q-analog of the Markoff injectivity conjecture

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Outline

1. Markoff injectivity conjecture

2. q-analog of a Markoff Injectivity Conjecture

3. Balanced sequences

4. Main result

Markoff Injectivity Conjecture

Markoff numbers

- Markoff triple : A positive solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

- Positive solutions of Equation 1 can be computed recursively :

$$(x, y, z)$$

$$(x, 3xy - z, y) \qquad (y, 3yz - x, z)$$

- If $y > x$, $y > z$ and $x \neq z$, then $(x, 3xy - z, y) \neq (y, 3yz - x, z)$, $3xy - z > y$ and $3yz - x > y$.

Binary tree of proper Markoff triples

- $(1, 5, 2)$
  - $(1, 13, 5)$
    - $(1, 34, 13)$
      - $(1, 89, 34)$
      - $(34, 1325, 13)$
    - $(13, 194, 5)$
      - $(13, 7561, 194)$
      - $(194, 2897, 5)$
  - $(5, 29, 2)$
    - $(5, 433, 29)$
      - $(5, 6466, 433)$
      - $(433, 37666, 29)$
    - $(29, 169, 2)$
      - $(29, 14701, 169)$
      - $(169, 985, 2)$

Markoff numbers

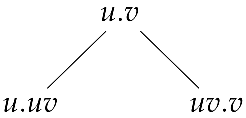
- Markoff numbers : An element of a Markoff triples

$$1, 2, 5, 13, 29, 34, 89, 169, 194, \ldots$$

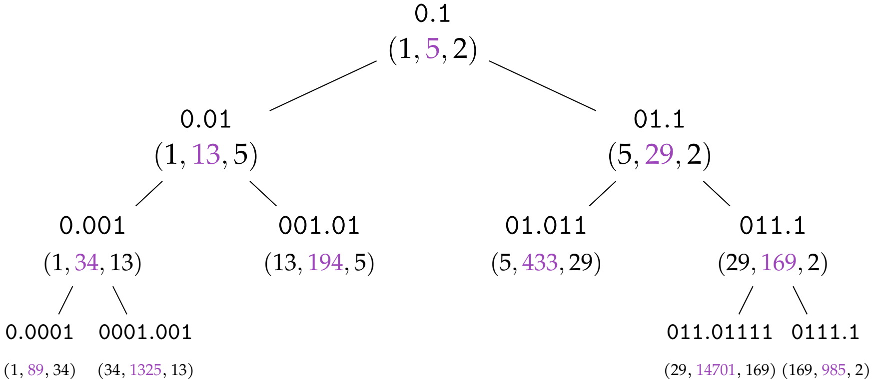
- OEIS sequence A002559

Christoffel words

- Christoffel word : A word over the alphabet $\{\mathtt{0}, \mathtt{1}\}$ such that $\mathtt{0}$, $\mathtt{1}$, $\mathtt{01}$ are Christoffel words and all words satisfying the following binary tree structure :



Binary tree of proper Christoffel words and proper Markoff numbers



A map between words and Markoff numbers

$\mu : \{0,1\}^* \to \mathrm{SL}_2(\mathbb{N})$

$$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Proposition (Reutenauer, 2009)

The function f from the set of Christoffel words to the set of Markoff triples defines as

$$f(w) = \{\mu(u)_{12}, \mu(w)_{12}, \mu(v)_{12}\},$$

where $u.v$ is the standard factorization of w, is a bijection.

Markoff Injectivity Conjecture

$\mu : \{0, 1\}^* \to \mathrm{SL}_2(\mathbb{N})$

$$\mu(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu(1) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Conjecture

The map $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel word.

- Uniqueness Conjecture, Frobenius conjecture, ...
- The Markoff Injectivity Conjecture has many other equivalent formulation (Aigner, 2013).

Partial Results

Proposition (L. and Reutenauer, 2021)

The map $w \mapsto \mu(w)_{12}$ is increasing over the language of factors of a Christoffel word, thus injective.

Corollary (L. and Reutenauer, 2021)

Moreover, the map is also increasing for all Christoffel words on an infinite path in the binary tree of Christoffel words.

