N-continued fractions and *S*-adic sequences

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Greedy NCF-expansions

Let $N \in \mathbb{N}_{n\geq 2}$ and $T_N : [0,1] \to [0,1]$ be defined as

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor.$$

We define $d_1(x) = \lfloor \frac{N}{x} \rfloor$ and $d_n(x) = d_1(T_N^{n-1}(x))$ for $n \geq 2$.

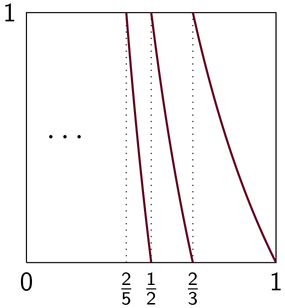


Figure: The map T_2.

For $x \in (0,1)$ we find

$$x = \frac{N}{d_1(x) + T_N(x)} = \frac{N}{d_1(x) + \frac{N}{d_2(x) + T_N^2(x)}} = \frac{N}{d_1(x) + \frac{N}{d_2(x) + \ddots}}$$

Some basic properties

- Let $([0,1], \mathcal{B}, \mu_N, T_N)$ be a dyn. sys. where $\mathcal{B}$ is the Borel σ-algebra and μ_A is given by $\mu_N(A) = \frac{1}{\log(\frac{N+1}{N})} \int_A \frac{1}{N+x}$ dx. Then $([0,1], \mathcal{B}, \mu_N, T_N)$ is ergodic and μ_N is a T_N-invariant probability measure.
- The greedy N-continued fraction expansion of x is finite if and only if $x \in \mathbb{Q}$.
- A number has many NCF-expansions (found by maps such as T_N, but only one with $d_i(x) \geq N$ when $x \in (0,1) \backslash \mathbb{Q}$. For rational numbers two.
- A lot of research went into the direction of periodicity of quadratic irrationals.
- Another direction of research are other NCF-expansions generated by different maps.

Substitutions and S-adic sequences

- Consider a finite alphabet $\mathcal{A}$ and let $\mathcal{A}^*$ be the set of all finite words with letters in this alphabet.
- A map $\sigma : \mathcal{A} \to \mathcal{A}^*$ is called a substitution.
- Extending the domain of σ to $\mathcal{A}^*$ by concatenating images of each letter allows for iteration.
- We can iterate over the same substitution or we can change the substitution in every step (using a directive sequence) taking them from a set $\mathcal{S}$.
- We might get a limiting word which we call an S-adic sequence.

Our directive sequences

Definition

Let $N \geq 2$ and let $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$.

1. For each $n \geq 1$, consider the substitutions

$$\sigma_n : \begin{cases} 0 \to 0^{d_n} 1^N, \\ 1 \to 0. \end{cases}$$

   We assign to x the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n \geq 1}$.

2. For each $n \geq 1$, consider the dual substitutions

$$\widehat{\sigma}_n : \begin{cases} 0 \to 0^{d_n} 1, \\ 1 \to 0^N. \end{cases}$$

   We assign to x the directive sequence $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n \geq 1}$.

Our S-adic sequences

Definition

Let $N \geq 2$ and $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$.

1. We define the *NCF sequence* $\omega(x, N)$ as the S-adic sequence of the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$. The finite words $(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1))_{n\geq 1}$ form a nested sequence of prefixes of $\omega(x, N)$ and they satisfy

$$\omega(x, N) = \lim_{n\to\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1).$$

2. We define the *dual NCF sequence* $\widehat{\omega}(x, N)$ as the S-adic sequence of the directive sequence $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n\geq 1}$. The finite words $(\widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0))_{n\geq 1}$ form a nested sequence of prefixes of $\widehat{\omega}(x, N)$ and they satisfy

$$\widehat{\omega}(x, N) = \lim_{n\to\infty} \widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0).$$

S-adic sequences ‘from above and below’

A sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ is an S*-adic sequence* (or *limit sequence*) of the directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1}$ if there exist $\omega^{(1)}, \omega^{(2)}, \ldots \in \mathcal{A}^{\mathbb{N}}$ such that

$$\omega^{(1)} = \omega, \quad \omega^{(n)} = \sigma_n(\omega^{(n+1)}) \quad \text{for all } n \geq 1.$$

Define the words

$$\Sigma_0 := 1, \quad \Sigma_n := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1) \quad \text{for } n \geq 1.$$

the words Σ_n satisfy the recurrence

$$\Sigma_{n+1} = \Sigma_n^{d_n} \Sigma_{n-1}^{N} \quad \text{for } n \geq 1. \tag{1}$$

Analogously, let

$$\widehat{\Sigma}_0 := 1, \quad \widehat{\Sigma}_1 := 0, \quad \widehat{\Sigma}_{n+1} := \widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0) \quad \text{for } n \geq 1.$$

Then we have that

$$\widehat{\Sigma}_{n+1} = \widehat{\Sigma}_n^{d_n} \widehat{\Sigma}_{n-1}^{N} \quad \text{for } n \geq 2. \tag{2}$$

An example

$$\sigma_n : \begin{cases} 0 \to 0^{d_n} 1^N, \\ 1 \to 0. \end{cases}$$

Let $x = [0; 7, 5, 2, 4, \ldots]_2$.

$$\Sigma_1 = \sigma_1(1) = 0$$
$$\Sigma_2 = \sigma_1 \circ \sigma_2(1) = \sigma_1(0) = 0^7 1^2$$
$$\Sigma_3 = \sigma_1 \circ \sigma_2 \circ \sigma_3(1) = \sigma_1 \circ \sigma_2(0) = \sigma_1(0^5 1^2) = (0^7 1^2)^5 0^2$$
$$\Sigma_4 = ((0^7 1^2)^5 0^2)^2 (0^7 1^2)^2$$

Note that indeed they satisfy the recurrence $\Sigma_{n+1} = \Sigma_n^{d_n} \Sigma_{n-1}^N$ also note that $|\Sigma_4| = 108$, these words grow fast. Let

$$\tau : \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 1^N. \end{cases} \tag{3}$$

Then

$$\omega(x, N) = \tau(\widehat{\omega}(x, N)). \tag{4}$$

Incidence matrices

The substitutions

$$\sigma_n : \begin{cases} 0 \to 0^{d_n}1^N, \\ 1 \to 0. \end{cases} \qquad \widehat{\sigma}_n : \begin{cases} 0 \to 0^{d_n}1, \\ 1 \to 0^N. \end{cases}$$

give the following incidence matrices

$$M_{\sigma_n} = \begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix}, \qquad M_{\widehat{\sigma}_n} = \begin{pmatrix} d_n & N \\ 1 & 0 \end{pmatrix}.$$

They allow you to calculate the amount of zero's and ones in the image of a word. For example $w = 0111$, gives vector ${}^t(1,3)$ then

$$\begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} d_n + 3 \\ N \end{pmatrix}$$

gives the frequencies for $\sigma(w)$. This is easy to check $\sigma(0111) = 0^{d_n}1^N000$.

Incidence matrices and convergents

Let $N \geq 2$ and consider the expansion $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$. Define the convergents $c_n = \frac{p_n}{q_n}$ for $n \geq 1$ as

$$\frac{p_n}{q_n} := [0; d_1, d_2, \ldots, d_n]_N,$$

and choose p_n and q_n so that they satisfy the following recurrence relations:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = 0, \quad p_n = d_n p_{n-1} + N p_{n-2}, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = d_n q_{n-1} + N q_{n-2}. \end{aligned} \tag{5}$$

Then we have $x = \lim_{n\to\infty} \frac{p_n}{q_n}$. Set $M_{[1,n]} = M_{\sigma_1} M_{\sigma_2} \cdots M_{\sigma_n}$. We find that

$$M_{[1,n]} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}.$$

Incidence matrices and convergents

In particular

$$M_{[1,n]} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} q_{n-1} \\ p_{n-1} \end{pmatrix}$$

