Critical values for the β-transformation with a hole at 0

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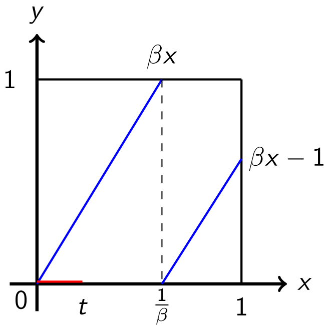
Given $\beta \in (1, 2]$ let

$$T_\beta : [0,1) \to [0,1); \quad x \mapsto \beta x \pmod 1.$$

For $t \in [0,1)$ we define the survivor set by

$$K_\beta(t) := \left\{ x \in [0,1) : T_\beta^n(x) \notin (0,t) \text{ for all } n \geq 0 \right\}.$$



Background

- Urbański (1986, 1987) considered the case for $\beta = 2$, and gave the Hausdorff dimension of $K_2(t)$. He showed that $\eta_2 : t \mapsto \dim_H K_2(t)$ is a non-increasing Devil's staircase, and studied its bifurcation set

$$B_2 := \{t \in [0,1) : \eta_2(t-\varepsilon) \neq \eta_2(t+\varepsilon) \ \forall \varepsilon > 0\}.$$

  In particular, he showed that

$$\lim_{\varepsilon \to 0^+} \dim_H (B_2 \cap (t-\varepsilon, t+\varepsilon)) = \dim_H K_2(t) \quad \forall t \in B_2.$$

- Carminati and Tiozzo (2017) studied the local Hölder exponent of η_2, and proved that

$$\liminf_{s \to t} \frac{\log |\eta_2(s) - \eta_2(t)|}{\log |s-t|} = \eta_2(t) \quad \Longleftrightarrow \quad t \in B_2.$$

- Glendinning and Sidorov (2015) considered an arbitrary hole $(a, b) \subset (0, 1)$, and showed that

$$K_2(a, b) = \{x \in [0, 1) : T_2^n(x) \notin (a, b) \ \forall n \geq 0\}$$

has positive Hausdorff dimension if $b - a < 0.175092$.

- Clark (2016) partially extended the work of Glendinning and Sidorov to $K_\beta(a, b)$ for $\beta \in (1, 2]$.
- Kalle, K., Langeveld and Li (2020) showed that

$$\dim_H K_\beta(t) = \frac{h(\mathbf{K}_\beta(t))}{\log \beta},$$

where

$$\mathbf{K}_\beta(t) := \left\{(d_i) \in \{0, 1\}^{\mathbb{N}} : b(t, \beta) \preccurlyeq \sigma^n((d_i)) \preccurlyeq a(1, \beta) \ \ \forall n \geq 0\right\}.$$

Furthermore, the dimension function $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ is a Devil's staircase.

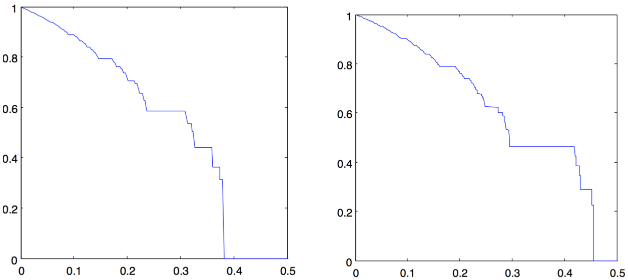


Figure: The graph of the dimension function $\eta_\beta : t \mapsto \dim_H K_\beta(t)$ with $\beta_1 = \frac{1+\sqrt{5}}{2}$ (Left) and $\beta_2 \approx 1.83929$ the tribonacci number (Right).

