Univoque bases of real numbers: local dimension, Devil's staircase and isolated points

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Numeration-OWNS, June 23, 2020

Motivation

Given $q \in (1, 2]$, for each $x \in I_q := [0, \frac{1}{q-1}]$ there exists a sequence $(d_i) \in \{0, 1\}^{\mathbb{N}}$ such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i} =: \pi_q((d_i)).$$

The sequence $(d_i) = d_1 d_2 \ldots$ is called a q-expansion of x.

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- For each $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1,2]$ and $x \in I_q$ such that x has k different q-expansions (Erdős, Joó and Komornik 1990; Erdős, Horváth and Joó 1991; Erdős and Joó 1992).
- Let $q \in (1,2)$. Then Lebesgue a.e. $x \in I_q$ has a continuum of q-expansions (Sidorov, 2003).

There is a great interest in unique q-expansions, due to their close connections with open dynamical systems.

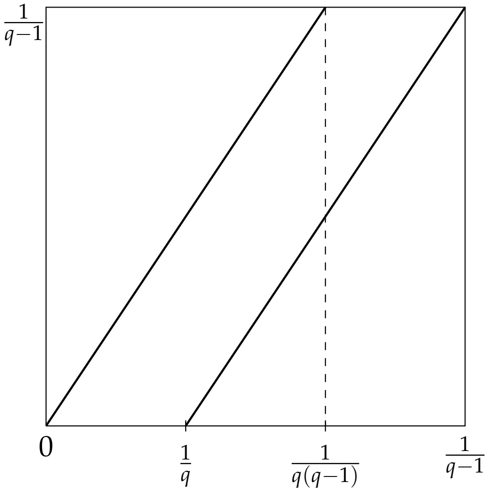


Figure: The overlapping graphs of $T_0 : x \mapsto qx$ and $T_1 : x \mapsto qx - 1$.

Univoque set

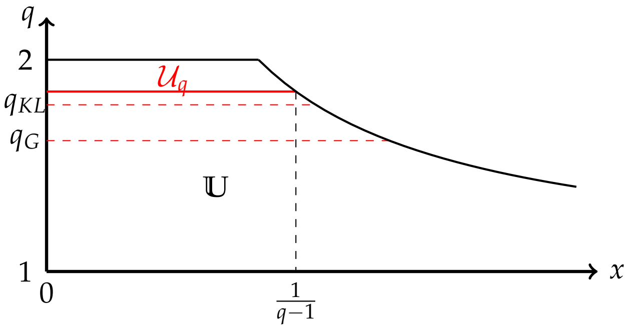
Let

$$\mathbb{U} := \{(x, q) : x \text{ has a unique } q \text{ expansion}\} .$$

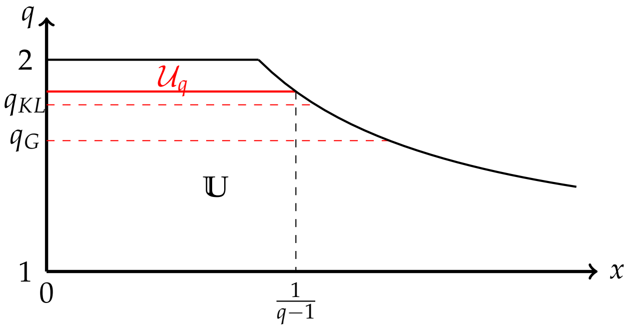
Then for each $q \in (1, 2]$ the horizontal slice

$$\mathcal{U}_q := \{x \in I_q : (x, q) \in \mathbb{U}\}$$

is the set of x having a unique q-expansion.



