Continued fractions with two non integer digits

Niels Langeveld

May 30, 2020

The set up

Let $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ and write them as $\alpha_1 = pz$ and $\alpha_2 = \frac{1}{z}$ such that $\alpha_1\alpha_2 = p$, note that $p \in (0, 1]$ and $z \in (0, \frac{1}{\sqrt{p}})$ to assure $\alpha_1 < \alpha_2$. We make (α_1, α_2)-expansions of the form

$$x = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \ddots}}$$

with $d_1, d_2, \ldots \in \{\alpha_1, \alpha_2\}^{\mathbb{N}}$. For such x we write $x = [0; d_1, d_2, \ldots]$. Let $\mathcal{R}_{\alpha_1,\alpha_2} = \{x \in \mathbb{R} : x = [0; d_1, d_2, \ldots], d_i \in \{\alpha_1, \alpha_2\} \text{ for all } i \in \mathbb{N}\}$

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Theorem (Dmytrenko, Kyurchev, Prats'ovytyi 2009)

- *if $p < \frac{1}{2}$ there exists an interval on which (Lebesgue) almost all points have uncountably many (α_1, α_2)-expansions.*
- *if $p = \frac{1}{2}$ there exists an interval on which (Lebesgue) almost all points have a unique (α_1, α_2)-expansions.*
- *if $p > \frac{1}{2}$ there exists no interval on which (Lebesgue) almost all points an (α_1, α_2)-expansions.*

Let $\Omega_z = [a, b]$ with $a = \frac{1}{2}z(\sqrt{p^2+4p} - p)$ and $b = \frac{1}{2}z^{-1}p^{-1}\left(\sqrt{p^2+4p} - p\right)$. We define $T_{p.z} : \Omega_z \to \Omega_z$ as

$$T_{p,z} = \begin{cases} \frac{1}{x} - \alpha_2 \text{ for } x \in \left[a, (a+z^{-1})^{-1}\right] \\ \frac{1}{x} - \alpha_1 \text{ for } x \in \left((b+pz)^{-1}, b\right] \end{cases}$$

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$$\begin{aligned} x &= \frac{1}{d_1(x) + T_z(x)} \\ &= \frac{1}{d_1(x) + \frac{1}{d_2(x) + T_z^2(x)}} \\ &\vdots \\ &= \frac{1}{d_1 + \frac{\ddots}{d_n(x) + T_z^n(x)}} \end{aligned}$$

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Explanation using pictures

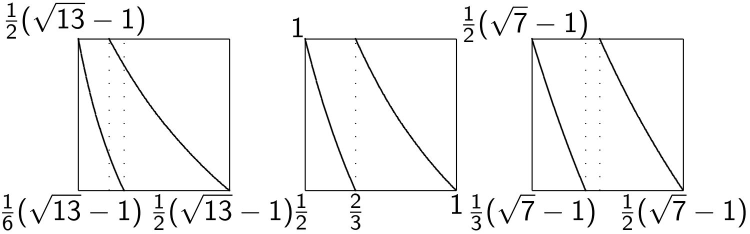

Figure: The three different cases where $z = 1$ and $p = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ from left to right.

Things of what you could study

- explore for which parameters you can make the natural extension of the system
- what about admissibility
- Diophantine approximation
- entropy
- for $p < \frac{1}{2}$, set of unique expansions
- for $p > \frac{1}{2}$, the Hausdorff dimension of $\mathcal{R}_{\alpha_1,\alpha_2}$
- sums of such sets
- are all systems ergodic
- regions in parameterspace where the attractor of the dynamical system is smaller than the interval its define on for greedy expansions
- relations with other expansions

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Natural extensions

Making a non-invertible system into an invertible one by adding dimensions. Advantages

- In many cases it is easier to guess the invariant density for the natural extension (in our case we can show that $\mu(A)\int_A \frac{1}{(1+xy)^2}dxdy$ is invariant)
- From this it is simple to find the invariant density for the original system (by projecting down to the original dimension)
- One can get also other information from the natural extension (often related to Diophantine approximation)

