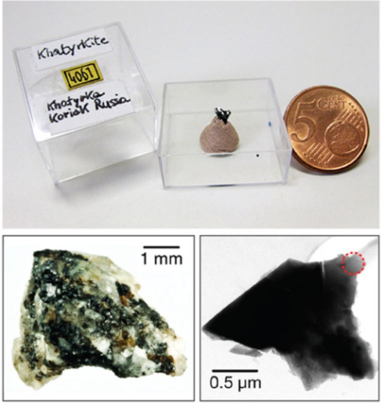


Quasicrystals

- Schechtman 1984 (Nobel prize 2011)
- Paul Steinhardt and Dov Levine 1984



Quasicrystals in nature



Paul Steinhardt, CC BY-SA 4.0

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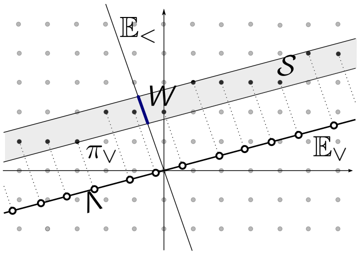
Patterns in cut and project sets

Patterns in cut and project sets

- $\mathbb{E}_\vee$ totally irrational d-subspace of $\mathbb{E}$
- $\mathbb{E}_<$ any transversal $(k-d)$-subspace
- Γ lattice
- W window
- $\mathcal{S} = W + \mathbb{E}_\vee$ slice
- $\pi_\vee$ projection to $\mathbb{E}_\vee$

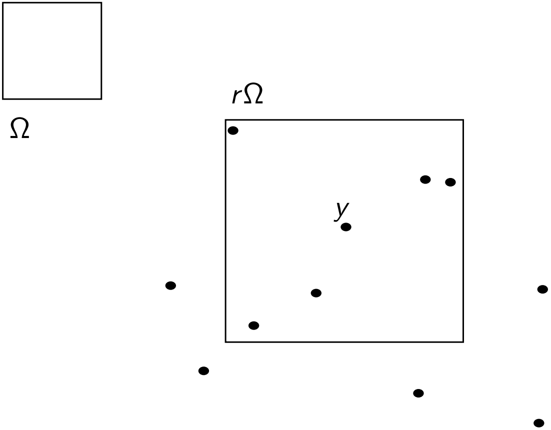


Cut and project set

$$\Lambda = \pi(\mathcal{S} \cap (\Gamma + s)) \subset \mathbb{E}_\vee$$

Patterns in cut and project sets

Size r neighbourhood of a point $y \in \Lambda$, of shape Ω:



Pattern

$$P(y, r) = \Lambda \cap (r\Omega + y)$$

Analysing patterns

Analysing patterns

- $P(y, r)$ equivalent to $P(y', r)$ if $P(y, r) = P(y', r) + y - y'$
- $[P(y, r)] = \mathcal{P}(y, r)$
- **complexity**: $p(r) =$ number of different patterns of size r
- **repetitivity function**: $\mathrm{Rep} : \mathbb{R}_+ \to \mathbb{R}_+$, any pattern of size $\mathrm{Rep}(r)$ contains all patterns of size r

Some previous work

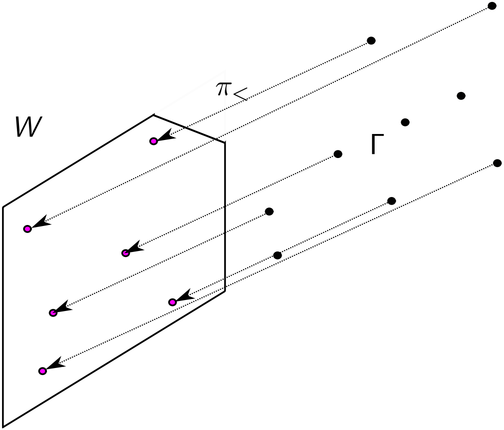
- growth rate of $p(r)$: ∼Julien 2010
- linearity of Rep?: Lagarias and Pleasants 2003