Some other partial results were proven recently by Rabideau and Schiffler (2020) and Lagisquet, Pelantová, Tavenas and Vuillon (2021).

q-analog of a Markoff Injectivity Conjecture

q-analog

A q-analog is a mathematical expression parametrized by a variable q that generalizes a known expression.

Examples :

- q-analog of a nonnegative integer n is

$$[n]_q = 1 + q + \cdots + q^{n-1}$$

- q-factorial

$$n!_q = [1]_q[2]_q \ldots [n]_q$$

- q-binomial coefficients

$$\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q k!_q}$$

q-analog of rational number

- The q-analog of a rational number $\frac{a}{b}$ is a polynomials over q defined from the continued fraction expansion (Morier-Genoud and Ovsienko, 2020)
- Their work is based on q-deformation of the generators of $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\pm Id$

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \quad S_q = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$$

q-analog of a Markoff number

The map μ can be expressed as of product of the matrix R and S :

$$\mu(\mathtt{0}) = R^2SR \qquad \mu(\mathtt{1}) = R^3SR^2SR$$

Hence, the q-analog of $\mu(\mathtt{0})$ and $\mu(\mathtt{1})$ are

$$\mu_q(\mathtt{0}) = R_q^2S_qR_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix},$$
$$\mu_q(\mathtt{1}) = R_q^3S_qR_q^2S_qR_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$$

and $\mu_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$ is a morphism of monoids.

q-analog of a Markoff number

If w is a Christoffel word, then the polynomial $\mu_q(w)_{12}$ is the q-analog of a Markoff number.

Example :

$$\texttt{001} \quad 13 \quad 1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5$$

q-analog of Markoff injectivity conjecture

Conjecture

The map $w \mapsto \mu_q(w)_{12}$ is injective on the set of Christoffel words.

The q-analog version is weaker than the original Markoff Injectivity Conjecture.
Example : 001101 and 010011 have different polynomials

$$\mu_q(001101)_{12} = 1 + 5q + 16q^2 + 37q^3 + 69q^4 + 107q^5 + 143q^6 + 166q^7 + 169q^8 + 151q^9 + 117q^{10} + 79q^{11} + 44q^{12} + 19q^{13} + 6q^{14} + q^{15}$$

$$\mu_q(010011)_{12} = 1 + 5q + 16q^2 + 38q^3 + 70q^4 + 109q^5 + 144q^6 + 166q^7 + 169q^8 + 150q^9 + 116q^{10} + 77q^{11} + 43q^{12} + 19q^{13} + 6q^{14} + q^{15}$$

but when $q = 1$: $\mu(001101)_{12} = \mu(010011)_{12} = 1130$.

q-analog of Markoff injectivity conjecture

Conjecture

The map $w \mapsto \mu_q(w)_{12}$ is injective on the set of Christoffel words.

The map $w \mapsto \mu_q(w)_{12}$ from $\{0,1\}^*$ to polynomials is not injective.

Example :

$$\begin{aligned}\mu_q(\mathtt{000111})_{12} &= 1 + 5q + 16q^2 + 38q^3 + 70q^4 + 109q^5 + 145q^6 \\ &\quad + 168q^7 + 171q^8 + 152q^9 + 118q^{10} + 79q^{11} \\ &\quad + 44q^{12} + 19q^{13} + 6q^{14} + q^{15} \\ &= \mu_q(\mathtt{011001})_{12}.\end{aligned}$$

Balanced sequences

Balanced sequences

- A biinfinite sequence $s \in \Sigma^{\mathbb{Z}}$ over a finite set Σ.
- The language of s is the set of factors occurring in s :

$$\mathcal{L}(s) = \{s_k s_{k+1} \dots s_{k+n-1} \mid k \in \mathbb{Z}, n \geq 0\}$$

Balanced sequences

A sequence $s \in \Sigma^{\mathbb{Z}}$ is balanced if for every $u, v \in \mathcal{L} \cap \Sigma^n$ and every letter $a \in \Sigma$, the number of a's occurring in u and v differ by at most 1.

In our case, $\Sigma = \{0, 1\}$.