Then it is interesting to determine the critical hole $(0, \tau(\beta))$, where

$$\begin{aligned}\tau(\beta) &:= \min\{t \in [0,1) : \dim_H K_\beta(t) = 0\} \\ &= \sup\{t \in [0,1) : \dim_H K_\beta(t) > 0\}.\end{aligned}$$

In the above graph, we have $\tau(\beta_1) = 1 - \frac{1}{\beta_1} \approx 0.382$ and $\tau(\beta_2) = 1 - \frac{1}{\beta_2} \approx 0.456$. The key is that

$$\mathbf{K}_{\beta_1}(\tau(\beta_1)) = \{(d_i) : 010^\infty \preccurlyeq \sigma^n((d_i)) \preccurlyeq (10)^\infty \ \forall n\} = \{(01)^\infty, (10)^\infty\}.$$
$$\mathbf{K}_{\beta_2}(\tau(\beta_2)) = \{(d_i) : 0110^\infty \preccurlyeq \sigma^n((d_i)) \preccurlyeq (110)^\infty \forall n\} = \{\sigma^n((011)^\infty)\}_{n=0}^2.$$

Proposition (Kalle, K., Langeveld and Li, 2020)

For any $\beta \in (1, 2]$ we have

$$\tau(\beta) \le 1 - \frac{1}{\beta},$$

and the equality holds for β in an uncountable set of zero Hausdorff dimension.

Question: can we determine $\tau(\beta)$ for all $\beta \in (1, 2]$?

In this talk we will answer this question completely.

Farey words, Lyndon words and substitutions

To describe the critical values $\tau(\beta)$ we need the Farey words:

$$F_0 = (0, 1);$$

$$F_1 = (0, 01, 1);$$

$$F_2 = (0, 001, 01, 011, 1);$$

$$F_3 = (0, 0001, 001, 00101, 01, 01011, 011, 0111, 1)$$

$$\dots$$

Let

$$\Omega_F^* := \bigcup_{n=1}^{\infty} F_n \setminus F_0.$$

Then each Farey word in Ω_F^* has length ≥ 2.

Remark: the Farey words correspond to the Farey tree of rational numbers in $[0, 1]$.

Let Ω_L^* be the set of Lyndon words $\mathbf{s} = s_1 \dots s_m \in \{0,1\}^*$ with $m \geq 2$ such that

$$s_{i+1} \dots s_m \succ s_1 \dots s_{m-i} \quad \forall\ 0 < i < m.$$

Then

$$\Omega_F^* \subset \Omega_L^*.$$

Remark

- any $\mathbf{s} = s_1 \dots s_m \in \Omega_L^*$ begins with $s_1 = 0$ and ends with $s_m = 1$.
- $\Omega_F^* \subsetneq \Omega_L^*$. For example, $\mathbf{s} = 00111 \in \Omega_L^* \setminus \Omega_F^*$.

For a word $w = w_1 \dots w_m \in \{0,1\}^*$ let $\mathbb{L}(w)$ and $\mathbb{S}(w)$ be the lexicographically largest and smallest words among

$$w_1 \dots w_m, \quad w_2 \dots w_m w_1, \quad w_3 \dots w_m w_1 w_2, \quad \dots, \quad w_m w_1 \dots w_{m-1}.$$

Then

$$\mathbb{S}(\mathbf{s}) = \mathbf{s} \quad \forall \mathbf{s} \in \Omega_L^*.$$

Given $\mathbf{s} = s_1 \dots s_m, \mathbf{r} = r_1 \dots r_\ell$, we define the substitution by

$$\mathbf{s} \bullet \mathbf{r} := c_1 \dots c_{\ell m},$$

where

$$c_1 \dots c_m = \begin{cases} \mathbf{s}^- & \text{if} \quad r_1 = 0 \\ \mathbb{L}(\mathbf{s})^+ & \text{if} \quad r_1 = 1, \end{cases}$$

and for $1 \le j < \ell$,

$$c_{jm+1} \dots c_{(j+1)m} = \begin{cases} \mathbb{L}(\mathbf{s}) & \text{if} \quad r_j r_{j+1} = 00 \\ \mathbb{L}(\mathbf{s})^+ & \text{if} \quad r_j r_{j+1} = 01 \\ \mathbf{s}^- & \text{if} \quad r_j r_{j+1} = 10 \\ \mathbf{s} & \text{if} \quad r_j r_{j+1} = 11. \end{cases}$$

