Topology of univoque sets in real base expansions

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Abstract

Non-integer base expansions have been constantly studied since Rényi's classical paper in 1957.

A new impetus was given around 1990 when Erdős, Horváth and Joó discovered curious uniqueness phenomena.

The field has been enriched by many important contributions by Alcaraz Barrera, Allaart, Baatz, Baiocchi, Baker, Y. Cai, Dajani, Daróczy, Dekking, Glendinning, Kátai, Kallós, D. Kong, Kraaikamp, Lai, J. Li, W. X. Li, J. Lu, Pedicini, Pethő, Sidorov, Steiner, L. Wang, Y. R. Zou,. . . .

Our talk is based on two papers in collaboration with M. de Vries and P. Loreti (Top. and Appl., 2016 and 2022). For simplicity we restrict ourselves to the two-digit alphabet $\{0, 1\}$.

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Part 1: Introduction and examples

Greedy expansions

Given a real number $q > 1$, the equality

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots =: (c_i)_q, \quad (c_i) \in \{0, 1\}^{\mathbb{N}}$$

is called an *expansion of x in base q*.

Theorem (Rényi, 1957)

If $1 < q \le 2$, then every $x \in [0, \frac{1}{q-1}]$ has a lexicographically largest (called greedy) expansion $b(x, q) = (b_i)$.

Proof.

Proceeding by induction, choose $b_n := 1$ unless

$$\left(\sum_{i=1}^{n-1} \frac{b_i}{q^i} \right) + \frac{1}{q^n} > x.$$

□

Finite and infinite sequences

We say that a sequence or expansion (c_i) is

- finite if it has a last digit $c_n = 1$, i.e., if it ends with 10^∞;
- infinite otherwise, i.e., if it contains infinitely many 1 digits, or if $(c_i) = 0^\infty$.
- doubly infinite if both (c_i) and its reflection $(1 - c_i)$ are infinite, i.e., if (c_i) contains both infinitely many 1 digits and infinitely many 0 digits, or if $(c_i) \in \{0^\infty, 1^\infty\}$.

Quasi-greedy expansions

Theorem (Daróczy and Kátai, 1993)

If $1 < q \leq 2$, then every $x \in [0, \frac{1}{q-1}]$ has a lexicographically largest infinite (called quasi-greedy) expansion $a(x,q) = (a_i)$.

Proof.

If $x = 0$ then the only expansion 0^∞ is infinite by definition.
Otherwise, proceeding by induction, choose $a_n := 1$ unless

$$\left(\sum_{i=1}^{n-1} \frac{a_i}{q^i}\right) + \frac{1}{q^n} \geq x.$$

Then (a_i) contains infinitely many 1 digits. □

Note. For $1 < q < 2$ the expansions $a(x,q)$ are in fact doubly infinite.

Examples: greedy expansions of $x = 1$

- For $q = 2$ we have $b(1,2) = 1^\infty$:

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

- For the *Tribonacci* base $q = \varphi_3 \approx 1.839$ we have $b(1, \varphi_3) = 111\,0^\infty$:

$$q^3 = q^2 + q + 1 \Longrightarrow 1 = \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3}.$$

- For the *Golden ratio* base $q = \varphi \approx 1.618$ we have $b(1, \varphi) = 11\,0^\infty$:

$$q^2 = q + 1 \Longrightarrow 1 = \frac{1}{q} + \frac{1}{q^2}.$$

Examples: quasi-greedy expansions of $x = 1$

- If $b(1,q) = (b_i)$ is infinite, then $a(1,q) = b(1,q)$; in particular, for $q = 2$ we have
$$a(1,2) = 1^\infty.$$
- If $b(1,q) = (b_i)$ is finite with a last $b_n = 1$, then
$$a(1,q) = (b_1 \cdots b_{n-1}(b_n - 1))^\infty .$$
In particular,
$$a(1,\varphi_3) = (110)^\infty \quad \text{and} \quad a(1,\varphi) = (10)^\infty.$$

