Odd-Odd Continued Fraction Algorithm

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One World Numeration Seminar

Joint work with Dong Han Kim and Lingmin Liao

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1. Regular Continued Fraction

Regular Continued Fraction (RCF)

For $d_0 \in \mathbb{Z}$, $d_n \in \mathbb{N}$,

$$d_0 + \cfrac{1}{d_1 + \cfrac{1}{\ddots + \cfrac{1}{d_n + \cfrac{1}{\ddots}}}} =: [d_0; d_1, d_2, \cdots, d_n, \cdots].$$

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For $x \in \mathbb{R}$, $\exists \{d_n\}$ such that

$$x = [d_0; d_1, d_2, \cdots, d_n, \cdots] \quad (\text{or} \quad [d_0; d_1, d_2, \cdots, d_n]).$$

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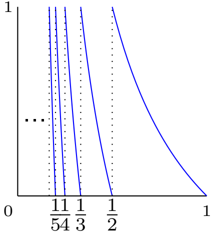
Partial quotients: $d_n(x) := d_n$

Gauss map: $G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \ x \in (0, 1]$

$$d_0 = \lfloor x \rfloor, \qquad d_n = \lfloor 1/G^{n-1}(x - d_0) \rfloor$$



Regular Continued Fraction (RCF)

$$x = [d_0; d_1, d_2, \cdots, d_n, \cdots] \quad (\text{or} \quad [d_0; d_1, d_2, \cdots, d_n]).$$

Principal convergents: $\dfrac{p_n^R}{q_n^R} = \dfrac{p_n^R(x)}{q_n^R(x)} := [d_0; d_1, d_2, \cdots, d_n] \in \mathbb{Q}$

$$p_n^R = d_n p_{n-1}^R + p_{n-2}^R \quad \text{and} \quad q_n^R = d_n q_{n-1}^R + q_{n-2}^R$$

Properties and Examples

1. $x \in \mathbb{Q} \iff x$ has exactly two finite RCF expansions.

   e.g. $\frac{4}{5} = \frac{1}{1+\frac{1}{4}} = [0; 1, 4] = \frac{1}{1+\frac{1}{3+\frac{1}{1}}} = [0; 1, 3, 1]$

2. **[Euler, Lagrange]** x quad. irr. $\iff$ its RCF is periodic.

   e.g. Golden ratio $\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \cdots]$.

Best Approximations

A Diophantine question:

For a given $x \notin \mathbb{Q}$ and a bounded integer q, which rational p/q minimizes $|qx - p|$?

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Definition p/q is *a best approximation of x* if

$$|qx - p| < |bx - a| \quad \text{for any } \frac{a}{b} \neq \frac{p}{q} \text{ s.t. } 0 < b \leq q.$$

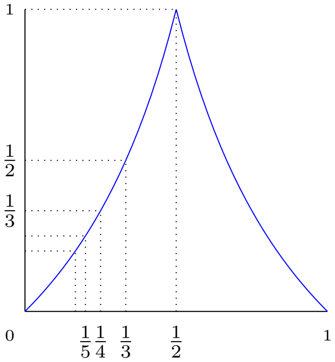
Theorem

For $x \notin \mathbb{Q}$,

$$\frac{p}{q} \text{ is a best approximation of } x \iff \frac{p}{q} \text{ is a principal convergent of } x.$$

Farey map

$$F(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1-x}{x}, & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$



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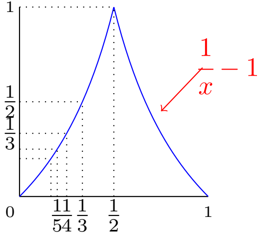
$$\cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}} \mapsto \cfrac{1}{d_1 - 1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}}$$



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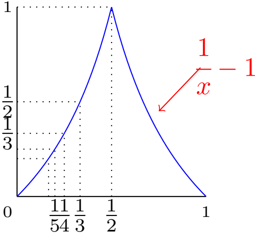

$$\cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}} \mapsto \cfrac{1}{d_1 - 1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}}$$

$$\cfrac{1}{1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}} \mapsto \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}$$

$F^{d_1(x)}(x) = G(x)$, $\quad d_1(x) - 1$: the first hitting time of x to $[\frac{1}{2}, 1]$

Jump Transformation

Let $T : [0,1] \to [0,1]$ be a map and E be a subinterval of $[0,1]$.

The first hitting time of x to E:

$$n_E(x) := \min\{i \geq 0 : T^i(x) \in E\}$$

Definition: Jump Transformation

We call a map $J : [0,1] \to [0,1]$ the *jump transformation* associated to T w.r.t. E if

$$J(x) = T^{n_E(x)+1}(x).$$

The Gauss map G is the jump transformation associated to F w.r.t. $[\frac{1}{2}, 1]$.

