k-regular Sequences: Asymptotics and Decidability

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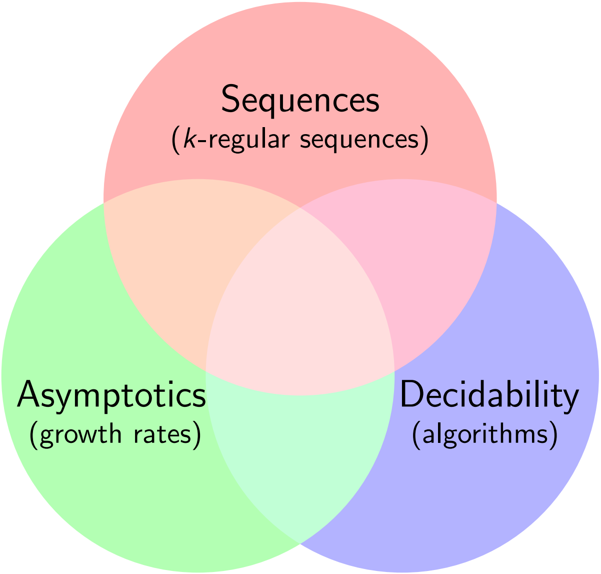
Sequences
(k-regular sequences)
Asymptotics
(growth rates)
Decidability
(algorithms)

Binary Sum-of-Digits Function

Example

$$s(28) = s\big((11100)_2\big) = 3$$

Recursive Description

even numbers: $s(2n) = s(n)$

odd numbers: $s(2n+1) = s(n) + 1$

generalizations:

$$s(2^j n) = s(n)$$
$$s(2^j n + r) = s(n) + s(r), \quad 0 \le r < 2^j$$

Rewriting as Linear Combinations

$s(2^j n + r) = 1 \cdot s(n) + c_{jr} \cdot 1$ for every $j \ge 0$, $0 \le r < 2^j$

k-regular Sequences

k-regular sequence $f(n)$

k-kernel $\{f(k^j n + r) \mid j \geq 0, \, 0 \leq r < k^j\}$
is contained in
finitely generated module

explicitly:

- there exist sequences $f_1(n), \ldots f_s(n)$ such that
- for all $j \geq 0$, $0 \leq r < k^j$
- there exist $c_1, \ldots, c_s$
- with

$$f(k^j n + r) = c_1 f_1(n) + \cdots + c_s f_s(n)$$

Binary Sum-of-Digits Function

Example

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Recursive Description

even numbers: $s(2n) = s(n)$

odd numbers: $s(2n+1) = s(n) + 1$

rewriting:

- set $v(n) = (s(n), 1)^\top$
- even $$v(2n) = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n)$$
- odd $$v(2n+1) = \begin{pmatrix} s(n)+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v(n)$$

k-regular Sequences

k-regular sequence $f(n)$

- square matrices $M_0, \ldots, M_{k-1}$
- vectors u and w
- k-linear representation

$$f(n) = u^T M_{n_0} M_{n_1} \ldots M_{n_{\ell-1}} w$$

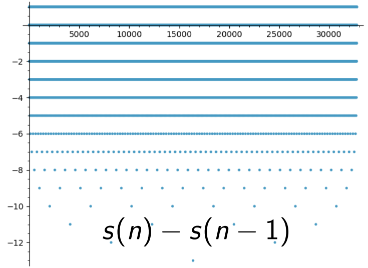
with standard k-ary expansion

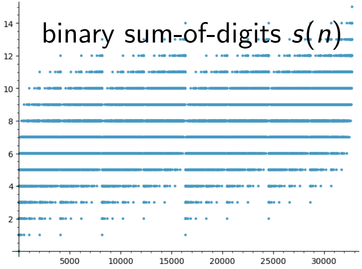
$$n = (n_{\ell-1} \ldots n_1 n_0)_k$$

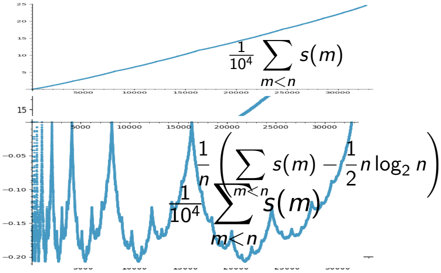
selected examples:

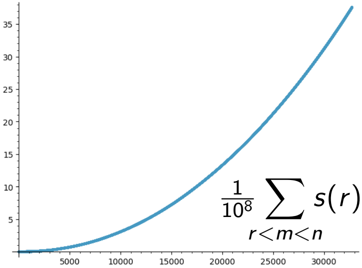
- k-ary sum of digits
- redundant systems: number of representations in base k
- k-automatic sequences
- output sum sequences of transducers
- completely k-additive functions

Some k-regular Sequences









Asymptotics of Partial Sums

- k-regular sequence $f(n)$
- partial sums $F(N) = \sum_{n<N} f(n)$

Theorem (Heuberger–K–Prodinger 2018, Heuberger–K 2020)

$$F(N) = \sum_{\substack{\lambda\in\sigma(M_0+\cdots+M_{k-1})\\ |\lambda|>\rho}} N^{\log_k \lambda} \sum_{0\le\ell<m(\lambda)} (\log_k N)^\ell \Phi_{\lambda\ell}(\{\log_k N\}) + O(N^{\log_k R}(\log N)^{\widehat{m}})$$

- *1-periodic (Hölder) continuous functions* $\Phi_{\lambda\ell}$
- *functional equation*

$$\Big(I - \frac{1}{k^s}(M_0+\cdots+M_{k-1})\Big)\mathcal{V}(s) = \sum_{n=1}^{k-1} \frac{v(n)}{n^s} + \frac{1}{k^s}\sum_{r=0}^{k-1} M_r \sum_{\ell\ge1} \binom{-s}{\ell}\Big(\frac{r}{k}\Big)^\ell \mathcal{V}(s+\ell)$$

- *meromorphic continuation on the half plane* $\Re s > \log_k R$
- *Fourier series* $\Phi_{\lambda\ell}(u) = \sum_{h\in\mathbb{Z}} \varphi_{\lambda\ell h} \exp(2\ell\pi i u)$

$$\varphi_{\lambda\ell h} = \frac{(\log k)^\ell}{\ell!} \operatorname{Res}\Bigg(\frac{\big(f(0)+\mathcal{F}(s)\big)\big(s - \log_k\lambda - \frac{2h\pi i}{\log k}\big)^\ell}{s}, s = \log_k \lambda + \frac{2h\pi i}{\log k}\Bigg)$$

Binary Sum-of-Digits Function: Analysis

- eigenvalues etc.
  - $C = M_0 + M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$
  - C has eigenvalue $\lambda = 2$ with multiplicity 2
  - joint spectral radius 1
  - $\|M_{r_1} \cdots M_{r_\ell}\| = O(R^\ell)$ for any $R > 1$
- $\rightsquigarrow$ analysis of summatory function:
  - $S(N) = N(\log_2 N)\,\Phi_{21}(\{\log_2 N\}) + N\,\Phi_{20}(\{\log_2 N\})$
  - 1-periodic continuous functions Φ_{21} and Φ_{20}
  - $\Phi_{21}(u) = \frac{1}{2}$ via functional equation
  - no error term
- recovering:

Summatory Binary Sum-of-Digits (Delange 1975)

$$S(N) = \sum_{n<N} s(n) = \tfrac{1}{2} N \log_2 N + N\,\Phi_{20}(\{\log_2 N\})$$

- explicit Fourier coefficients of $\Phi_{20}(u)$

Joint Spectral Radius

- finite set S of $n \times n$ matrices
- $\|\cdot\|$ any matrix norm

Joint Spectral Radius

$$\rho(S) = \lim_{\ell \to \infty} \max\{\|F_1 \cdots F_\ell\|^{1/\ell} \mid F_i \in S\}$$

- example $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$
  - $\rho(S) \leq 1$ (maximal spectral norm)
  - $\rho(S) \geq 1$ (joint eigenvalue 1)
  - $\rightsquigarrow \rho(S) = 1$
- approximation algorithms available

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Decidability

Decision Problem

problem with a yes/no answer

Recursively Solvable, Solvable, Decidable

for decision problem
there exists an algorithm (or Turing machine)
that unerringly solves it on all inputs

Is Prime?

"Given a natural number,
is it prime?"