Examples :

- biinfinite periodic ${}^{\infty}w^{\infty}$ repetition of Christoffel words
- Sturmian sequences

Fibonacci word

- The right-infinite Fibonacci Word

$$F = 0100101001001010010 1 \ldots \in \Sigma^{\mathbb{N}}$$

- $\widetilde{F} \cdot 01 \cdot F$

$$\ldots 1010010100100101001 0 \cdot 01 \cdot 0100101001001010010 1 \ldots$$

- $\widetilde{F} \cdot 10 \cdot F$

$$\ldots 1010010100100101001 0 \cdot 10 \cdot 0100101001001010010 1 \ldots$$

Fibonacci words are balanced sequences

- $\widetilde{F} \cdot \mathtt{01} \cdot F$:

$$\ldots \mathtt{1010010100100101001 0} \cdot \mathtt{01} \cdot \mathtt{0100101001001010 0101} \ldots$$

n	$\mathcal{L}(\widetilde{F}\mathtt{01}F) \cap \Sigma^n$	number of $\mathtt{0}$'s	number of $\mathtt{1}$'s
0	$\{\varepsilon\}$	0	0
1	$\{\mathtt{0}, \mathtt{1}\}$	0 or 1	0 or 1
2	$\{\mathtt{00}, \mathtt{01}, \mathtt{10}\}$	1 or 2	0 or 1
3	$\{\mathtt{001}, \mathtt{010}, \mathtt{100}, \mathtt{101}\}$	1 or 2	1 or 2
4	$\{\mathtt{0010}, \mathtt{0100}, \mathtt{0101}, \mathtt{1001}, \mathtt{1010}\}$	2 or 3	1 or 2
5	$\{\mathtt{00100}, \mathtt{00101}, \mathtt{01001}, \mathtt{01010}, \mathtt{10010}, \mathtt{10100}\}$	3 or 4	1 or 2

Binary balanced sequence

If $u \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence, then it falls into exactly one of the following classes :

(MH_1) u is a purely periodic word ${}^{\infty}w^{\infty}$ for some Christoffel word w,

(MH_2) u is a generic aperiodic Sturmian word

(MH_3) u is a characteristic aperiodic Sturmian word

(MH_4) u is an ultimately periodic word but not purely periodic :

$$\cdots xxyxx \cdots \text{ or } \cdots (ymx)(ymx)(ymy)(xmy)(xmy) \cdots ,$$

where $\{x,y\} = \{0,1\}$ and $0m1$ is a Christoffel word.

Markoff property

We say that a biinfinite word $s \in \{0, 1\}^{\mathbb{Z}}$ satisfies the Markoff property if for any factorization $s = uxyv$, where $\{x, y\} = \{0, 1\}$, one has

- either $\widetilde{u} = v$,
- or there is a factorization $u = u'ym$, $v = \widetilde{m}xv'$.

Four classes of balanced sequences

If a biinfinite sequence $u \in \{0, 1\}^{\mathbb{Z}}$ satisfies the Markoff property, then it falls into exactly one of the following classes :

(M_1) u cannot be written as $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$ and the lengths of the Christoffel words occurring in u are bounded ;

(M_2) u cannot be written as $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$ and the lengths of the Christoffel words occurring in u are unbounded ;

(M_3) u has a unique factorization $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$;

(M_4) u has at least two factorizations $u = \widetilde{p}xyp$ where $\{x, y\} = \{0, 1\}$.

Equivalence between classes

Proposition (Reutenauer, 2006)

Let $u \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence. For every $i \in \{1, 2, 3, 4\}$, u satisfies (M_i) if and only if u satisfies (MH_i).

Main result

Radix order

- Let $\{\mathtt{0},\mathtt{1}\}$ such that $\mathtt{0} < \mathtt{1}$.
- The radix order is defined for every $u, v \in \{\mathtt{0},\mathtt{1}\}^*$ as

$$u <_{radix} v \qquad \text{if} \qquad \begin{cases} |u| < |v| & \text{or} \\ |u| = |v| & \text{and} \quad u <_{lex} v. \end{cases}$$

Main Result

Proposition (L. and Reutenauer, 2021)

The map $w \mapsto \mu(w)$ is increasing over the language of factors of a Christoffel word, thus injective.

Theorem

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence and $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients.