- Critical values $q_G = \frac{1+\sqrt{5}}{2}$ and $q_{KL} \approx 1.78723$ (Erdős, Joó and Komornik 1990; Glendinning and Sidorov 2001);



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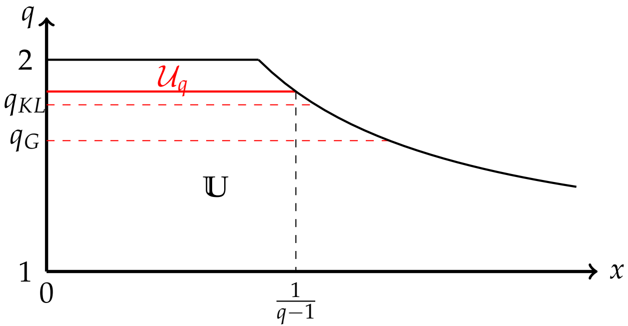

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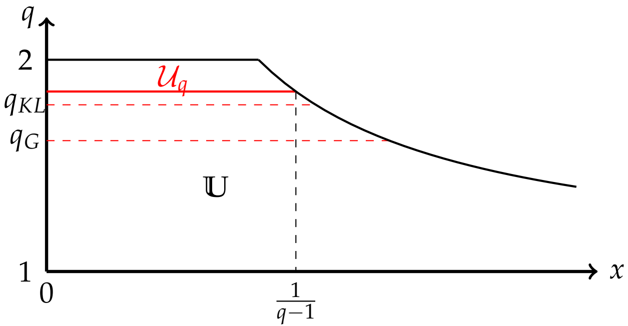

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What about the vertical slice of $\mathbb{U}$?

Univoque bases

For $x \geq 0$ let

$$\mathcal{U}(x) := \{q \in (1,2] : (x,q) \in \mathbb{U}\}\,.$$

- If $x = 0$, then $\mathcal{U}(0) = (1,2]$ (trivial!).

Univoque bases

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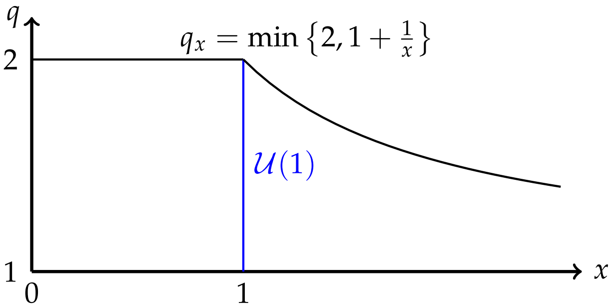
- If $x = 0$, then $\mathcal{U}(0) = (1,2]$ (trivial!).
- If $x > 0$, then the largest element of $\mathcal{U}(x)$ is

$$q_x := \min\left\{2, 1 + \frac{1}{x}\right\}.$$

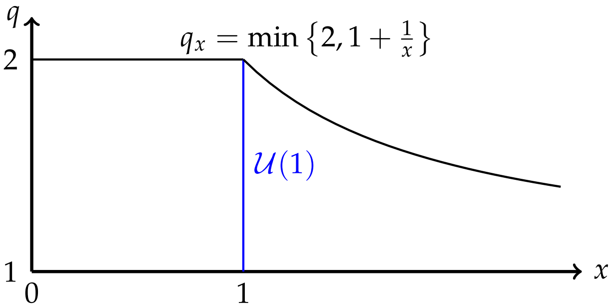
  - If $x \in (0,1]$, then $q_x = 2$.
  - If $x \in (1,\infty)$, then $q_x = 1 + \frac{1}{x}$, and in this case,

$$x = \sum_{i=1}^{\infty} \frac{1}{q_x^i}.$$



For $x = 1$ the set $\mathcal{U} = \mathcal{U}(1)$ was well-studied:

- $\mathcal{L}(\mathcal{U}) = 0$ (Erdős, Joó and Komornik 1990) and $\dim_H \mathcal{U} = 1$ (Daróczy and Kátai 1995);