Challenge: to find the right domain for the system.In our case the 2-dimensional map is given by

$$\mathcal{T}_{p,z}(x,y) = (T_{p,z}(x), \frac{1}{d(x)+y})$$

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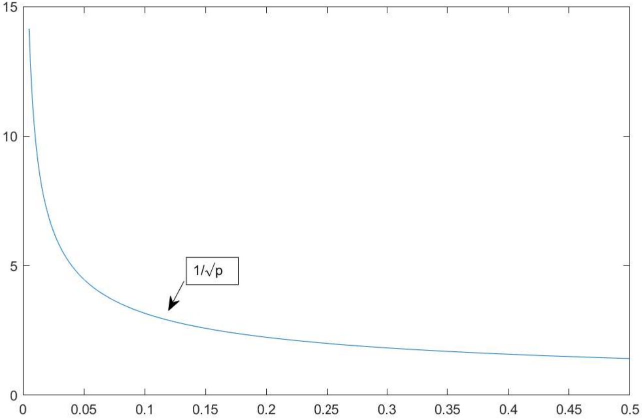
Picture of parameterspace



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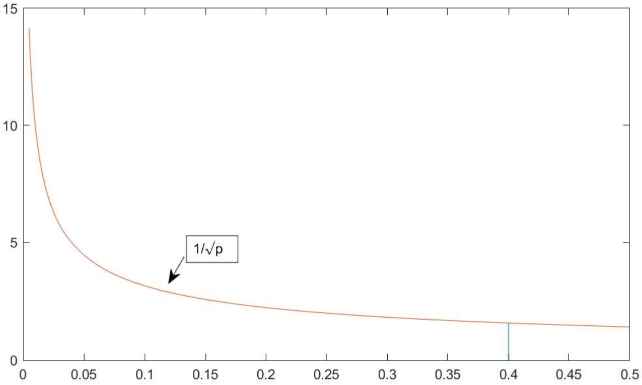

Movietime

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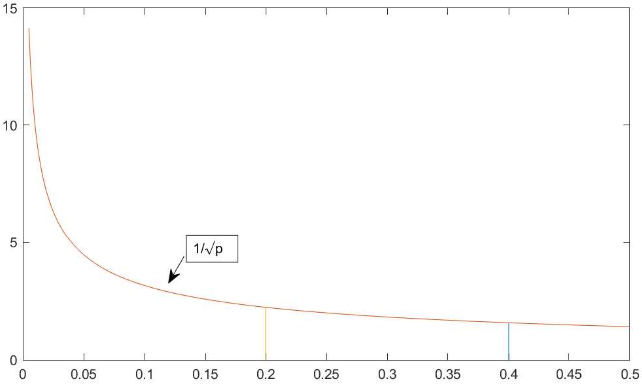

$p = 1/5$ and $z = \sqrt{2 + \sqrt{\frac{7}{3}}}$

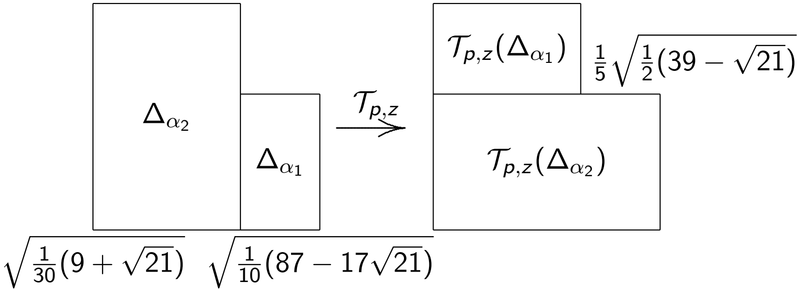

Figure: The natural extension for $p = \frac{1}{5}$ and $z = \sqrt{2 + \sqrt{\frac{7}{3}}}$

similar expansions and admissibility

Solving $(a+z^{-1})^{-1} = a + z^{-1} - pz$ gives that for $p < \sqrt{5} - 2$ we find $z = \frac{1}{\sqrt{p+1-\sqrt{p^2+4p}}}$ and ensures that α_1 is never followed by α_1 in any α_1, α_2-expansion on the corresponding interval (note that the left hand side is the discontinuity point and the right hand side its image under the branch $\frac{1}{x} - \alpha_1$).