Main Result

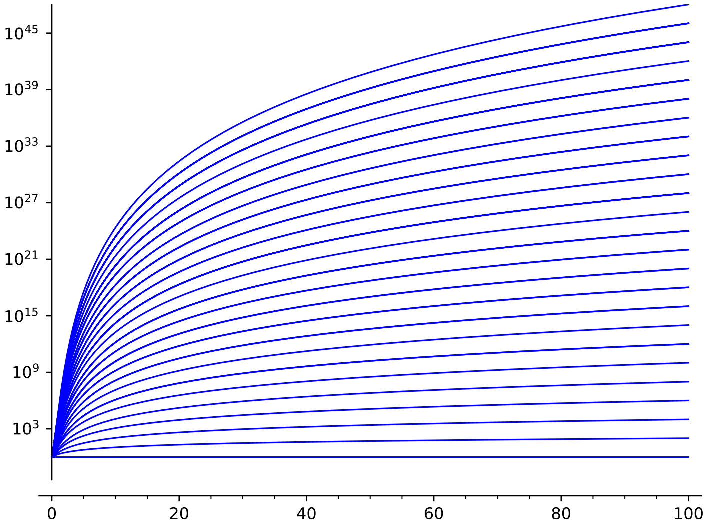
Theorem

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence and $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients.

Corollary

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence. For every $q > 0$, the map $\{0,1\}^* \to \mathbb{R}$ defined by $w \mapsto \mu_q(w)_{12}$ is injective over the language $\mathcal{L}(s)$ of factors occurring in s.

The graph of $\mu_q(w)_{12}$ for $0 \leq q \leq 100$ for all 55 factors w of lenght $|w| < 10$ in the Fibonacci word



Difference between two polynomials of same degree

The difference between two polynomials of the same degree is very small since the y axis uses a logarithmic scale.
Example :

$$\begin{array}{ll} \texttt{010} & 1 + 2q + 4q^2 + 4q^3 + 3q^4 + 2q^5 + q^6 \\ \texttt{100} & 1 + 3q + 4q^2 + 4q^3 + 4q^4 + 2q^5 + q^6 \end{array}$$

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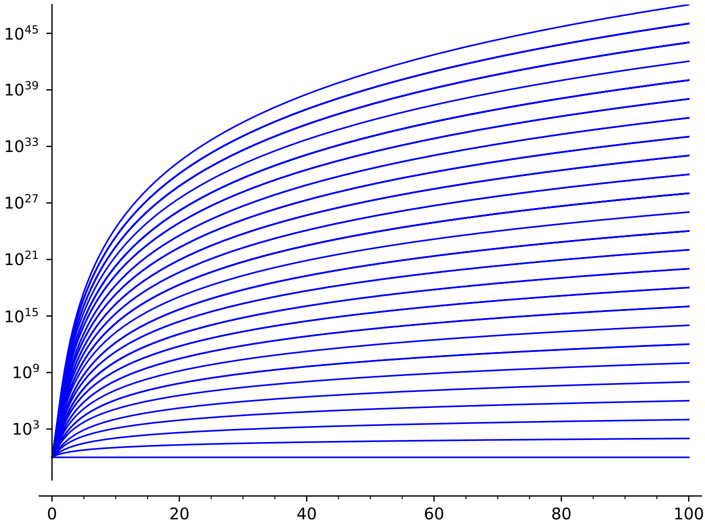

What does it means to be sorted in radix order ?

Proposition (Borel and Reutenauer, 2006)

Let $w = w_0 <_{lex}< w_1 <_{lex} \cdots <_{lex} w_n - 1$ be the conjugates of a Christoffel word of lenght n. Then, for each $i = 1, \ldots, n-1$, one has

$$w_{i-1} = u01v \qquad w_i = u10v$$

for some words $u, v \in \{0,1\}^*$.

Example : The conjugates of the Christoffel words 00100101 :

0010<u>01</u>01
0<u>01</u>01001
010010<u>01</u>
01<u>01</u>010
<u>01</u>010010
10010<u>01</u>0
10<u>01</u>0100
10100100

What does it means to be sorted in radix order ?