Examples: general expansions of $x = 1$

- If $q = 2$, then 1^∞ is the only expansion of 1.
- For $q = \varphi_3$ we have $a(1, \varphi_3) = (110)^\infty$, and for every $n = 3k$ there is a finite expansion of 1, first differing from $(110)^\infty$ at the nth digit:

$$(110)^{k-1}111\,0^\infty, \quad k = 1, 2, \ldots .$$

- For $q = \varphi$ we have $a(1, \varphi) = (10)^\infty$, and for every $n \geq 1$ there is a unique expansion of 1, first differing from $(10)^\infty$ at the nth digit:

$$(10)^{k-1}11\,0^\infty \quad \text{and} \quad (10)^{k-1}0\,1^\infty, \quad k = 1, 2, \ldots .$$

There are no other expansions in the above cases; hence 1 has a unique expansion in base 2, a unique infinite expansion in base φ_3, and a unique doubly infinite expansion in base φ.

Three more examples of Erdős et al.

- If $1 < q < \varphi$, then $x = 1$ has $2^{\aleph_0}$ expansions (Erdős–Joó–K., 1990).
- If $q > 1$ is defined by the equation $(1(10)^\infty)_q = 1$, then $x = 1$ has no other expansions (Erdős–Horváth–Joó, 1991).
- If $q > 1$ is defined by the equation

$$\left(1^9(0^8 1)^{m-1}(0^4 1)^\infty\right)_q = 1, \quad m = 1, 2, \ldots,$$

  then $x = 1$ has exactly m expansions (Erdős–Joó, 1992).

The above examples motivate the investigation of the number of expansions of a given x in a given base q.

In this talk we study the unique expansions, and we start with the unique expansions of $x = 1$.

Part 2: Univoque bases

The sets $\mathcal{U}$, $\widetilde{\mathcal{U}}$ and $\mathcal{V}$

Let us introduce the following sets:

- $q \in \mathcal{U} \Longleftrightarrow x = 1$ has a unique expansion;
- $q \in \widetilde{\mathcal{U}} \Longleftrightarrow x = 1$ has a unique infinite expansion;
- $q \in \mathcal{V} \Longleftrightarrow x = 1$ has a unique doubly infinite expansion.

Then

$$\mathcal{U} \subsetneqq \widetilde{\mathcal{U}} \subsetneqq \mathcal{V};$$

furthermore,

$$2 \in \mathcal{U}, \quad \varphi_3 \in \widetilde{\mathcal{U}} \setminus \mathcal{U}, \quad \varphi \in \mathcal{V} \setminus \widetilde{\mathcal{U}} \quad \text{and} \quad \min \mathcal{V} = \varphi.$$

The size of $\mathcal{U}$

We recall the definition:

$$q \in \mathcal{U} \iff x = 1 \quad \text{has a unique expansion.}$$

Theorem (Erdős–Horváth–Joó, 1991)

- *$\mathcal{U}$ has zero Lebesgue measure;*
- $\operatorname{card} \mathcal{U} = 2^{\aleph_0}$*;*
- *$\mathcal{U}$ is of the first category.*

Theorem (Daróczy–Kátai, 1995)

- *$\mathcal{U}$ has maximal Hausdorff dimension.*

Lexicographic characterization

We denote by $\mathcal{U}' \subseteq \{0,1\}^{\mathbb{N}}$ the set of unique expansions. Writing $\overline{0} := 1$ and $\overline{1} := 0$ a Parry type characterization holds:

Theorem (Erdős–Joó–K., 1990)

A sequence $(c_i) \in \{0,1\}^{\infty}$ *belongs to* $\mathcal{U}'$ *if and only if*

$$c_{n+1}c_{n+2}\cdots < c_1c_2\cdots \quad \textit{whenever} \quad c_n = 0,$$

and

$$\overline{c_{n+1}c_{n+2}\cdots} < c_1c_2\cdots \quad \textit{whenever} \quad c_n = 1.$$

Example

$1(10)^{\infty} \in \mathcal{U}'$ because the left hand sides start with 10 or 01, while the right hand sides start with 11.