In general one can show the following

Let $\mathcal{O}_{p,z}(c)$ be the quasigreedy expansion of $c = (a + z^{-1})^{-1}$. The set of all possible sequences then is given by

$$\{x \in \{\alpha_1, \alpha_2\}^{\mathbb{N}} : \sigma^n(x) \leq \alpha_2\alpha_2\alpha_1 \text{ or } \sigma^n(x) \geq \mathcal{O}_{p,z}(c) \text{ for all } \in \mathbb{N}\}.$$

The set of possible expansions for the discontinuity is given by

$$\mathcal{Q} = \{x \in \{0,1\}^{\mathbb{N}} : \sigma^n(x) \leq \alpha_2\alpha_2\alpha_1 \text{ or } \sigma^n(x) \geq x \text{ for all } n \in \mathbb{N}\}.$$

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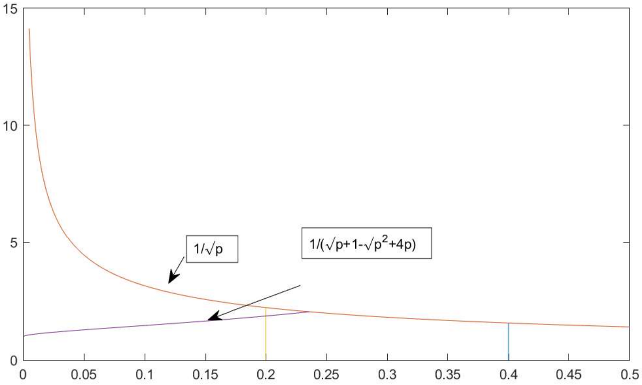
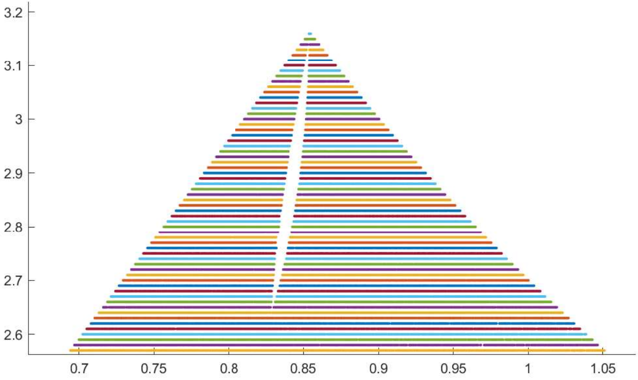
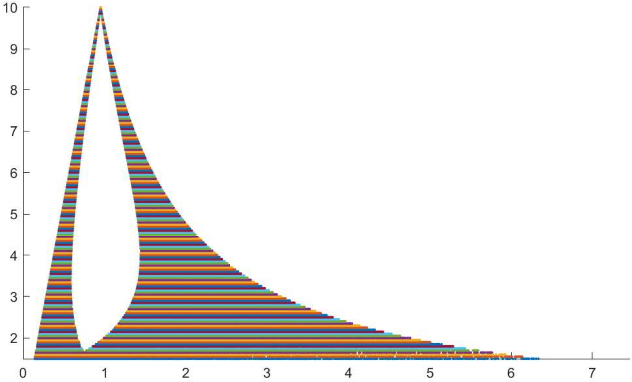


