On the distribution of sequences of the form $(q_n y)$

Joint work with Tomas Persson

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Some rather old work

- About 10 years ago, with Haynes and Jensen, we studied sequences of the form $(q_n y)$.
- Our approach was aimed at saying something about Littlewood type problems. The q_n were typically denominators of convergents to other numbers.
- We were able to get discrepancy estimates for typical y, which allowed for conclusions for rapidly increasing sequences.
- It subsequently appeared natural to consider sets of the form

$$W_{y,\alpha} = \left\{x : \|q_n y - x\| < n^{-\alpha} \text{ i.o.}\right\}$$

Old work, new problems

$$W_{y,\alpha} = \{x : \|q_n y - x\| < n^{-\alpha} \text{ i.o.}\}$$

- If (q_n) is lacunary and $\alpha < 1/2$, [HKJ] implies that this set is almost surely full.
- By contrast, the Borel–Cantelli lemma will not give a null set until until $\alpha > 1$.
- The gap was closed for lacunary sequences in 2023 by Chow and Technau after many developments.
- For non-lacunary (q_n), there are still questions. Note however, that [HJK] does give a discrepancy estimate without lacunarity.

Old work, new problems

$$W_{y,\alpha} = \left\{x : \|q_n y - x\| < n^{-\alpha} \text{ i.o.}\right\}$$

- If $\alpha > 1$, the set is null, but the Hausdorff dimension appears to not have been investigated.
- Using the Hausdorff–Cantelli lemma, it is not terribly difficult to prove that $1/v$ is an upper bound.

Progress on the problems

$$W_{y,\alpha} = \left\{x : \|q_n y - x\| < n^{-\alpha} \text{ i.o.}\right\}$$

- We have considered the case $\alpha < 1$, but with more general sequences than lacunary ones. Less precise results than those of Chow and Technau have been obtained.
- Additionally, we have studied the Hausdorff dimension when $\alpha > 1$. Our results are almost surely sharp for lacunary sequences; and for arbitrary sequences when $\alpha \geq 3$.

Almost surely?

- I have said ‘almost surely’ a few times. This implies a measure on the y’s.
- For us, the measures considered are Radon probability measures on $[0, 1]$ with positive Fourier exponent $\eta > 0$, i.e. measures μ for which

$$|\hat{\mu}(t)| = \left| \int e^{2\pi i x} \mathrm{d}\mu(x) \right| \ll |t|^{-\eta}.$$

- Such measures occur thoughout Diophantine approximation, e.g. as Lebesgue measure, Kaufmann’s measure on badly approximable numbers og Kaufmann-Bluhm measures on w-approximable numbers.

Discrepancy?

- For a sequence $(x_n)_{n=1}^{\infty}$ in $[0,1)$ and $N \in \mathbb{N}$, recall that

$$D_N(x_n) = \sup_{I \subseteq [0,1)} |\#\{ n \leq N : x_n \in I \} - N\lambda(I)|.$$

- The sequence $(x_n)_{n=1}^{\infty}$ is said to be uniformly distributed in $[0,1)$ if $D_N(x_n) = o(N)$.

And a key result!

Lemma (Persson and Reeve, 2015)

Let (μ_n) be a sequence of probability measures with $\operatorname{supp}\mu_n \subset E_n \subset [0,1]$, and such that μ_n converges weakly. If

$$I_s(\mu_n) := \iint |x-y|^{-s}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \ll C$$

uniformly in n, then $E = \limsup E_n$ has Hausdorff dimension at least s and is Falconer intersective.

- Falconer intersective sets have stable Hausdorff dimension under intersection with sets from the same class.

A first result

Theorem

Suppose that μ is a probability measure of positive Fourier exponent with decay $|\hat{\mu}(\xi)| = O(|\xi|^{-\tau})$. If $(q_n)_{n=1}^{\infty}$ is a sequence of integers with $|q_n - q_m| > c|n - m|^{\frac{1}{\tau}+\varepsilon}$ for some $c, \varepsilon > 0$, and $\alpha < 1$, then $\lambda(W_{y,\alpha}) > 0$ for μ-almost every y.

Note that if μ is the Lebesgue measure, we can set $\tau = 1$, so sequences of quadratic growth will work.