This gives us the following

$$\lim_{n\to\infty} \frac{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_1}{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_0} = \lim_{n\to\infty} \frac{p_{n-1}}{q_{n-1}} = x \tag{6}$$

as well as $|\Sigma_n| = p_{n-1} + q_{n-1}$. We define the *frequency* of a letter $a \in \mathcal{A}$ in the sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ as

$$f_a := \lim_{|p|\to\infty} \frac{|p|_a}{|p|},$$

provided that the limit, which is taken over the prefixes p of ω, exists. Equation (6) implies that the N-continued fraction sequence $\omega(x, N)$ has letter frequencies and the *frequency vector* is given by $(f_0, f_1) = \left(\frac{1}{x+1}, \frac{x}{x+1}\right)$.

Growth rate

For the growth rate of the lengths of the words Σ_n we obtain the formula

$$\lim_{n\to\infty} \frac{1}{n} \log(|\Sigma_n|) = -\frac{1}{2} \left(h(T_N) + \log(N)\right). \tag{7}$$

Here $h(T_N)$ is the measure theoretic entropy of the map T_N and is given by

$$h(T_N) = \frac{\frac{\pi^2}{3} + 2Li_2(N+1) + \log(N+1)\log(N)}{\log\left(\frac{N+1}{N}\right)}$$

where Li_2 denotes the dilogarithm function defined by

$$Li_2(x) = \int_0^x \frac{\log(t)}{1-t} dt.$$

Growth rate

Recall that $|\Sigma_n|_1 = p_{n-1}$ and $|\Sigma_n|_0 = q_{n-1}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log(|\Sigma_n|) &= \lim_{n\to\infty} \frac{1}{n} \log(p_{n-1} + q_{n-1}) \\
&= \lim_{n\to\infty} \frac{1}{n} \log\left(q_{n-1}\left(\frac{p_{n-1}}{q_{n-1}} + 1\right)\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log(q_n).
\end{aligned}$$

Thus it remains to show that

$$\lim_{n\to\infty} \frac{1}{n} \log(q_n) = -\frac{1}{2}\left(h(T_N) + \log(N)\right). \tag{8}$$

Growth rate

Let

$$\Delta_n(x) = \{y \in [0,1] \,:\, y = [0; d_1(x), d_2(x), \ldots, d_n(x), \ldots]_N\}.$$

We have

$$\lim_{n\to\infty} -\frac{1}{n} \log\left(\mu_N(\Delta_n(x))\right) = h(T_N) \tag{9}$$

by using the Shannon-McMillan-Breiman-Chung Theorem and a Theorem of Kolmogorov and Sinai. Here μ_N is the absolutely continuous invariant measure. Since the measure μ_N and the Lebesgue measure λ are equivalent we have

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mu_N(\Delta_n(x))\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\lambda(\Delta_n(x))\right).$$

Growth rate

Now, similar to the cylinders of the regular continued fraction, we have

$$\lambda(\Delta_n(x)) = \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{N^n}{q_n(q_n + q_{n-1})}.$$

We find

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log\left(\lambda(\Delta_n(x))\right) &= \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{N^n}{q_n(q_n+q_{n-1})}\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{1}{n} \log\left(q_n(q_n+q_{n-1})\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{1}{n} \log\left(q_n^2\left(1 + \frac{q_{n-1}}{q_n}\right)\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{2}{n} \log(q_n)
\end{aligned}$$

We conclude $\lim_{n\to\infty} \frac{1}{n} \log(|\Sigma_n|) = -\frac{1}{2}\left(h(T_N) + \log(N)\right)$.