Analysing patterns

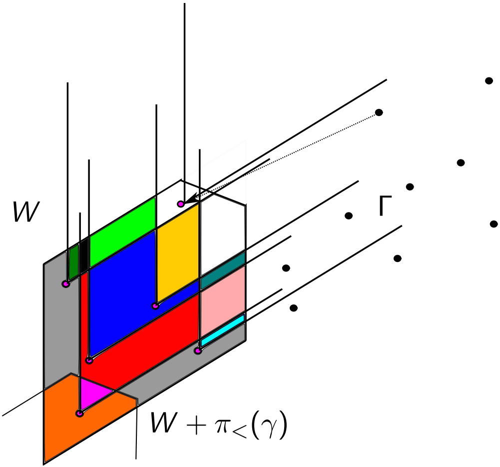
Lemma

There is a one-to-one correspondence between

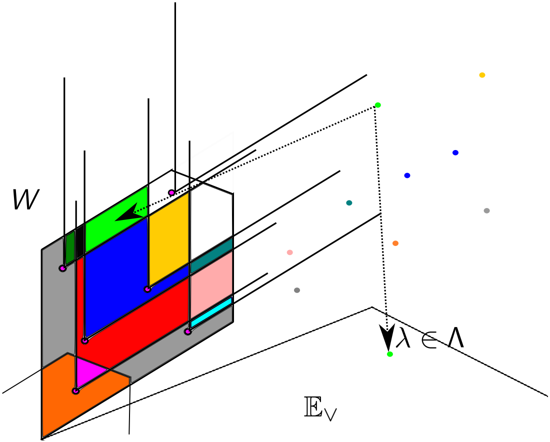
- *patterns* $[P]$, *and*
- **acceptance domains** $A \subset W$.



W
$\pi_<$
Γ



W
Γ
$W + \pi_<(\gamma)$


W
$\lambda \in \Lambda$
$\mathbb{E}_V$

Complexity and repetitivity

Lemma

The complexity function can be evaluated by counting the number of acceptance domains.

Lemma

The repetitivity function can be evaluated by finding $Rep(r)$ so large that every acceptance domain contains a point from $P(y, Rep(r))$. This is equivalent to acceptance domains being of 'even size'.

Results for cubical cut and project sets

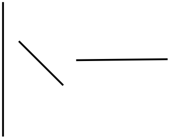
- Characterisation of cut and project sets with linear repetitivity for a cube W
- Number of different frequences at which patches are observed as well as the speed at which they converge
- Equivalent form of a famous open problem in Diophantine approximation

(subsets of $\{$Haynes, Julien, K., Sadun, Walton$\}$)

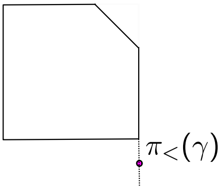
Our results for polytopal windows

Polytopes and stabilisers

- W intersection of positive half-spaces $H^+ \in \mathcal{H}^+$
- **sides** $H \in \mathcal{H}$ affine hyperplanes in $\mathbb{E}_<$
- **stabiliser** $\Gamma^H = \{\gamma \in \Gamma \mid \pi_<(\gamma) + H = H\}$

$\mathcal{H} =$ 

$\gamma \in \Gamma^H$:



Our results: Complexity

Theorem (K., Walton, *Erg. Theory Dynam. Systems*, 2020)

The complexity $p(r)$ of a cut and project set with a polytope W grows as $p(r) \asymp r^{\alpha}$, where the number $\alpha \in \{d, \ldots, d(k-d)\}$ depends on the shape of W relative to $\pi_<(\Gamma)$.

Theorem

*For $H \in \mathcal{H}$, let β_H be the dimension of the linear span of $\pi_<(\Gamma^H)$. Given a **flag** $f \subset \mathcal{H}$, let*

$$\alpha_f = \sum_{H \in f} d - rk(\Gamma^H) + \beta_H .$$

Then $\alpha = \max_f \alpha_f$.

Our results: Linear repetitivity

Theorem (K., Walton, 2020?)

A polytopal, **homogeneous**, **hyperplane spanning** *cut and project set has a linear repetitivity function if and only if*

(i) *it has minimal complexity, and*

(ii) *it decomposes into* **subsystems** *satisfying a Diophantine condition.*

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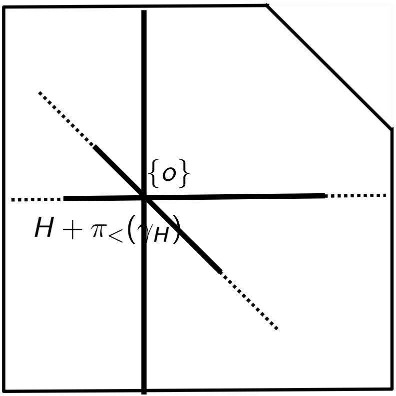
- homogeneous and hyperplane spanning: window W is a polytope that is 'well-aligned' with respect to $\pi_<(\Gamma)$
- condition (ii): $\Gamma_<$ is 'not too dense' in a quantifiable way

