k-regular Sequences: Asymptotics and Decidability

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Binary Sum-of-Digits Function

Example

\[ s(28) = s((11100)_2) = 3 \]

Recursive Description

- even numbers: \[ s(2n) = s(n) \]
- odd numbers: \[ s(2n + 1) = s(n) + 1 \]

Generalizations:

- \[ s(2^j n) = s(n) \]
- \[ s(2^j n + r) = s(n) + s(r), \quad 0 \leq r < 2^j \]

Rewriting as Linear Combinations

\[ s(2^j n + r) = 1 \cdot s(n) + c_{jr} \cdot 1 \quad \text{for every} \quad j \geq 0, \quad 0 \leq r < 2^j \]
$k$-regular Sequences

$k$-regular sequence $f(n)$

$k$-kernel \( \{ f(k^j n + r) \mid j \geq 0, 0 \leq r < k^j \} \)

is contained in

finitely generated module

explicitly:

- there exist sequences $f_1(n), \ldots, f_s(n)$ such that
- for all $j \geq 0, 0 \leq r < k^j$
- there exist $c_1, \ldots, c_s$
- with

\[
f(k^j n + r) = c_1 f_1(n) + \cdots + c_s f_s(n)
\]
Binary Sum-of-Digits Function

Example

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Recursive Description

- even numbers: \[ s(2n) = s(n) \]
- odd numbers: \[ s(2n + 1) = s(n) + 1 \]

rewriting:

- set \[ v(n) = (s(n), 1)^T \]
- even \[ v(2n) = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n) \]
- odd \[ v(2n + 1) = \begin{pmatrix} s(n) + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v(n) \]
**k-regular Sequences**

A *k*-regular sequence $f(n)$ is defined by the following properties:

- **Square Matrices**: $M_0, \ldots, M_{k-1}$
- **Vectors**: $u$ and $w$
- **k-linear Representation**
  
  $$f(n) = u^T M_{n_0} M_{n_1} \ldots M_{n_{\ell-1}} w$$

with standard *k*-ary expansion

$$n = (n_{\ell-1} \ldots n_1 n_0)_k$$

### Selected Examples:

- *k*-ary sum of digits
- Redundant systems: number of representations in base *k*
- *k*-automatic sequences
- Output sum sequences of transducers
- Completely *k*-additive functions
Some $k$-regular Sequences

\[ s(n) - s(n - 1) \]

\[ \frac{1}{10^4} \sum_{m < n} s(m) \]

\[ \frac{1}{n} \left( \sum_{m < n} s(m) - \frac{1}{2} n \log_2 n \right) \]

\[ \frac{1}{10^8} \sum_{r < m < n} s(r) \]
Asymptotics of Partial Sums

- $k$-regular sequence $f(n)$
- partial sums $F(N) = \sum_{n<N} f(n)$

**Theorem (Heuberger–K–Prodinger 2018, Heuberger–K 2020)**

$$F(N) = \sum_{\lambda \in \sigma(M_0 + \cdots + M_{k-1})} N^{\log_k \lambda} \sum_{|\lambda| > \rho, 0 \leq \ell < m(\lambda)} (\log_k N)^\ell \Phi_{\lambda \ell}(\{\log_k N\}) + O(N^{\log_k R} (\log N)^{\hat{m}})$$

- 1-periodic (Hölder) continuous functions $\Phi_{\lambda \ell}$
- functional equation

$$\left(I - \frac{1}{k^s}(M_0 + \cdots + M_{k-1})\right)\mathcal{V}(s) = \sum_{n=1}^{k-1} \frac{v(n)}{n^s} + \frac{1}{k^s} \sum_{r=0}^{k-1} M_r \sum_{\ell \geq 1} \left(-\frac{s}{\ell}\right) \left(\frac{r}{k}\right)^\ell \mathcal{V}(s+\ell)$$

- meromorphic continuation on the half plane $\Re s > \log_k R$
- Fourier series $\Phi_{\lambda \ell}(u) = \sum_{h \in \mathbb{Z}} \varphi_{\lambda \ell h} \exp(2\ell\pi i u)$

$$\varphi_{\lambda \ell h} = \frac{(\log k)^\ell}{\ell!} \text{Res} \left( \frac{(f(0) + F(s))(s - \log_k \lambda - \frac{2h\pi i}{\log k})^\ell}{s}, s = \log_k \lambda + \frac{2h\pi i}{\log k} \right)$$
Binary Sum-of-Digits Function: Analysis