The NCF algorithm and the projective space

Define $\mathbb{P}_{<} = \{[1:x] : 0 < x < 1\}$. Let $\mathbf{x} = [1:x] \in \mathbb{P}_{<}$ and consider the matrix

$$C_N(\mathbf{x}) = \begin{pmatrix} \lfloor \frac{N}{x} \rfloor & N \\ 1 & 0 \end{pmatrix} = M_{\widehat{\sigma}_n}.$$

Then the map

$$G_N : \mathbb{P}_{<} \to \mathbb{P}_{<}, \quad \mathbf{x} \mapsto {}^t C_N(\mathbf{x})^{-1}\mathbf{x}$$

is called the *linear multiplicative N-continued fraction algorithm*.

$$G_N([1:x]) = \begin{pmatrix} 0 & \frac{1}{N} \\ 1 & -\frac{1}{N}\lfloor \frac{N}{x} \rfloor \end{pmatrix} \cdot [1:x]$$
$$= \left[\frac{x}{N} : 1 - \frac{x}{N}\left\lfloor \frac{N}{x} \right\rfloor\right] = \left[1 : \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor\right]$$

The NCF algorithm and the projective space

If $x = [0; d_1, d_2, \ldots]_N \notin \mathbb{Q}$ then we have for each $n \geq 1$ that

$$
{}^t C_N(G_N^{n-1}(\mathbf{x})) = {}^t C_N([1 : T_N^{n-1}(x)]) = \begin{pmatrix} \left\lfloor \frac{N}{T_N^{n-1}(x)} \right\rfloor & 1 \\ N & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix} = M_{\sigma_n}.
$$

Iteration yields

$$
\begin{aligned}
\mathbf{x} &= {}^tC_N(\mathbf{x}) G_N(\mathbf{x}) = {}^tC_N(\mathbf{x})\, {}^tC_N(G_N(\mathbf{x})) G_N^2(\mathbf{x}) = \ldots \\
&= {}^tC_N(\mathbf{x}) \cdots {}^tC_N(G_N^{n-1}(\mathbf{x})) G_N^n(\mathbf{x})
\end{aligned}
$$

and therefore

$$
\mathbf{x} = M_{\sigma_1} G_N(\mathbf{x}) = M_{\sigma_1} M_{\sigma_2} G_N^2(\mathbf{x}) = \cdots = M_{[1,n]} G_N^n(\mathbf{x}).
$$

The NCF algorithm and the projective space

Let $\mathbf{x} = [1 : x]$ and $\mathbf{y} = [1 : y]$ be elements of $\mathbb{P}_<$. A natural extension of the map G_N is given by $\widetilde{G}_N : \mathbb{P}^2_< \to \mathbb{P}^2_<$

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \mapsto \begin{pmatrix} {}^t C_N(\mathbf{x})^{-1} & 0 \\ 0 & C_N(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \left[1 : \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor\right] \\ \left[1 : \frac{1}{N \cdot y + \left\lfloor \frac{N}{x} \right\rfloor}\right] \end{pmatrix}. \quad (10)$$

A natural extension of the map $T_N : [0,1] \to [0,1]$ is given by

$$\widetilde{T}_N : [0,1] \times \left[0, \frac{1}{N}\right] \to [0,1] \times \left[0, \frac{1}{N}\right],$$

$$(x, y) \mapsto \begin{cases} \left(T_N(x), \frac{1}{N \cdot y + \left\lfloor \frac{N}{x} \right\rfloor}\right) & x \neq 0, \\ (0,0) & x = 0, \end{cases}$$

with $\frac{dxdy}{(1+xy)^2}$ as invariant measure.

Swapping labels?