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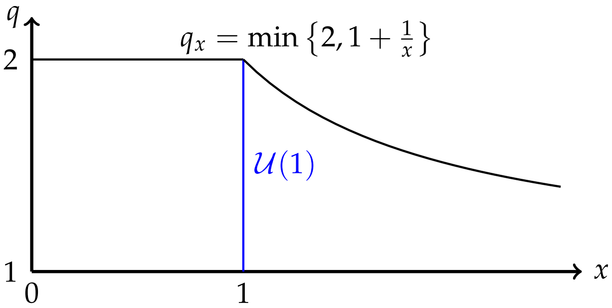

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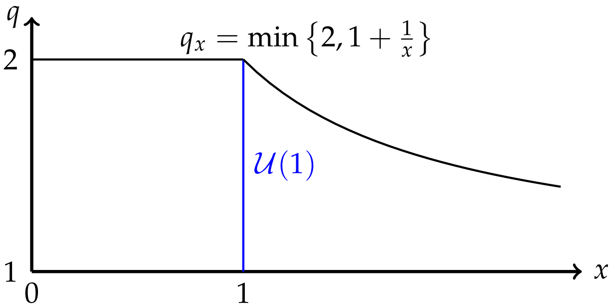

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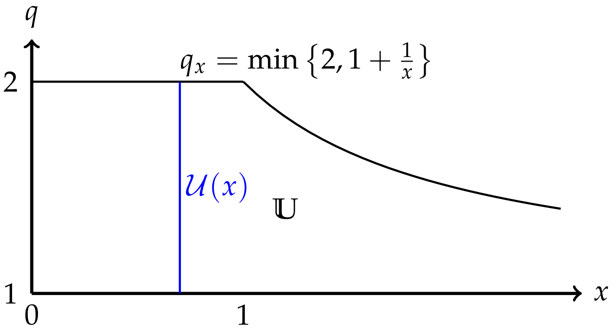
- Local dimension (Allaart and K. 2020).



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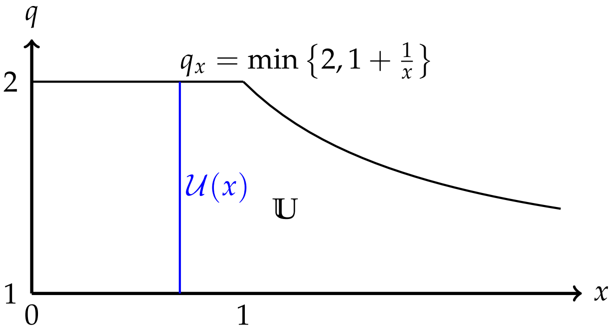

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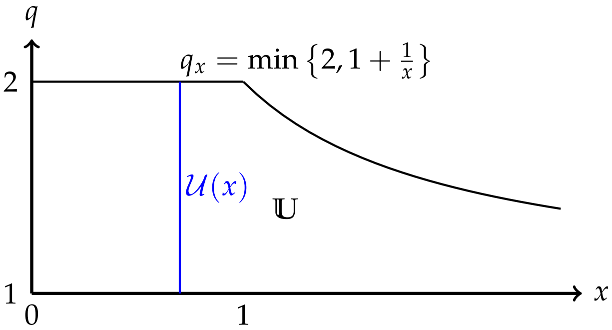

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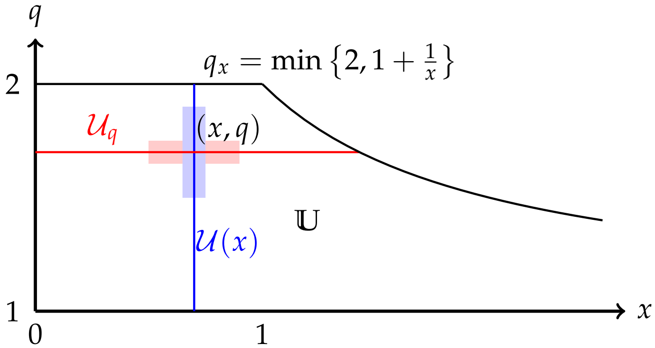
- The smallest element of $\mathcal{U}(x)$ (K. 2016; Allaart and K. 2020).