Example : The 9 factors of lenght 8, the largest factor of lenght 7 and the smallest factors of lenght 9 of the Fibonacci words :

1010010
00100101
00101001
01001001
01001010
01010010
10010010
10010100
10100100
10100101
001001010

$$\widetilde{u}01v \mapsto \widetilde{u}10v, \qquad w0 \mapsto w1, \qquad 1w \mapsto 0w0, \qquad 1w \mapsto 0w1$$

where $u, v, w \in \{0, 1\}^*$ and u is a prefix of v or vice versa.

Factors in radix order

Lemma (Labbé and L., 2021)

Let $s \in \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}$ be a balanced sequence having at least one factorization $s = \widetilde{p}xyp$ where $\{x, y\} = \{\mathtt{0}, \mathtt{1}\}$. Let $n \geq 1$ and $u_0, \dots, u_n$ be the $n+1$ factors of length n of s such that

$$u_0 <_{lex} \cdots <_{lex} u_n.$$

If w is the prefix of length $n-1$ of p, we have

- $u_0 = \mathtt{0}w$ *and* $u_n = \mathtt{1}w$,
- *there exists* $i \in \{0, \dots, n-1\}$ *such that* $u_i = \widetilde{w}\mathtt{0}$ *and* $u_{i+1} = \widetilde{w}\mathtt{1}$,
- *for all* $j \in \{0, \dots, n-1\} \setminus \{i\}$, *there exist prefixes* x, y *of* w *such that* $u_j = \widetilde{x}\mathtt{01}y$ *and* $u_{j+1} = \widetilde{x}\mathtt{10}y$.

Increasing over small local changes

Proposition (Labbé and L., 2021)

For every $w \in \{0,1\}^*$,

- $\mu_q(w1)_{12} - \mu_q(w0)_{12}$,
- $\mu_q(0w0)_{12} - \mu_q(1w)_{12}$
- $\mu_q(0w1)_{12} - \mu_q(0w0)_{12}$

are nonzero polynomials with nonnegative coefficients.

Proposition (Labbé and L., 2021)

Let $u, v \in \{0,1\}^*$ *such that* u *is a prefix of* v *or vice versa. Then*

$$\mu_q(\widetilde{u}10v)_{12} - \mu_q(\widetilde{u}01v)_{12}$$

is a nonzero polynomial with nonnegative coefficients.

Factors of Sturmian words or ultimately periodic sequences

Proposition (Labbé and L., 2021)

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence having at least one factorization $s = \widetilde{p}xyp$ where $\{x,y\} = \{0,1\}$. Let $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial with nonnegative coefficients.

Theorem (Labbé and L., 2021)

Let $s \in \{0,1\}^{\mathbb{Z}}$ be a balanced sequence and $u, v \in \mathcal{L}(s)$ be two factors in the language of s. If $u <_{radix} v$, then $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomial of indeterminate q with nonnegative integer coefficients.

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where $\{x, y\} = \{0, 1\}$ and $0m1$ is a Christoffel word.

Back the Markoff Injectivity Conjecture

Language of all balanced sequences

The following statement is false : If u, v in the language of all balanced sequences such that $u <_{radix} v$, then $\mu_q(v)_{12} - \mu_q(u)_{12}$ is a nonzero polynomials with nonnegative coefficients.

Example :

$$\mu_q(\texttt{00001})_{12} = 1 + 4q + 8q^2 + 13q^3 + 16q^4 + 17q^5 + 14q^6 + 10q^7 + 5q^8 + q^9,$$

$$\mu_q(\texttt{0111})_{12} = 1 + 3q + 9q^2 + 16q^3 + 24q^4 + 29q^5 + 29q^6 + 25q^7 + 18q^8 + 10q^9 + 4q^{10} + q^{11},$$

and their difference

$$\mu_q(\texttt{00001})_{12} - \mu_q(\texttt{0111})_{12} = q - q^2 - 3q^3 - 8q^4 - 12q^5 - 15q^6 - 15q^7 - 13q^8 - 9q^9 - 4q^{10} - q^{11}$$

has negative coefficients.

Open problem

Conjecture (Markoff Injectivity Conjecture)

The map $w \mapsto \mu(w)_{12}$ is injective on the set of Christoffel word.

Conjecture (q-analog of Markoff Injectivity Conjecture)

The map $w \mapsto \mu_q(w)_{12}$ is injective on the set of Christoffel word.