Homogeneous

- there is some $o \in \mathbb{E}_<$ such that
- all $H \in \mathcal{H}$ have some $\gamma_H \in \Gamma$,

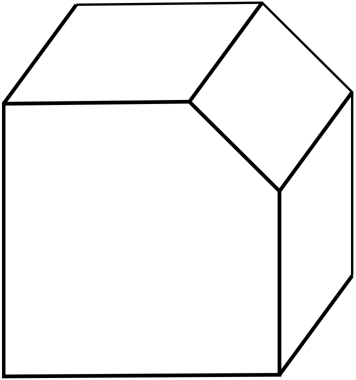
$$\bigcap_{H \in \mathcal{H}} (H + \pi_<(\gamma_H)) = \{o\}$$



Hyperplane spanning

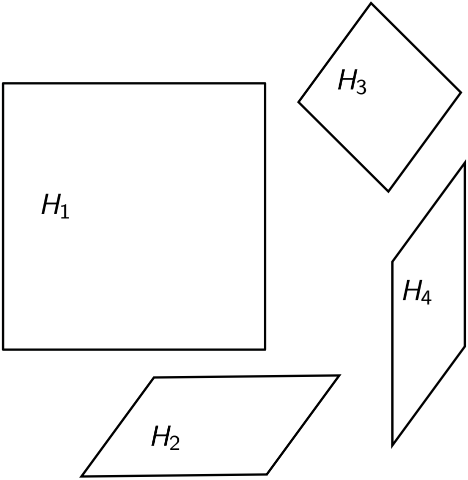
- $H \in \mathcal{H}$
- the linear span of stabiliser $\pi_<(\Gamma^H)$ is the subspace generated by H
- $\beta_H = k - d - 1$ (from calculating complexity)

Decomposing schemes: Example



W

$\mathcal{H} =$



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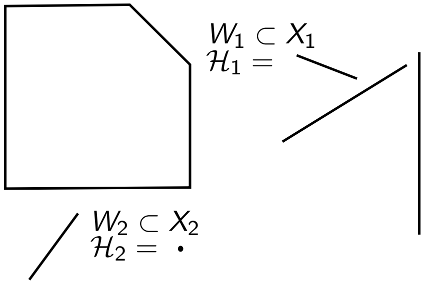

- $\mathcal{H}_i$ define W_i
- Γ ‘splits’ into subgroups $\Gamma^1 + \Gamma^2$ with $\pi_<(\Gamma^i) \subset X_i$
- $H \in \mathcal{H}_i$ define stabilisers $\Gamma^i(H) = \{\gamma \in \Gamma^i \mid \pi_<(\gamma) + H = H\}$

Decomposing schemes

Find a maximal decomposition into (X_i, W_i, Γ^i)

- $\mathbb{E}_< = X_1 + \cdots + X_m$
- $W = W_1 + \cdots + W_m$, $W_i \subset X_i$
- Γ has $\Gamma^1 + \cdots + \Gamma^m$ as a finite index subgroup, $\pi_<(\Gamma^i) \subset X_i$ densely
- (X_i, W_i, Γ^i) are not cut and project schemes, however
- $\pi_<(\Gamma^i)$-translates of W_i define ‘acceptance domains’ etc.

Exact statement

Theorem

A polytopal, homogeneous, hyperplane spanning cut and project set has a linear repetitivity function if and only if

(i) *for any choice of flag $f \subset \mathcal{H}$*

$$\sum_{H \in f} k - rk(\Gamma^H) - 1 = d, \text{ and}$$

(ii) *each subsystem (X_i, W_i, Γ^i) satisfies: there is $c > 0$ such that for each $\gamma \in \Gamma^i(r) \setminus \{0\}$,*

$$|\pi_<(\gamma)| \geq \frac{c}{r^{rk\,\Gamma^i - rk\,\Gamma^i(H) - 1}}.$$

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Lemma

For $H, K \in \mathcal{H}_i$, *the ranks of the stabilisers* $rk\Gamma^i(H) = rk\Gamma^i(K)$.

Remarks

- strict generalisation of [HKW] (cubes W)
- canonical windows
- sharpness?
- how common is linear repetitivity?