. . . decidable? (Quiz: A)

Integer Roots of Polynomials

"Given a univariate polynomial
with integer coefficients,
does it have an integer root?"

. . . decidable? (Quiz: B)

Rational Roots of Polynomials

"Given a univariate polynomial
with rational coefficients,
does it have a rational root?"

. . . decidable? (Quiz: C)

Hilbert's tenth problem; variant (MRDP 1949–1970)

"Given a multivariate polynomial p with integer coefficients,
do there exist natural numbers x_1, x_2, . . . , x_t
such that $p(x_1, \dots, x_t) = 0$?"

. . . decidable? (Quiz: D)

Unit Spectral Radius

"Given a square matrix F with rational entries,
is its spectral radius $\rho(F) \leq 1$?"

...decidable? (Quiz: A)

Bounded Matrix Powers

"Given a square matrix with rational entries,
is the set of all powers of this matrix bounded?"

...decidable? (Quiz: B)

Unit Joint Spectral Radius (Blondel–Tsitsiklis 2000)

"Given a finite set S of $n \times n$ matrices with rational entries,
is the joint spectral radius $\rho(S) \leq 1$?"

...decidable? (Quiz: C)

Bounded Matrix Products ($\mathbb{N}_0$) (Mandel–Simon 1977)

"Given a finite set of $n \times n$ matrices
with nonnegative integer entries,
is the set of all matrix products bounded?"

…decidable? (Quiz: A)

Bounded Matrix Products ($\mathbb{Z}$) (Jacob 1977)

"Given a finite set of $n \times n$ matrices
with integer entries,
is the set of all matrix products bounded?"

…decidable? (Quiz: B)

Bounded Matrix Products ($\mathbb{Q}$) (Blondel–Tsitsiklis 2000)

"Given a finite set of $n \times n$ matrices
with rational entries,
is the set of all matrix products bounded?"

…decidable? (Quiz: C)

Equality of k-regular Sequences

Theorem (Krenn–Shallit 2020)

"Given two k-regular sequences $(f(n))_{n\geq 0}$ and $(g(n))_{n\geq 0}$ over $\mathbb{Q}$, does $f(n) = g(n)$ for all n hold?"

... recursively solvable

- Proof:
  - compute linear representation of $f(n) - g(n)$
  - apply minimization algorithm
  - rank 0 iff $f(n) - g(n) = 0$ for all n

Zero Terms

Theorem (Allouche–Shallit 1992)

"Given a k-regular sequence over $\mathbb{N}$, does it have a zero term?"

... recursively unsolvable

- Proof:
  - multivariate polynomial p in t variables over $\mathbb{Z}$
  - choose $r \in \mathbb{N}$ large enough
  - $f(n) = p(|z|_1, |z|_2, \dots, |z|_t)$
    - z equals k^r-representation of n
    - $|z|_d$ is number of occurrences of letter d in z
  - $(f(n))_{n \geq 0}$ is k^r-regular and consequently k-regular

Images

Theorem (Krenn–Shallit 2020)

"Given a k-regular sequence $(f(n))_{n\geq 0}$ over $\mathbb{N}_0$, is $\{f(n) \mid n \in \mathbb{N}_0\} = \mathbb{N}_0$?"

... recursively unsolvable

- Proof:
  - multivariate polynomial p in t variables over $\mathbb{Z}$
  - choose $r \in \mathbb{N}$ large enough
  - $$f(n) = \begin{cases} n/2+1 & \text{if } n \text{ is even} \\ \big(p(|z|_1, |z|_2, \dots, |z|_t)\big)^2 & \text{if } n \text{ is odd} \end{cases}$$
    - z equals k^r-representation of $(n-1)/2$
    - $|z|_d$ is number of occurrences of letter d in z
  - $(f(n))_{n\geq 0}$ is k^r-regular and consequently k-regular

More (Un-)Decidability Results

- recursively unsolvable:
  - Image of sequence equals $\mathbb{Z}$?
  - Images of two sequences coincide?
  - Sequences takes same value twice?
  - Sequence contains a square?
  - Sequence contains a palindrom?
  - Preimages can be recognized by a deterministic finite automaton?