Variation principle

Theorem (K., Li, Lü, Wang and Xu, 2020)

For any $x > 0$ *and for any* $q \in (1, q_x] \setminus \overline{\mathcal{U}}$ *we have*

$$\lim_{\delta \to 0} \dim_H(\mathcal{U}(x) \cap (q-\delta, q+\delta)) = \lim_{\delta \to 0} \dim_H(\mathcal{U}_q \cap (x-\delta, x+\delta)).$$



Proof

The proof is based on the local bi-Hölder continuity of the map

$$\Phi_x : \mathcal{U}(x) \to \mathbf{U}(x); \quad q \mapsto x_1(q)x_2(q)\dots,$$

where $\mathbf{U}(x)$ is the set of all unique expansions of x for some $q \in \mathcal{U}(x)$.

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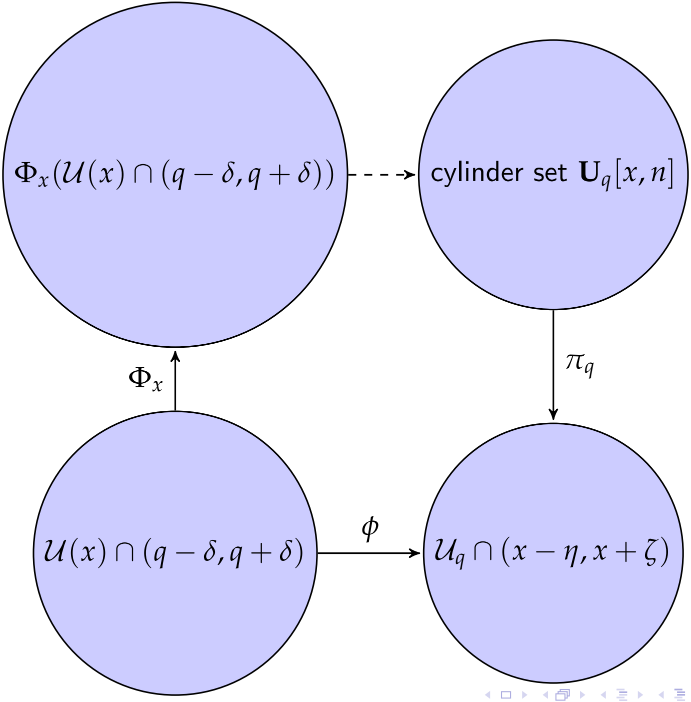
We also need the local bi-Hölder continuity of the projection map

$$\pi_q : \mathbf{U}_q \to \mathcal{U}_q; \quad (d_i) \mapsto \sum_{i=1}^{\infty} \frac{d_i}{q^i},$$

where $\mathbf{U}_q$ is the set of all unique q-expansions.


$\Phi_x(\mathcal{U}(x) \cap (q-\delta, q+\delta))$
cylinder set $\mathbf{U}_q[x,n]$
Φ_x
π_q
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ϕ
$\mathcal{U}_q \cap (x-\eta, x+\zeta)$

Proof conti

Let $q \in (1, q_x] \setminus \overline{\mathcal{U}}$ and $x = \pi_q(\Phi_x(q))$. Then $\exists\ \delta > 0$ such that $(q - \delta, q + \delta) \cap \overline{\mathcal{U}} = \emptyset$.

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$$\begin{array}{rccc} \phi: & \mathcal{U}(x) \cap (q-\delta, q+\delta) & \to & \mathcal{U}_q \cap (x-\eta, x+\zeta) \\ & p & \mapsto & \pi_q(\Phi_x(p)). \end{array}$$

Note that $\delta \to 0$ implies $\eta, \zeta \to 0$.