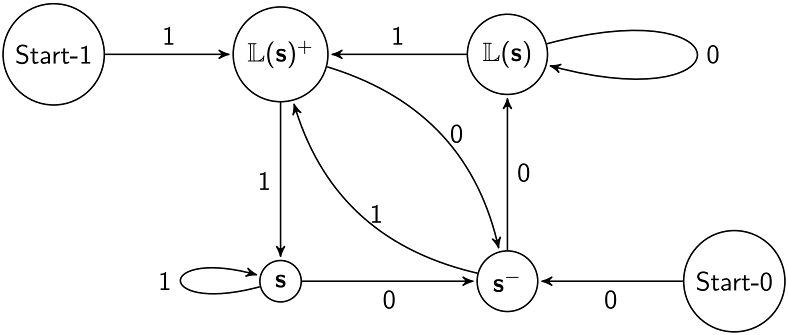
Example

Let $\mathbf{s} = 01, \mathbf{r} = 001 \in \Omega_F^*$. Then $\mathbb{L}(\mathbf{s}) = 10, \mathbb{L}(\mathbf{r}) = 100$. So,

$$\mathbf{s} \bullet \mathbf{r} = \mathbf{s}^- \mathbb{L}(\mathbf{s}) \mathbb{L}(\mathbf{s})^+ = 00\,10\,11 \in \Omega_L^* \setminus \Omega_F^*,$$

$$\mathbf{r} \bullet \mathbf{s} = \mathbf{r}^- \mathbb{L}(\mathbf{r})^+ = 000\,101 (\neq \mathbf{s} \bullet \mathbf{r}).$$

A dynamical view of the substitution $\mathbf{s} \bullet \mathbf{r}$



Example. Let $\mathbf{r} = 01011, \mathbf{t} = 10010$. Then

$$\mathbf{s} \bullet \mathbf{r} = \mathbf{s} \bullet 01011 = \mathbf{s}^- \mathbb{L}(\mathbf{s})^+ \mathbf{s}^- \mathbb{L}(\mathbf{s})^+ \mathbf{s},$$

$$\mathbf{s} \bullet \mathbf{t} = \mathbf{s} \bullet 10010 = \mathbb{L}(\mathbf{s})^+ \mathbf{s}^- \mathbb{L}(\mathbf{s}) \mathbb{L}(\mathbf{s})^+ \mathbf{s}^-.$$

Proposition (Allaart and K., 2022)

$(\Omega_L^*, \bullet)$ *forms a non-Abelian semi-group.*

Let

$$\Lambda := \bigcup_{k=1}^{\infty} \{\mathbf{S} = \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k : \quad \mathbf{s}_i \in \Omega_F^* \text{ for any } 1 \le i \le k\}.$$

Then

$$\Omega_F^* \subsetneqq \Lambda \subsetneqq \Omega_L^*.$$

The set Λ will be our basic bricks for the critical values $\tau(\beta)$.

Basic intervals and critical values

For a word $\mathbf{S} \in \Lambda$ we call the closed interval $I^{\mathbf{S}} = [\beta_\ell, \beta_*] \subset (1,2]$ a basic interval generated by $\mathbf{S}$, if

$$(\mathbb{L}(\mathbf{S})^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (\mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_{\beta_*} = (\mathbf{S} \bullet 10^\infty)_{\beta_*} = 1.$$

Theorem (Allaart and K., 2022)

For any $\mathbf{S} \in \Lambda$ we have

$$\tau(\beta) = (\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_\beta = (\mathbf{S} \bullet 0^\infty)_\beta \quad \forall \beta \in I^{\mathbf{S}}.$$

Example. Take $\mathbf{s} = 000101 = 001 \bullet 01 \in \Lambda$. Then the basic interval $I^{000101} = [\beta_\ell, \beta_*] \approx [1.5385, 1.5526]$ satisfies

$$((101000)^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (101001\,000100\,(101000)^\infty)_{\beta_*} = 1.$$

And $\tau(\beta) = (000100\,(101000)^\infty)_\beta$ for all $\beta \in I^{000101}$.