We can relabel
$$\sigma_n : \begin{cases} 0 \to 0^{d_n} 1^N \\ 1 \to 0 \end{cases} \qquad \widehat{\sigma}_n : \begin{cases} 0 \to 0^{d_n} 1 \\ 1 \to 0^N \end{cases}$$

and get
$$\sigma_n : \begin{cases} 0 \to 1 \\ 1 \to 1^{d_n} 0^N \end{cases} \qquad \widehat{\sigma}_n : \begin{cases} 0 \to 1^N \\ 1 \to 1^{d_n} 0 \end{cases}$$

This would give

$$M_{\sigma_n} = \begin{pmatrix} 0 & N \\ 1 & d_n \end{pmatrix}, \qquad M_{[1,n]} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}.$$

$$M_{\sigma_n} = \begin{pmatrix} 0 & N \\ 1 & d_n \end{pmatrix} \cdot (x) = \frac{N}{x + d_n}$$

Now the incidence matrices can be seen as Möbius transformations acting like the inverse branches. We chose the substitutions as it is so that the dual substitutions $\widehat{\sigma}_n$ are a particular instance of β-substitutions for simple Parry numbers.

Balancedness

Definition (Balanced)

We say that a pair (u, v) of words over the alphabet $\mathcal{A}$ is *balanced* if $|u| = |v|$ and

$$-1 \leq |u|_a - |v|_a \leq 1 \qquad \text{for every } a \in \mathcal{A}.$$

We say that a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ is *balanced* if every pair (u, v) of factors of ν with $|u| = |v|$ is balanced.

Figure: The letter a weights 1 Kg. Other letters weigh nothing.

C-Balancedness

Definition (Balance)

Given $C > 0$, we say that a pair (u, v) of words over the alphabet $\mathcal{A}$ is *C-balanced* if $|u| = |v|$ and

$$-C \leq |u|_a - |v|_a \leq C \quad \text{for every } a \in \mathcal{A}.$$

We say that a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ is *C-balanced* if every pair (u, v) of factors of ν with $|u| = |v|$ is C-balanced. We say that ν is *finitely balanced* if it is C-balanced for some $C > 0$.

Figure: Now we are allowed any extra weight between 1 and C Kg.

Results on Ballancedness upper bound

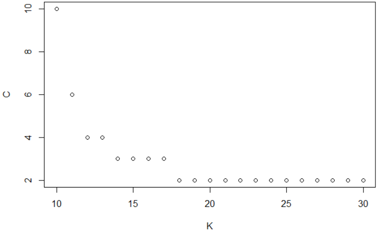
Theorem (L., Rossi, and Thuswaldner)

Let $N \geq 2$ be fixed and set $K \geq N$ and $C = \left\lfloor \frac{K-1}{K+1-N} \right\rfloor + 1$. If we set

$$W_{K,N} := \{[0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q} \,:\, d_n \geq K \text{ for all } n \geq 1\}$$

then the following assertions hold.

1. *For all $x \in W_{K,N}$ the dual NCF sequence $\widehat{\omega}(x, N)$ is C-balanced.*
2. *For all $x \in W_{K,N}$ the NCF sequence $\omega(x, N)$ is $N \cdot C$-balanced.*



Results on Ballancedness upper bound

Idea of proof

1. Prove the statement for $\widehat{\omega}(x, N)$.
2. Prove by contradiction, take the smallest minimal pair (u, v) of not C-balanced words among all sequences generated by $x \in W_{K,N}$ and find a smaller pair.
3. For the statement $\omega(x, N)$ use that if $\widehat{\omega}(x, N)$ is C-balanced for some $C > 0$, then $\omega(x, N)$ is $N \cdot C$-balanced.

Corollary

1. *For all $x \in [0,1] \setminus \mathbb{Q}$ we have that $\widehat{\omega}(x, N)$ is N-balanced. Furthermore, for every $N \geq 2$ there are uncountable many $x \in [0,1] \setminus \mathbb{Q}$ such that $\widehat{\omega}(x, N)$ is 2-balanced.*
2. *For all $x \in [0,1] \setminus \mathbb{Q}$ we have that $\omega(x, N)$ is N^2-balanced. Furthermore, for every $N \geq 2$ there are uncountable many $x \in [0,1] \setminus \mathbb{Q}$ such that $\omega(x, N)$ is $2N$-balanced.*

Results on Ballancedness lower bound

Theorem (L., Rossi, and Thuswaldner)

Let $N \geq 2$ and $x \in [0,1] \backslash \mathbb{Q}$. Then the following assertions hold.