The topological structure of $\mathcal{U}$

Theorem (Loreti–K., 1998, 2007, Loreti–Pethő–K., 2003)

- *$\mathcal{U}$ has a smallest element $q_{KL} \approx 1.787$.*
- *$\mathcal{U}$ is closed from above, but not from below, and $\overline{\mathcal{U}} = \widetilde{\mathcal{U}}$.*
- *$\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable, and dense in $\overline{\mathcal{U}}$.*
- *$\overline{\mathcal{U}}$ is a Cantor set of zero Lebesgue measure.*
- *$\mathcal{U}$ is nowhere dense.*

Remark

$b(1, q_{KL})$ is the truncated Thue–Morse sequence 1101 0010 $\cdots$.

The finer structure of $\mathcal{U}$

Theorem (de Vries–K., 2009)

- $(1,2]\setminus\overline{\mathcal{U}}$ *is an infinite union of disjoint open intervals* (q_0, q_0^*).
- *The left endpoints are algebraic integers, running over* $1\cup(\overline{\mathcal{U}}\setminus\mathcal{U})$.
- *The right endpoints run over a proper subset* $\mathcal{U}^*$ *of* $\mathcal{U}$.

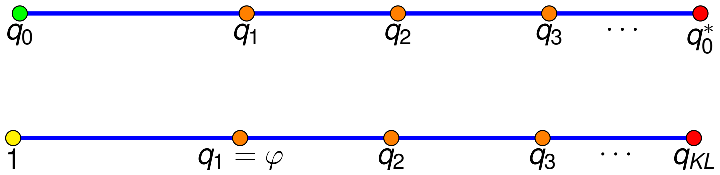
Theorem (Allouche–Cosnard, 2000, Kong–Li, 2014)

The elements of $\mathcal{U}^*$ *are transcendental.*

The structure of $\mathcal{V}$

Theorem

- *The elements of* $\mathcal{V} \setminus \overline{\mathcal{U}}$ *are algebraic integers.*
- *In each connected component* (q_0, q_0^*) *of* $(1, 2] \setminus \overline{\mathcal{U}}$*, the elements of* $\mathcal{V}$ *form a sequence* $q_n \nearrow q_0^*$*:*



- $\mathcal{V}$ *is closed.*

Part 3: Univoque sets

Univoque sets in fixed bases

For a fixed base $1 < q \le 2$ we define

- $x \in \mathcal{U}_q \iff x$ has a unique expansion;
- $\mathcal{U}_q' := \{b(x,q) \ : \ x \in \mathcal{U}_q\}$.

Parry's classical theorem implies the following characterization:

Theorem

A sequence $(c_i) \in \{0,1\}^\infty$ belongs to $\mathcal{U}_q'$ if and only if

$$c_{n+1}c_{n+2}\cdots < a(1,q) \quad \text{whenever} \quad c_n = 0,$$

and

$$\overline{c_{n+1}c_{n+2}\cdots} < a(1,q) \quad \text{whenever} \quad c_n = 1.$$

Monotonicity and size of the sets $\mathcal{U}_q'$

Corollary (Daróczy–Kátai, 1993, 1995)

If $p < q$, then $\mathcal{U}_p' \subsetneqq \mathcal{U}_q'$.

The values $\min \mathcal{V} = \varphi$ and $\min \mathcal{U} = q_{KL}$ play an important role here:

Theorem (Glendinning–Sidorov, 2001)

- $1 < q \le \varphi \Longrightarrow \mathcal{U}_q' = \{0^\infty, 1^\infty\}$.
- $\varphi < q < q_{KL} \Longrightarrow \operatorname{card} \mathcal{U}_q = \aleph_0$.
- $q_{KL} \le q \le 2 \Longrightarrow \operatorname{card} \mathcal{U}_q = 2^{\aleph_0}$.