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$$C_1|p_1 - p_2|^{1+\varepsilon} \leq |\phi(p_1) - \phi(p_2)| \leq C_2|p_1 - p_2|^{1-\varepsilon}.$$

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This implies

$$\lim_{\delta \to 0} \dim_H(\mathcal{U}(x) \cap (q-\delta, q+\delta)) = \lim_{\eta \to 0} \dim_H(\mathcal{U}_q \cap (x-\eta, x+\eta)).$$

Devil's staircase

Recall that

$$\mathbf{U}(x) = \{(d_i) : (d_i) \text{ is the unique expansion of } x \text{ in some base}\},$$
$$\mathbf{U}_q = \{(d_i) : (d_i) \text{ is a unique } q \text{ expansion of some point}\}.$$

Theorem (K., Li, Lü, Wang and Xu, 2020)

For any $x > 0$ we have

$$\dim_H \mathbf{U}(x) = \dim_H \mathbf{U}_{q_x},$$

where $q_x = \max \mathcal{U}(x) = \min\left\{2, 1 + \frac{1}{x}\right\}$.
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In general, we are not able to calculate $\dim_H \mathcal{U}(x)$.

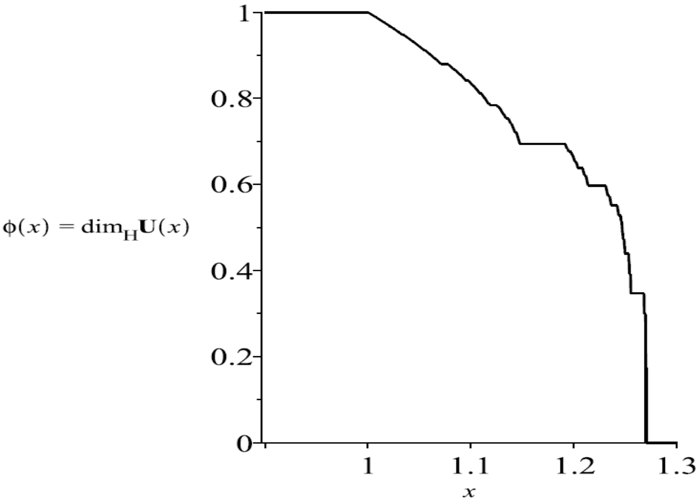

Figure: The graph of $D(x) = \dim_H \mathbf{U}(x)$.

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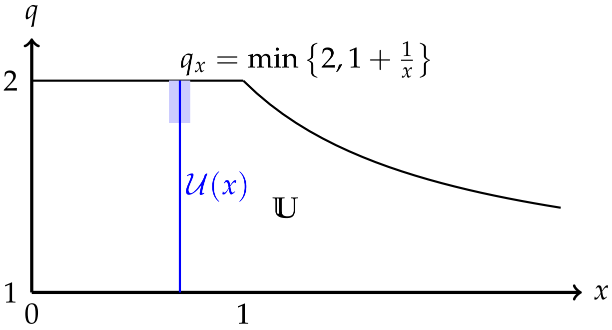

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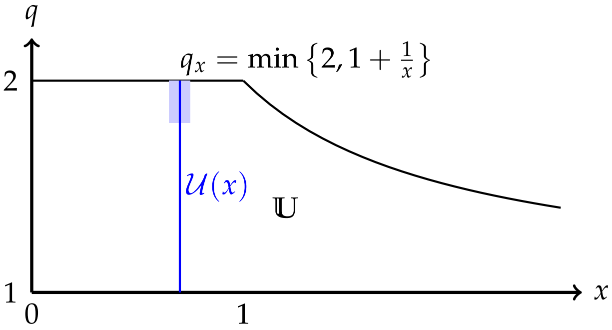


- $\mathbf{U}(x) \subseteq \mathbf{U}_{q_x}$, and then $\dim_H \mathbf{U}(x) \leq \dim_H \mathbf{U}_{q_x}$;
- For any $s < \dim_H \mathbf{U}_{q_x}$ we can construct a subset $\Gamma \subset \mathbf{U}(x)$ close to $\Phi_x(q_x)$ such that $\dim_H \Gamma \geq s$.