1. $\widehat{\omega}(x, N)$ *is not* 1*-balanced.*
2. $\omega(x, N)$ *is not* $(2N-1)$*-balanced.*

Corollary

For $N = 2$*, for all* $x \in [0,1] \setminus \mathbb{Q}$ *we have*

- $\widehat{\omega}(x, N)$ *is not* 1*-balanced but it is* 2*-balanced.*
- $\omega(x, N)$ *is not* 3*-balanced but it is* 4*-balanced.*

With other words, the bounds are sharp for $N = 2$*.*

Proof of (1)

Let $x = [0; d_1, d_2, \dots]_N$ and recall

$$\widehat{\Sigma}_{n+1} = \widehat{\Sigma}_n^{d_n} \widehat{\Sigma}_{n-1}^N. \tag{11}$$

On the one hand we have

$$\begin{aligned}
\widehat{\Sigma}_4 &= \widehat{\Sigma}_3^{d_3}\, \widehat{\Sigma}_2^N \\
&= \widehat{\Sigma}_2^{d_2}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_2^{d_2}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_3^{d_3-2}\, \widehat{\Sigma}_2^N \\
&= \widehat{\Sigma}_2^{d_2}\, 0^N\, 0^{d_1}\, 1\, \widehat{\Sigma}_2^{d_2-1}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_3^{d_3-2}\, \widehat{\Sigma}_2^N \\
&= \widehat{\Sigma}_2^{d_2} 0^{N-2}\, 0^{d_1+2}\, 1\, \widehat{\Sigma}_2^{d_2-1}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_3^{d_3-2}\, \widehat{\Sigma}_2^N
\end{aligned} \tag{12}$$

which gives us the factor $u = 0^{d_1+2}$ of $\widehat{\omega}(x, N)$. On the other hand we have

$$\begin{aligned}
\widehat{\Sigma}_4 &= \widehat{\Sigma}_3^{d_3}\, \widehat{\Sigma}_2^N \\
&= \widehat{\Sigma}_2^{d_2}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_3^{d_3-1}\, \widehat{\Sigma}_2^N \\
&= 0^{d_1}\, 1\, 0^{d_1}\, 1\, \widehat{\Sigma}_2^{d_2-2}\, \widehat{\Sigma}_1^N\, \widehat{\Sigma}_3^{d_3-1}\, \widehat{\Sigma}_2^N
\end{aligned} \tag{13}$$

giving us the factor $v = 10^{d_1}1$.

Complexity

Definition (Factor complexity function)

Given a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ over the finite alphabet $\mathcal{A}$ and $n \in \mathbb{N}$, set

$$\mathcal{L}_n(\nu) := \{u \in \mathcal{A}^* \,:\, |u| = n,\ u \text{ is a factor of } \nu\}.$$

Define the *factor complexity function* $p_\nu : \mathbb{N} \to \mathbb{N}$ as $p_\nu(n) = |\mathcal{L}_n(\nu)|$.

Define, for each $k \geq 1$, the words

$$\widehat{S}_k := \widehat{\Sigma}_k^{d_k} \widehat{\Sigma}_{k-1}^{d_{k-1}} \cdots \widehat{\Sigma}_1^{d_1}, \qquad \widehat{T}_k := \widehat{\Sigma}_k^{N-1} \widehat{S}_k$$

$$S_k := \Sigma_k^{d_k} \Sigma_{k-1}^{d_{k-1}} \cdots \Sigma_1^{d_1}, \qquad T_0 := 1^{N-1}, \qquad T_k := \Sigma_k^{N-1} S_k.$$

Define the numbers $\widehat{t}_0 < \widehat{s}_1 < \widehat{t}_1 < \widehat{s}_2 < \widehat{t}_2 < \cdots$ as

$$\widehat{s}_k := |\widehat{S}_k|, \quad \widehat{t}_0 = 0, \quad \widehat{t}_k := |\widehat{T}_k| \qquad (k \geq 1).$$