Stability intervals

A more complete result involves the whole set $\mathcal{V}$:

Theorem (de Vries–K., 2009)

Let $1 < p < q \leq 2$. *Then*

$$\mathcal{U}'_p = \mathcal{U}'_q \iff p, q \in (q_{n-1}, q_n]$$

for some connected component (q_{n-1}, q_n) *of* $(1, 2] \setminus \mathcal{V}$.

Topological structure of $\mathcal{U}_q$

For $1 < q \le 2$ we define

$$\mathcal{V}_q := \{x \ : \ x \text{ has at most one doubly infinite expansion}\}.$$

The following result contrasts with the relations $\mathcal{U} \subsetneqq \overline{\mathcal{U}} \subsetneqq \mathcal{V}$, and it also shows that

$$\mathcal{U}_q \text{ is closed } \iff q \notin \overline{\mathcal{U}}.$$

Theorem (de Vries–K., 2009)

$$\begin{aligned} q \in \overline{\mathcal{U}} &\Longrightarrow \mathcal{U}_q \subsetneqq \overline{\mathcal{U}_q} = \mathcal{V}_q; \\ q \in \mathcal{V} \setminus \overline{\mathcal{U}} &\Longrightarrow \mathcal{U}_q = \overline{\mathcal{U}_q} \subsetneqq \mathcal{V}_q; \\ q \notin \mathcal{V} &\Longrightarrow \mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q; \end{aligned}$$

The finer structure of $\mathcal{U}_q$

Theorem (de Vries–Loreti–K., 2022)

Let $\varphi \le q < 2$, and consider the disjoint union

$$\left[0, \frac{1}{q-1}\right] \setminus \overline{\mathcal{U}_q} = \cup^*(x_L, x_R).$$

- *If $q \in \overline{\mathcal{U}}$, then x_L and x_R run over the infinite set $\mathcal{V}_q \setminus \mathcal{U}_q$.*
- *Otherwise, the elements of $\mathcal{V}_q$ form in each (x_L, x_R) an increasing sequence $(x_k)_{k=-\infty}^{\infty}$ satisfying*

$$x_{-k} \searrow x_L \quad \text{and} \quad x_k \nearrow x_R \quad \text{as} \quad k \to \infty.$$

Remark

The case $1 < q < \varphi$ is trivial because $\mathcal{U}_q = \overline{\mathcal{U}_q} = \{0, 1/(q-1)\}$.

Cantor sets

We recall that $\overline{\mathcal{U}}$ is a Cantor set. The case of $\overline{\mathcal{U}_q}$ is more complex:

Theorem

Let $1 < q < 2$.

- *If* $q \in \overline{\mathcal{U}}$, *then* $\overline{\mathcal{U}_q}$ *is a Cantor set.*
- *If* $q \notin \overline{\mathcal{U}}$, *then* $\mathcal{U}_q$ *is a Cantor set* $\Longleftrightarrow q \in (q_0, q_1]$ *with the usual notation for some* $q_0 \in \overline{\mathcal{U}} \setminus \mathcal{U}$.

Remarks and open problems

- An important application of the topological results is the study of the Hausdorff dimension of the sets $\mathcal{U}_q$.
- Many further results are given in de Vries–Loreti–K., *Topology of univoque sets in real base expansions,* Topology and its Applications, 312 (2022) 108085:
  - description of the isolated, accumulation and condensation points of $\mathcal{U}_q$;
  - description of all expansions of the numbers $x \in \mathcal{V}_q \setminus \mathcal{U}_q$;
  - characterization of the subshift property of $\mathcal{U}_q'$;
  - study of the two-dimensional univoque set

$$\mathbf{U} := \left\{(x, q) \in \mathbb{R}^2 \ : \ x \in \mathcal{U}_q\right\}.$$

- Sidorov, Baker, Kong, Zou and others proved various important results of the set of numbers having exactly N expansions, where $N > 1$. However, the complete picture is still far away.