Critical values

Theorem (K., Li, Lü, Wang and Xu, 2020)

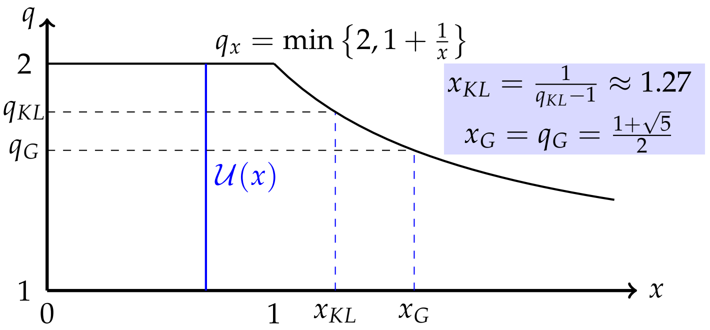
The set $\mathcal{U}(x)$ *has zero Lebesgue measure for any* $x > 0$.

(i) *If* $x \in (0,1]$, *then* $\dim_H \mathcal{U}(x) = 1$;

(ii) *If* $x \in (1, x_{KL})$, *then* $0 < \dim_H \mathcal{U}(x) < 1$;

(iii) *If* $x \in [x_{KL}, x_G)$, *then* $|\mathcal{U}(x)| = \aleph_0$;

(iv) *If* $x \geq x_G$, *then* $\mathcal{U}(x) = \{q_x\}$.



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Let

$$X_{iso} := \{x \in (0, \infty) : \mathcal{U}(x) \text{ contains isolated points}\}.$$

Theorem (K., Li, Lü, Wang and Xu, 2020)

X_{iso} is dense in $(0, \infty)$. Furthermore, $\mathcal{U}(x)$ contains isolated points for any $x > 1$.

proof

Recall that $\mathbf{U}_q = \{(d_i) : (d_i) \text{ is the unique } q\text{-expansion}\}$. Then $\mathbf{U}_p \subseteq \mathbf{U}_q$ for any $p < q$.

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$$\mathcal{V} := \left\{q \in (1,2] : \mathbf{U}_r \neq \mathbf{U}_q \ \forall\ r > q\right\}.$$

Then $\mathcal{U} \subset \mathcal{V}$ and $\#(\mathcal{V} \setminus \mathcal{U}) = \aleph_0$.

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Observe that

$$(1,2] \setminus \overline{\mathcal{U}} = \bigcup (q_0, q_0^*).$$

For each (q_0, q_0^*) we have $\mathcal{V} \cap (q_0, q_0^*) = \{q_n\}_{n=1}^{\infty}$ such that

$$q_0 < q_1 < q_2 < \cdots < q_n < q_{n+1} < \cdots, \quad \text{and} \quad q_n \nearrow q_0^*.$$

So the map $q \mapsto \mathbf{U}_q$ is constant on each interval $(q_n, q_{n+1}]$.

Proof conti

Set $\mathbf{U}^*_{q_{n+1}} := \mathbf{U}_{q_{n+1}} \setminus \mathbf{U}_{q_n}$. Then $\mathbf{U}^*_{q_{n+1}}$ is dense in $\mathbf{U}_{q_{n+1}}$.

Lemma

For any

$$x \in \bigcup_{n=1}^{\infty} \bigcup_{p \in (q_n, q_{n+1})} \pi_p(\mathbf{U}^*_{q_{n+1}})$$

the set $\mathcal{U}(x)$ *contains at least one isolated point.*

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Conjecture

$\mathcal{U}(x)$ contains isolated points $\iff$ $x \neq 1$.

Open questions

1. When is $\mathcal{U}(x)$ a closed set for $x \in (0, x_G)$?
2. What is the Hausdorff dimension of $\mathcal{U}(x)$ for $x \in (1, x_{KL})$?

Thank you!
And welcome to Chongqing!