Sketch of proof

- Let $A_k = B(q_k y, k^{-v})$, so that $|A_k| = 2k^{-v}$.
- The intersections are controlled on average over y,

$$\int \lambda(A_k \cap A_l)\,\mathrm{d}\mu \leq 8l^{-\alpha}(k^{-\alpha} + c_1(l-k)^{-1-\tau\varepsilon}).$$

- Consider mean and variance in blocks,

$$S_{m,n}(y) = \sum_{k=m}^{n} \lambda(A_k),\ C_{m,n}(y) = \sum_{m \leq k, l \leq n} \lambda(A_k(y) \cap A_l(y)).$$

Sketch of proof

- Consider for $p \geq 1$ the sets
$$\left\{ y : C_{m,n}(y) \leq p \int C_{m,n}(y)\,\mathrm{d}\mu(y) \right\}.$$
- The limsup set of these along an appropriate subsequence, $G(p)$ say, will have μ-measure at least $1 - 1/p$.
- Using Chung–Erdős (and some calculation), for $y \in G(p)$, the Lebesgue measure of $W_{y,\alpha}$ is at least $1/(4p)$.
- Finally, the union of these sets is μ-full.

A better result for Lebesgue measure

Theorem

Suppose that $(q_n)_{n=1}^{\infty}$ is a sequence of integers with $|q_n - q_m| > c|n-m|^{1+\varepsilon}$. For $\alpha < 1$, the set $W_{y,\alpha}$ is Lebesgue full for Lebesgue-almost every y.

This follows from the preceding theorem together with Fubini and an 'inflation trick' of Cassels.

Sketch of proof

- Fix $\alpha' \in (\alpha, 1)$.
- By the preceeding theorem, $W_{y,\alpha}$ has positive Lebesgue measure for almost every y.
- By Fubini,

$$\overline{W}_{\alpha'} = \{(x, y) : \|q_n y - x\| < n^{-\alpha'} \text{i.o}\}$$

has positive measure.

Sketch of proof

- Now, let $\epsilon > 0$.
- A lemma in Cassels allows us to chose an integer T, so that if $[0,1)^2$ is split into T^2 disjoint boxes of side length $1/T$, such that one box B has

$$\lambda(B \cap \overline{W}_{\alpha'}) > (1-\epsilon)\lambda(B)$$

- Now, scale the set by T and project back onto the unit square, so that for (x,y) outside a set of measure $< \epsilon$, there is a point $(x',y') \in \overline{W}_{\alpha'}$ with $(x,y) = T(x',y')$.

Sketch of proof

- For such a point (x, y),

$$\|q_n y - x\| = \|T(q_n y' - x')\| \leq T\|q_n y' - x'\|.$$

- Since $(x', y') \in \overline{W}_{\alpha'}$, for infinitely many n,

$$T\|q_n y' - x'\| < Tn^{-\alpha'} = Tn^{\alpha-\alpha'} n^{-\alpha}.$$

- Since $\alpha' > \alpha$, the point (x, y) lies in $\overline{W}_{\alpha}$ which in turn has Lebesgue measure $> 1 - \epsilon$.
- Since $\epsilon > 0$ was arbitrary, $\overline{W}_{\alpha}$ is full.
- Another application of Fubini shows that almost all fibers are full.

Assuming lacunarity

Theorem

Let μ be a probability measure on $[0,1]$ with positive Fourier exponent, and let $(q_n)_{n=1}^{\infty}$ be a lacunary sequence of integers. For any $\alpha \geq 1$, for μ-almost all x,

$$\dim(W_{y,\alpha}) = \frac{1}{\alpha}.$$

In fact, the set is intersective in the sense of Falconer.

Sketch of proof

- The upper bound is a straightforward covering argument.
- For the lower bound, we use Persson and Reeve, so define probability measures μ_N by

$$\frac{\mathrm{d}\mu_N}{\mathrm{d}x} = \frac{1}{N}\sum_{k=N+1}^{2N} \frac{1}{2(2N)^{-\alpha}} \mathbb{1}_{B(q_k x, r_N)}, \qquad r_N = (2N)^{-\alpha}.$$

- We must bound the t-energies uniformly. As always,

$$I_t(\mu_N) = \iint |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) = c_0 \int |\hat{\mu}_N(\xi)|^2 |\xi|^{t-1}\,\mathrm{d}\xi.$$

Sketch of proof

- The challenge is in the Fourier-transform. First,

$$\hat{\mu}_N(\xi) = (2N)^{\alpha-1} \sum_{k=N+1}^{2N} \widehat{\mathbb{1}_{B(0,r_N)}}(\xi) e^{-i2\pi\xi q_k x},$$

- Now, in any textbook we find

$$|\widehat{\mathbb{1}_{B(0,r_N)}}(\xi)|^2 \leq m_N(\xi) := \min\{2r_N, \pi|\xi|^{-1}\}^2.$$