Picture of parameterspace

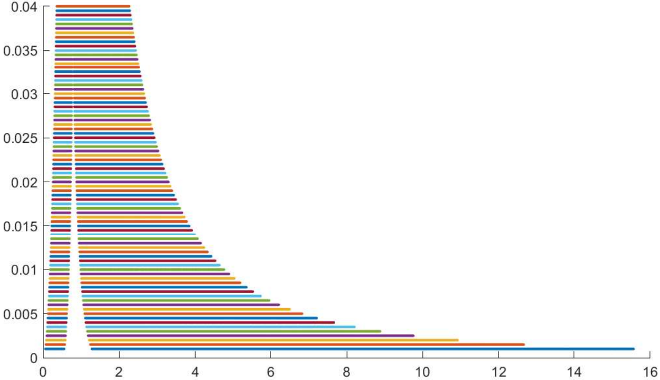
smaller attractors ($p = \frac{1}{10}$)



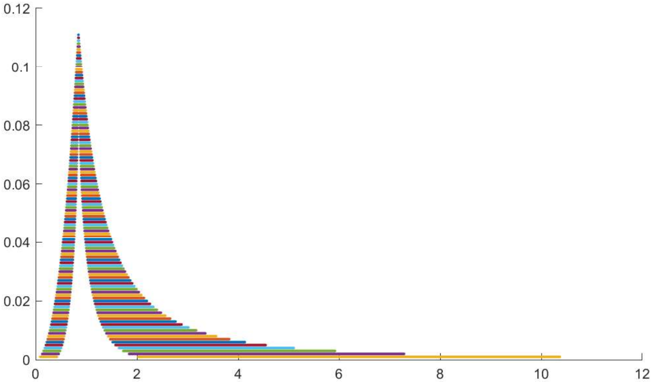
smaller attractors ($p = \frac{1}{100}$)



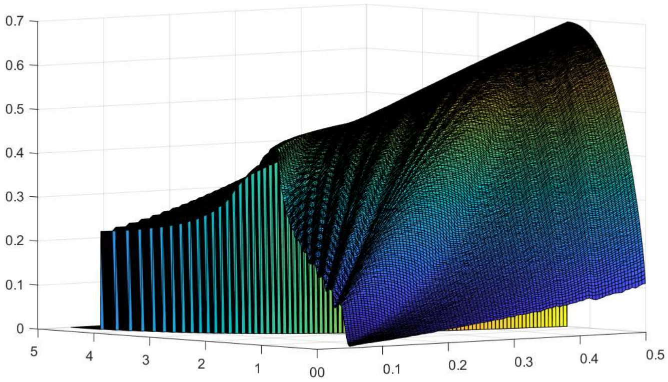
smaller attractors ($z = 2$)



smaller attractors ($z = 3$)



entropy



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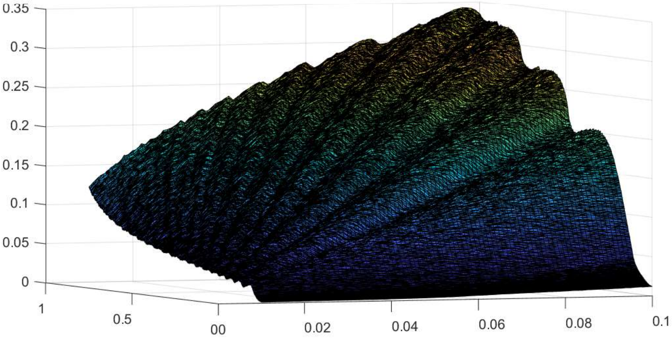

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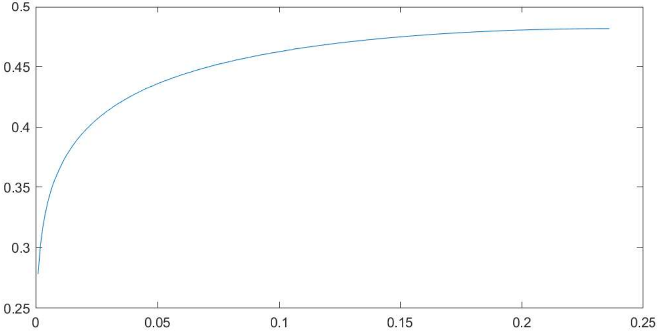

some of the problems

- admissibility (are there more 'lines' on which the symbolic space remains the same)
- the attractors
- sums of cantor sets
- unique expansions

Thank you for your time