- eigenvalues etc.
  - $C = M_0 + M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$
  - $C$ has eigenvalue $\lambda = 2$ with multiplicity 2
  - joint spectral radius 1
  - $\|M_{r_1} \cdots M_{r_\ell}\| = O(R^\ell)$ for any $R > 1$
- ↲ analysis of summatory function:
  - $S(N) = N(\log_2 N) \Phi_{21}(\{\log_2 N\}) + N \Phi_{20}(\{\log_2 N\})$
  - 1-periodic continuous functions $\Phi_{21}$ and $\Phi_{20}$
  - $\Phi_{21}(u) = \frac{1}{2}$ via functional equation
  - no error term

- recovering:
  **Summatory Binary Sum-of-Digits (Delange 1975)**
  \[
  S(N) = \sum_{n < N} s(n) = \frac{1}{2} N \log_2 N + N \Phi_{20}(\{\log_2 N\})
  \]
  - explicit Fourier coefficients of $\Phi_{20}(u)$
Joint Spectral Radius

- finite set $S$ of $n \times n$ matrices
- $\| \cdot \|$ any matrix norm

Joint Spectral Radius

$$\rho(S) = \lim_{\ell \to \infty} \max \left\{ \frac{\| F_1 \cdots F_\ell \|^{1/\ell}}{\ell} \mid F_i \in S \right\}$$

- example $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$
  - $\rho(S) \leq 1$ (maximal spectral norm)
  - $\rho(S) \geq 1$ (joint eigenvalue 1)
  - $\lim \rho(S) = 1$

- approximation algorithms available
Decidability

**Decision Problem**

Problem with a *yes/no* answer

**Recursively Solvable, Solvable, Decidable**

For decision problem, there *exists an algorithm* (or Turing machine) that unerringly solves it on all inputs
Is Prime?
“Given a natural number, is it prime?”
...decidable? (Quiz: A)

Integer Roots of Polynomials
“Given a univariate polynomial with integer coefficients, does it have an integer root?”
...decidable? (Quiz: B)

Rational Roots of Polynomials
“Given a univariate polynomial with rational coefficients, does it have a rational root?”
...decidable? (Quiz: C)

Hilbert’s tenth problem; variant
(MRDP 1949–1970)
“Given a multivariate polynomial $p$ with integer coefficients, do there exist natural numbers $x_1, x_2, \ldots, x_t$ such that $p(x_1, \ldots, x_t) = 0$?”
...decidable? (Quiz: D)
### Unit Spectral Radius

“Given a square matrix $F$ with rational entries, is its spectral radius $\rho(F) \leq 1$?”

... decidable? (Quiz: A)

### Bounded Matrix Powers

“Given a square matrix with rational entries, is the set of all powers of this matrix bounded?”

... decidable? (Quiz: B)

### Unit Joint Spectral Radius

(Blondel–Tsitsiklis 2000)

“Given a finite set $S$ of $n \times n$ matrices with rational entries, is the joint spectral radius $\rho(S) \leq 1$?”

... decidable? (Quiz: C)
Bounded Matrix Products ($\mathbb{N}_0$) (Mandel–Simon 1977)

“Given a finite set of $n \times n$ matrices with nonnegative integer entries, is the set of all matrix products bounded?”

...decidable? (Quiz: A)

Bounded Matrix Products ($\mathbb{Z}$) (Jacob 1977)

“Given a finite set of $n \times n$ matrices with integer entries, is the set of all matrix products bounded?”

...decidable? (Quiz: B)

Bounded Matrix Products ($\mathbb{Q}$) (Blondel–Tsitsiklis 2000)

“Given a finite set of $n \times n$ matrices with rational entries, is the set of all matrix products bounded?”

...decidable? (Quiz: C)
Equality of $k$-regular Sequences

Theorem (Krenn–Shallit 2020)

“Given two $k$-regular sequences $(f(n))_{n \geq 0}$ and $(g(n))_{n \geq 0}$ over $\mathbb{Q}$, does $f(n) = g(n)$ for all $n$ hold?”

... recursively solvable

Proof:
- compute linear representation of $f(n) - g(n)$
- apply minimization algorithm
- rank 0 iff $f(n) - g(n) = 0$ for all $n$
Zero Terms

Theorem (Allouche–Shallit 1992)

“Given a $k$-regular sequence over $\mathbb{N}$, does it have a zero term?”