Sketch of the proof

Take $\beta \in [\beta_\ell, \beta_*]$ and $t^* = (\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty)_\beta$. Then

$$\dim_H K_\beta(t^*) = \frac{h(\mathbf{K}_\beta(t^*))}{\log \beta}, \quad \text{where}$$

$$\begin{aligned} \mathbf{K}_\beta(t^*) &:= \{(d_i) : b(t^*, \beta) \preccurlyeq \sigma^n((d_i)) \preccurlyeq a(1, \beta)\ \forall n \geq 0\} \\ &\subset \left\{(d_i) : \mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty \preccurlyeq \sigma^n((d_i)) \preccurlyeq \mathbb{L}(\mathbf{S})^+\mathbf{S}^-\mathbb{L}(\mathbf{S})^\infty\ \forall n \geq 0\right\} \\ &= \{(d_i) : \mathbf{S} \bullet 0^\infty \preccurlyeq \sigma^n((d_i)) \preccurlyeq \mathbf{S} \bullet 10^\infty\ \forall n \geq 0\} =: \Gamma(\mathbf{S}). \end{aligned}$$

We can prove that $\Gamma(\mathbf{S})$ is at most countable for any $\mathbf{S} \in \Lambda$ (This needs more effort! It is also the reason why we define the set Λ by the substitution $\bullet$.). From this we obtain $\tau(\beta) \leq t^*$.

On the other hand, let

$$t_N := (\mathbf{S}^-\mathbb{L}(\mathbf{S})^N w)_\beta \quad \text{for some word } w \text{ depending on } \mathbf{S}.$$

Then we can show that $t_N \nearrow t^*$ as $N \to \infty$, and $\dim_H K_\beta(t_N) > 0$ for all $N \geq 1$. So, $\tau(\beta) \geq t^*$.

Geometrical structure of basic intervals

For each word $\mathbf{S} \in \Lambda$ the closed interval $J^{\mathbf{S}} = [\beta_\ell, \beta_r] \subset (1,2]$ is called a Lyndon interval generated by $\mathbf{S}$, if

$$(\mathbb{L}(\mathbf{S})^\infty)_{\beta_\ell} = 1 \quad \text{and} \quad (\mathbb{L}(\mathbf{S})^+\mathbf{S}^\infty)_{\beta_r} = (\mathbf{S} \bullet 1^\infty)_{\beta_r} = 1.$$

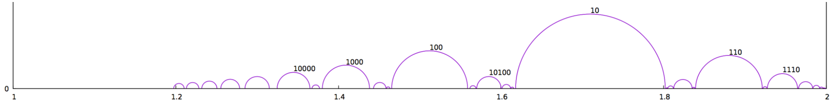
If in particular $\mathbf{S} \in \Omega_F^*$, we call $J^{\mathbf{S}}$ a Farey interval.

Proposition (Kalle, K., Langeveld and Li, 2020)

All of these Farey intervals $J^{\mathbf{S}}, \mathbf{S} \in \Omega_F^*$ are pairwise disjoint, and the bifurcation set

$$E := (1,2] \setminus \bigcup_{\mathbf{s} \in \Omega_F^*} J^{\mathbf{s}}$$

has zero packing dimension. Furthermore, $\tau(\beta) = 1 - 1/\beta$ for $\beta \in E$.



Relative bifurcation sets

Proposition (Allaart and K., 2022)

For any $\mathbf{S} \in \Lambda$ *the Lyndon intervals* $J^{\mathbf{S}\bullet\mathbf{r}}, \mathbf{r} \in \Omega_F^*$ *are pairwise disjoint subsets of* $J^{\mathbf{S}} \setminus I^{\mathbf{S}}$*, and the relative bifurcation set*

$$E^{\mathbf{S}} := (J^{\mathbf{S}} \setminus I^{\mathbf{S}}) \setminus \bigcup_{\mathbf{r} \in \Omega_F^*} J^{\mathbf{S}\bullet\mathbf{r}}$$

is uncountable and has zero box-counting dimension. Furthermore,

$$\tau(\beta) = (\mathbf{S} \bullet (0\delta_2\delta_3\dots))_\beta \quad \forall \beta \in E^{\mathbf{S}},$$

where $(\delta_2\delta_3\dots)$ *satisfies* $(\mathbf{S} \bullet (1\delta_2\delta_3\dots))_\beta = 1$.