- Hence,

$$\int |\hat{\mu}_N(\xi)|^2 \, \mathrm{d}\mu \leq (2N)^{2\alpha-2} m_N(\xi) \sum_{k=N+1}^{2N} \sum_{l=N+1}^{2N} \hat{\mu}((q_k - q_l)\xi).$$

Sketch of proof

- Since $|\hat{\mu}_N(\xi)|$ is uniformly bounded, we always have

$$\sum_{k=N+1}^{2N} \sum_{l=N+1}^{2N} \hat{\mu}((q_k - q_l)\xi) \leq cN^2.$$

- If $|\xi| \geq 1$, lacunarity allows us to beat this, so

$$\sum_{k=N+1}^{2N} \sum_{l=N+1}^{2N} \hat{\mu}((q_k - q_l)\xi) \leq CN.$$

- No matter what, we find that

$$\int I_t(\mu_N) \, \mathrm{d}\mu \leq c_4 + c_5 N^{-1+\alpha t},$$

which is uniformly bounded in N provided $t < 1/\alpha$.

Sketch of proof

- For $t < 1/\alpha$ fixed,

$$\int I_t(\mu_N)\,\mathrm{d}\mu \leq c_4 + c_5 N^{-1+\alpha t},$$

- Thus, for μ-almost every y, we can find a sequence (N_k) along which the integrand is uniformly bounded.
- Appealing to Persson and Reeve, the proof is complete.

Non-lacunary sequences

- The first non-lacunary sequence, one can think of is $q_n = n$. For this sequence, everything is known!
- If y is rational, the $W_{y,\alpha}$ is trivially finite.
- Kim (2007) proved that $W_{y,\alpha}$ has full measure if $\alpha \leq 1$ and y is irrational.
- Bugeaud (2003) and independently Schmeling and Troubetskoy (2003) proved that the Hausdorff dimension is equal to $1/\alpha$ for y irrational.
- Kim's result is best possible by Bugeaud, Harrap, K. and Velani (2010).

Non-lacunary sequences

- The results for $q_n = n$ depends on the sequence (ny) having a lot of strucure. By contrast, the results in this talk are more random in nature.
- In [HJK], we considered not the sequences $(q_n y)$ directly, but rather derived a bound on their discrepancy.
- It appears natural to look at the set

$$W = \{ \gamma \in [0,1) : \|x_n - \gamma\| < n^{-\alpha} \text{ infinitely often} \}.$$

- [HKJ] tells us, that if $|\hat{\mu}(\xi)| = O(|\xi|^{-2\eta})$, then $D_N(q_n y) = O(N^{1-\eta'})$ for any $\eta' < \eta$.

General results

Theorem

Let $(x_n)_{n=1}^{\infty}$ be a sequence in $[0,1)$ with $D_N((x_n)) = O(N^{1-\eta})$ for some $\eta \in (0,1)$. Then $\frac{\eta}{\alpha} \leq \dim W \leq \frac{1}{\alpha}$ for $\alpha \geq 1$.

Corollary

Let μ be a probability measure on $[0,1]$ with Fourier decay $|\hat{\mu}(\xi)| = O(|\xi|^{-2\eta})$, and let $(q_n)_{n=1}^{\infty}$ be an increasing sequence of integers. For any $\alpha \geq 1$, for μ-almost all y,

$$\frac{\eta}{\alpha} \leq \dim W_{y,\alpha} \leq \frac{1}{\alpha}.$$

Closing the gap

- Even for Lebesgue measure, this leaves an annoying gap!
- Adding the additional assumption to the theorem that

$$\inf_{1 \leq k < l \leq N} \|x_k - x_l\| \geq cN^{-\beta}, \quad 1 \leq \beta \leq \eta(\alpha - 1) + 1,$$

  the gap can be closed for Lebesgue measure, so that $\dim W = 1/\alpha$.
- For $W_{y,\alpha}$, this means that the gap is closed whenever $\alpha \geq 3$ and (q_n) is increasing, at least for Lebesgue almost every y.

More problems

- It would be natural to replace $n \mapsto n^{-\alpha}$ by some other function tending to 0.
- Non-monotonic functions appear to be far beyond reach!
- Even for monotonic functions, the problem of just Lebesgue measure becomes the classical one studied by Kurzweil in the Fifties.
- For Hausdorff dimension, Fan and Wu (2006) constructed an example, where the dimension does not behave as expected.

Thank you!