... recursively unsolvable

Proof:
- multivariate polynomial $p$ in $t$ variables over $\mathbb{Z}$
- choose $r \in \mathbb{N}$ large enough
- $f(n) = p(|z|_1, |z|_2, \ldots, |z|_t)$
  - $z$ equals $k^r$-representation of $n$
  - $|z|_d$ is number of occurrences of letter $d$ in $z$
- $(f(n))_{n \geq 0}$ is $k^r$-regular and consequently $k$-regular
Theorem (Krenn–Shallit 2020)

"Given a k-regular sequence \((f(n))_{n \geq 0}\) over \(\mathbb{N}_0\), is \(\{f(n) \mid n \in \mathbb{N}_0\} = \mathbb{N}_0\)?

\[\ldots\text{recursively unsolvable}\]

Proof:
- multivariate polynomial \(p\) in \(t\) variables over \(\mathbb{Z}\)
- choose \(r \in \mathbb{N}\) large enough
- \(f(n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ (p(|z|_1, |z|_2, \ldots, |z|_t))^2 & \text{if } n \text{ is odd} \end{cases}\)
  - \(z\) equals \(k^r\)-representation of \((n - 1)/2\)
  - \(|z|_d\) is number of occurrences of letter \(d\) in \(z\)
- \((f(n))_{n \geq 0}\) is \(k^r\)-regular and consequently \(k\)-regular
More (Un-)Decidability Results

- **recursively unsolvable:**
  - Image of sequence equals $\mathbb{Z}$?
  - Images of two sequences coincide?
  - Sequences takes same value twice?
  - Sequence contains a square?
  - Sequence contains a palindrome?
  - Preimages can be recognized by a deterministic finite automaton?

(Krenn–Shallit 2020)

- **quasi-universal $k$-regular sequences:**
  - “no nontrivial property is decidable”

(Honkala 2021)

- **recursively solvable** for $k$-automatic sequences:
  - Sequence contains a square?
  - Sequence contains a palindrome?
  - ...

(e.g. Charlier–Rampersad–Shallit 2012)
Theorem (Krenn–Shallit 2020)

“Given a $k$-regular sequence $(f(n))_{n \geq 0}$ over $\mathbb{N}$, is $f(n)$ in $\Omega(n^\sigma \log n^\ell)$?”

... recursively unsolvable

Proof (for $\Omega(n)$):
- multivariate polynomial $p$ in $t$ variables over $\mathbb{Z}$
- choose $r \in \mathbb{N}$ large enough
- $f(n) = (n+1)(p(|z_1|, |z_2|, \ldots, |z_t|)(|z|_{t+1} + 1)$
  - $z$ equals $k^r$-representation of $n$
  - $|z|_d$ is number of occurrences of letter $d$ in $z$
- $(f(n))_{n \geq 0}$ is $k^r$-regular and consequently $k$-regular
Theorem (Krenn–Shallit 2020)

"Given a $k$-regular sequence $(f(n))_{n \geq 0}$ over $\mathbb{Q}$, is $f(n)$ in $O(n^\sigma (\log n)^\ell)$?"

... recursively unsolvable

Proof (for $O(1)$, i.e., boundedness):

- set of matrices $\{F_0, \ldots, F_{k-1}\}$
- split $F_{n_0} \cdots F_{n_{s-1}} = F_{n_0} \cdots F_{n_{j-1}} F_0^{s-j}$
- deciding boundedness of matrix products
  - boundedness of the $F_0^{s-j}$ can be decided (by Jordan decomposition)
  - boundedness of all products of matrices recursively unsolvable
    (Blondel–Tsitsiklis 2000)
- sequences of numbers $\rightsquigarrow$ sequences of matrices
- decidable for linear representation with
  - nonnegative integer matrices (Mandel–Simon 1977)
  - integer matrices (Jacob 1977)
Theorem (Krenn–Shallit 2020)

“Given a k-regular sequence \((f(n))_{n \geq 0}\) over \(\mathbb{Q}\), does \(f(n)\) have at least polynomial growth?”

... recursively unsolvable

Proof:

- minimal representation of \(f(n)\) with matrices \(F_0, \ldots, F_{k-1}\)
- equivalent statements:
  1. joint spectral radius of \(F_0, \ldots, F_{k-1}\) is \(\rho\)
  2. for all \(\varepsilon > 0\) we have \(f(n) \in O(n^{(\log_k \rho) + \varepsilon})\) and \(F_0^s \in O((k^s)^{(\log_k \rho) + \varepsilon})\)

\[ f(n) \notin O(n^{(\log_k \rho) - \varepsilon}) \quad \text{as } s \to \infty \]

- not at least polynomial growth \(\iff\) for all \(\varepsilon > 0\), \(f(n) \in O(n^\varepsilon)\)
- joint spectral radius \(\rho \leq 1\) recursively unsolvable

(Blondel–Tsitsiklis 2000)
Regular Sequences
Asymptotics
Decidability
Decidability of Sequence-Properties
Decidability of Asymptotics

Sequences
(k-regular sequences)

Asymptotics
(growth rates)

Decidability
(algorithms)