  (Krenn–Shallit 2020)
- quasi-universal k-regular sequences:
  "no nontrivial property is decidable"

  (Honkala 2021)
- recursively solvable for k-automatic sequences:
  - Sequence contains a square?
  - Sequence contains a palindrom?
  - ...

  (e.g. Charlier–Rampersad–Shallit 2012)

Lower Bounds

Theorem (Krenn–Shallit 2020)

"Given a k-regular sequence $(f(n))_{n\geq 0}$ over $\mathbb{N}$, is $f(n)$ in $\Omega\big(n^\sigma (\log n)^\ell\big)$?"

. . . recursively unsolvable

- Proof (for $\Omega(n)$):
  - multivariate polynomial p in t variables over $\mathbb{Z}$
  - choose $r \in \mathbb{N}$ large enough
  - $f(n) = (n+1)\big(p(|z|_1, |z|_2, \ldots, |z|_t)\big)(|z|_{t+1} + 1)$
    - z equals k^r-representation of n
    - $|z|_d$ is number of occurrences of letter d in z
  - $(f(n))_{n\geq 0}$ is k^r-regular and consequently k-regular

Upper Bounds

Theorem (Krenn–Shallit 2020)

"Given a k-regular sequence $(f(n))_{n\geq 0}$ over $\mathbb{Q}$, is $f(n)$ in $O(n^\sigma(\log n)^\ell)$?"

. . . recursively unsolvable

- Proof (for $O(1)$, i.e., boundedness):
  - set of matrices $\{F_0, \ldots, F_{k-1}\}$
  - split $F_{n_0} \cdots F_{n_{s-1}} = F_{n_0} \cdots F_{n_{j-1}} F_0^{s-j}$
  - deciding boundedness of matrix products
    - boundedness of the F_0^{s-j} can be decided (by Jordan decomposition)
    - boundedness of all products of matrices recursively unsolvable (Blondel–Tsitsiklis 2000)
  - sequences of numbers $\leftrightsquigarrow$ sequences of matrices
- decidable for linear representation with
  - nonnegative integer matrices (Mandel–Simon 1977)
  - integer matrices (Jacob 1977)

Polynomial Growth

Theorem (Krenn–Shallit 2020)

"Given a k-regular sequence $(f(n))_{n\geq 0}$ over $\mathbb{Q}$, does $f(n)$ have at least polynomial growth?"

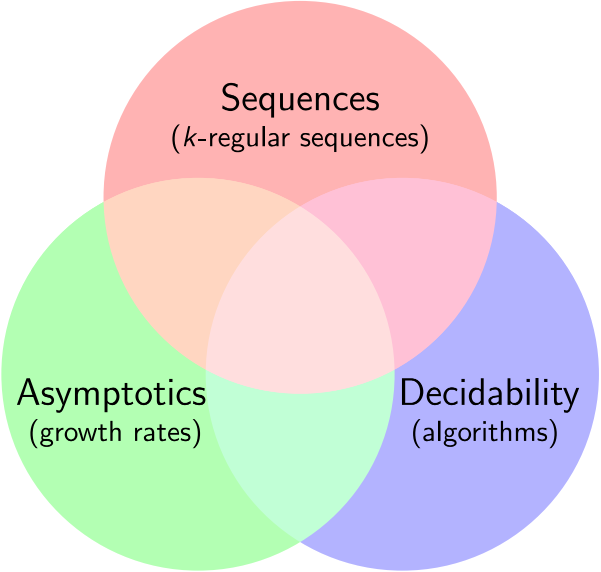
... recursively unsolvable

n^2

- Proof:
  - minimal representation of $f(n)$ with matrices $F_0, \ldots, F_{k-1}$
  - equivalent statements:
    1. joint spectral radius of $F_0, \ldots, F_{k-1}$ is ρ
    2. for all $\varepsilon > 0$ we have $f(n) \in O(n^{(\log_k \rho)+\varepsilon})$ and $F_0^s \in O((k^s)^{(\log_k \rho)+\varepsilon})$

       $f(n) \notin O(n^{(\log_k \rho)-\varepsilon})$ as $s \to \infty$
  - not at least polynomial growth $\iff$ for all $\varepsilon > 0$, $f(n) \in O(n^\varepsilon)$
  - joint spectral radius $\rho \leq 1$ recursively unsolvable

    (Blondel–Tsitsiklis 2000)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