Tree structure

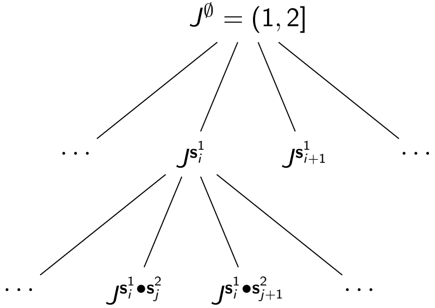


Figure: The tree structure of the Lyndon intervals $J^S, S = \mathbf{s}^1 \dots s^k, k \in \mathbb{N}$ with each $\mathbf{s}^i \in \Omega_F^*$. The root is $(1, 2]$, it has countably infinitely many offsprings (pairwise disjoint subsets) $J^{\mathbf{s}_i^1}, i \in \mathbb{Z}$. The bifurcation set $E = (1, 2] \setminus \bigcup_{i \in \mathbb{Z}} J^{\mathbf{s}_i^1}$. Furthermore, each offspring $J^{\mathbf{s}_i^1}$ also has countably infinitely many offsprings $J^{\mathbf{s}_i^1 \bullet \mathbf{s}_j^2}, j \in \mathbb{Z}$, and the relative bifurcation set $E^{\mathbf{s}_i^1} = (J^{\mathbf{s}_i^1} \setminus I^{\mathbf{s}_i^1}) \setminus \bigcup_{j \in \mathbb{Z}} J^{\mathbf{s}_i^1 \bullet \mathbf{s}_j^2}$, where $I^{\mathbf{s}_i^1}$ is a basic interval. The infinite bifurcation set is then given by $E_\infty = \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{s}_1, \dots, \mathbf{s}^k \in \Omega_F^*} J^{\mathbf{s}^1 \bullet \dots \bullet \mathbf{s}^k}$.

The tree structure of Lyndon intervals $J^{\mathbf{S}}, \mathbf{S} \in \Lambda$ gives rise to the infinite bifurcation set

$$E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{S} \in \Lambda(k)} J^{\mathbf{S}}, \qquad (1)$$

where $\Lambda(k) := \{\mathbf{S} = \mathbf{s}_1 \bullet \cdots \bullet \mathbf{s}_k : \ \mathbf{s}_i \in \Omega_F^* \ \forall 1 \le i \le k\}$.

Proposition (Allaart and K., 2022)

E_∞ is an uncountable set with zero Hausdorff dimension. Furthermore, for any $\beta \in E_\infty$ there exists a unique sequence $(\mathbf{s}_k) \subset \Omega_F^*$ such that

$$\{\beta\} = \bigcap_{n=1}^{\infty} J^{\mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_n},$$

and then $\tau(\beta) = \lim_{n \to \infty} (\mathbf{s} \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_n 0^\infty)_\beta$.

Now we have the partition:

$$(1, 2] = E \cup E_\infty \cup \bigcup_{\mathbf{S} \in \Lambda} E^{\mathbf{S}} \cup \bigcup_{\mathbf{S} \in \Lambda} I^{\mathbf{S}}.$$

Main Result

Theorem (Allaart and K., 2022)

(i) *The function* $\tau : \beta \mapsto \tau(\beta)$ *is left continuous on* $(1, 2]$ *with right-hand limits everywhere* (càdlàg), *and as a result has only countably many discontinuities;*

(ii) τ *has no downward jumps;*

(iii) *There is an open set* $O \subset (1, 2]$, *whose complement* $(1, 2] \setminus O$ *has zero Hausdorff dimension, such that* τ *is real-analytic, convex and strictly decreasing on each connected component of* O.