Define the numbers $t_0 < s_1 < t_1 < s_2 < t_2 < \cdots$ as

$$s_k := |S_k| \quad (k \geq 1), \qquad t_k := |T_k| \quad (k \geq 0).$$

Result on complexity

Theorem (L., Rossi, and Thuswaldner)

The factor complexity functions of $\omega = \omega(x, N)$ *and* $\widehat{\omega} = \widehat{\omega}(x, N)$ *satisfy* $p_\omega(n) \le 2n$ *and* $p_{\widehat{\omega}}(n) \le 2n$ $(n \ge 1)$.
In particular, they are given by

$$p_\omega(n) = \begin{cases} 1, & n = 0, \\ 2n, & 1 \le n \le N-1, \\ n + 1 + \sum_{j=-1}^{k-1}(p_j + q_j)(N-1), & t_k < n \le s_{k+1}, \\ 2n + 1 + \sum_{j=-1}^{k-1}(p_j + q_j)(N-1) - s_k, & s_k < n \le t_k \end{cases} \tag{14}$$

and

$$p_{\widehat{\omega}}(n) = \begin{cases} 1, & n = 0, \\ n + 1 + \sum_{j=0}^{k-2}(\frac{p_j}{N} + q_j)(N-1), & \widehat{t}_{k-1} < n \le \widehat{s}_k, \\ 2n + 1 + \sum_{j=0}^{k-2}(\frac{p_j}{N} + q_j)(N-1) - \widehat{s}_k, & \widehat{s}_k < n \le \widehat{t}_k. \end{cases} \tag{15}$$

Corollaries

Corollary

The topological dynamical systems (X_ω, Σ) and $(X_{\widehat{\omega}}, \Sigma)$ are uniquely ergodic.

Definition (Uniform word and letter frequency)

Consider a sequence $\nu = \nu_0\nu_1 \cdots \in \mathcal{A}^{\mathbb{N}}$ with $\nu_i \in \mathcal{A}$. We say that ν has uniform word frequency if for each $u \in \mathcal{A}^*$ there exists $f_\nu(u) \in \mathbb{R}$ which does not depend on k such that

$$\lim_{l \to \infty} \frac{|\nu_k \cdots \nu_{k+l-1}|_u}{l} = f_\nu(u),$$

and this limit is uniform on k.

Corollary

$\omega(x, N)$ and $\widehat{\omega}(x, N)$ have uniform word frequency. Moreover, this holds for all the elements of X_ω and $X_{\widehat{\omega}}$.

Non-greedy NCF's

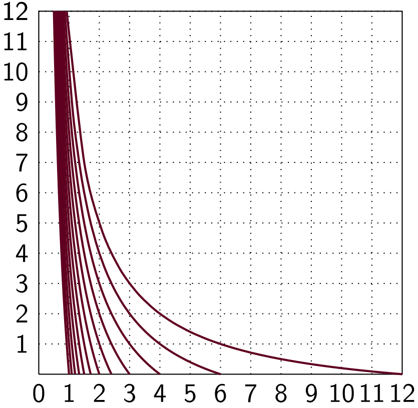


Figure: All possible maps in case $N = 12$.

$$T_d(x) = \frac{N}{x} - d \quad \text{gives} \quad x = \frac{N}{d + T_d(x)}.$$

Remarks

For general NCF sequences we have the following.

1. Suppose $(d_n) = (d)$ for some $d < N$. Then the associated substitution σ is not Pisot, and hence the sequence $\widetilde{\omega}(x, N)$ is imbalanced, that is, it is not C-balanced for any $C > 0$.
2. S-adic sequences corresponding to eventually greedy NCF expansions are finitely balanced.
3. For general NCF sequences we also have

$$\lim_{n\to\infty} \frac{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_1}{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_0} = \lim_{n\to\infty} \frac{p_{n-1}}{q_{n-1}} = x \qquad (16)$$

but convergence will be slower.

Thank you for your attention!