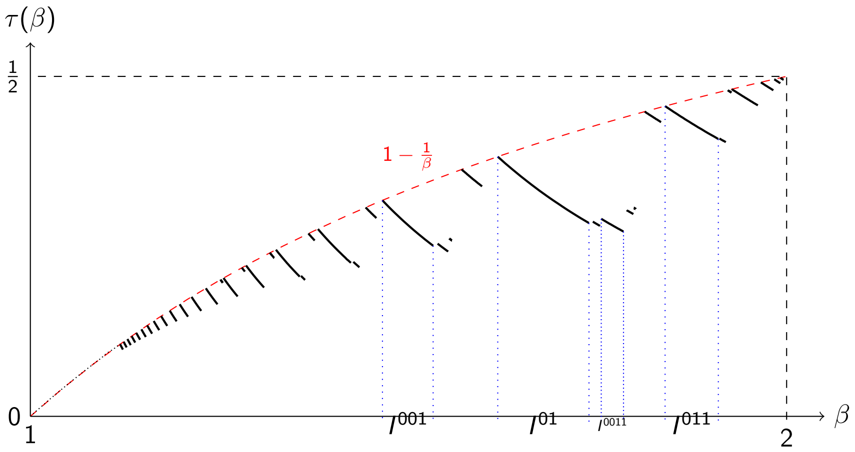


Figure: The graph of the critical value function $\tau(\beta)$ for $\beta \in (1, 2]$. We see that $\tau(\beta) \leq 1 - 1/\beta$ for all $\beta \in (1, 2]$, and the function τ is strictly decreasing in each basic interval $I^{\mathbf{S}}$.

Proposition (Allaart and K., 2022)

Given $\mathbf{s} = s_1 \dots s_m = 0\mathbf{c}1 \in \Omega_F^*$, *let* $\beta \in (1,2]$ *such that*

$$(\theta_1 \mathbf{c} \theta_2 \, \theta_3 \mathbf{c} \theta_4 \ \dots \ \theta_{2k+1} \mathbf{c} \theta_{2k+2} \dots)_\beta = 1,$$

where $(\theta_i)_{i=0}^\infty = 01101001\dots$ *is the Thue-Morse sequence. Then* $\beta \in E_\infty \cap J^{\mathbf{s}}$ *is transcendental, and*

$$\tau(\beta) = \frac{2\sum_{j=2}^m s_j \beta^{m-j} + \beta^{m-1} - \beta^m}{\beta^m - 1}.$$

Key idea.

Consider $\mathbf{S}_k = \mathbf{s}_1 \bullet \mathbf{s}_2 \bullet \cdots \bullet \mathbf{s}_k$ with $\mathbf{s}_1 = \mathbf{s}$ and $\mathbf{s}_i = 01 \in \Omega_F^*$ for all $i \geq 2$. Then $\beta \in J^{\mathbf{S}_k}$ for all $k \geq 1$. Furthermore, $\mathbb{L}(\mathbf{S}_k) \to \theta_1 \mathbf{c} \theta_2 \, \theta_3 \mathbf{c} \theta_4 \ \dots$ and $\mathbf{S}_k \to \overline{\theta_1} \mathbf{c} \overline{\theta_2} \, \overline{\theta_3} \mathbf{c} \overline{\theta_4} \ \cdots$ as $k \to \infty$. □

Conjecture: each $\beta \in E_\infty$ is transcendental.

Future work

- Extend this work to $\beta > 2$.
- Consider a hole $(a, a+t)$ for $a \in [0,1)$ and $t \in [0, 1-a)$. Here a is fixed.
- Consider a random hole $(0, t_n), n \geq 0$, so the survivor set is given by

$$K_\beta((t_n)) := \left\{ x \in [0,1) : T_\beta^n(x) \notin (0, t_n) \ \forall n \geq 0 \right\}.$$

Thank you!