2. Even-Integer Continued Fraction

Even-Integer Continued Fractions (EICF)

For $x \in (0,1]$,

$$x = \cfrac{1}{2k_1 + \cfrac{\eta_1}{2k_2 + \cfrac{\eta_2}{2k_3 + \ddots}}}$$

where $k_n \in \mathbb{N}$, $\eta_n \in \{1,-1\}$.

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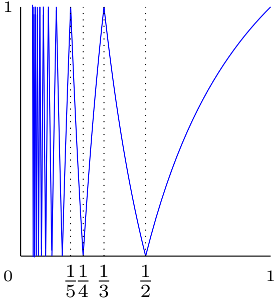


Figure: Graph of T_{EICF}

$$T_{\mathsf{EICF}}(x) = \left| \frac{1}{x} - 2k \right|, \text{ where } 2k \text{ is the nearest even integer of } \frac{1}{x}$$

$$\cfrac{1}{2k_1 + \cfrac{\eta_1}{2k_2 + \ddots}} \xrightarrow{T_{\mathsf{EICF}}} \cfrac{1}{2k_2 + \cfrac{\eta_2}{2k_3 + \ddots}}$$

EICF

$$\Theta = \left\{ \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \text{ or } \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \right\}$$

∞-rationals: $\Theta(\infty) = \{\text{even/odd or odd/even}\}$,

1-rationals: $\Theta(1) = \{\text{odd/odd}\}$

EICF principal convergent: $(p_0^E/q_0^E = 0/1$ and $p_1^E/q_1^E = 1/2k_1)$

$$\frac{p_n^E}{q_n^E} = \cfrac{1}{2k_1 + \cfrac{\eta_1}{2k_2 + \cfrac{\eta_2}{\ddots + \cfrac{\eta_{n-1}}{2k_n}}}}$$

$$p_n^E = 2k_n p_{n-1}^E + \eta_n p_{n-2}^E \quad \text{and} \quad q_n^E = 2k_n q_{n-1}^E + \eta_n q_{n-2}^E$$
$$p_n^E/q_n^E \in \Theta(\infty)$$

Best ∞-rational approximations

Definition An ∞-rational p/q is *a best ∞-**rational** approximation of x* if

$$|qx - p| < |bx - a| \quad \text{for any } \infty\text{-}\textbf{rational } a/b \neq p/q \text{ s.t. } 0 < b \leq q.$$

Theorem [Short-Walker, 2014]

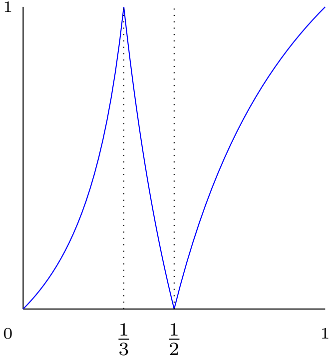
For $x \notin \mathbb{Q}$,

p/q is a best ∞-rational approximation of x

$\iff p/q$ is an EICF principal convergent of x.

Romik map

$$R(x) = \begin{cases} \frac{x}{1-2x}, & 0 \le x \le \frac{1}{3}, \\ \frac{1}{x} - 2, & \frac{1}{3} \le x \le \frac{1}{2}, \\ 2 - \frac{1}{x}, & \frac{1}{2} \le x \le 1. \end{cases}$$



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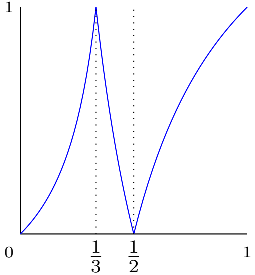

$$\cfrac{1}{2k_1 + \cfrac{\varepsilon_1}{2k_2 + \cfrac{\varepsilon_2}{2k_3 + \cdots}}} \mapsto \cfrac{1}{2k_1 - 2 + \cfrac{\varepsilon_1}{2k_2 + \cfrac{\varepsilon_2}{2k_3 + \cdots}}}$$

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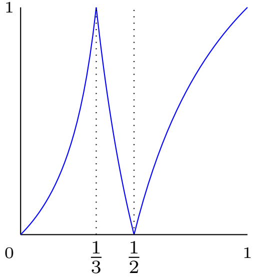

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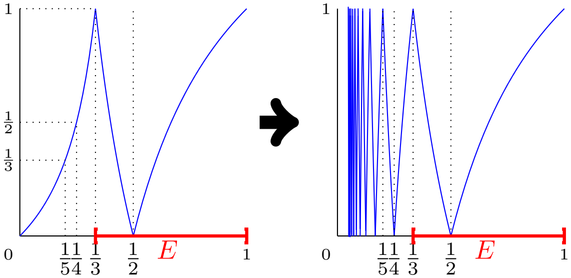
$$\cfrac{1}{2 + \cfrac{\varepsilon_1}{2k_2 + \cfrac{\varepsilon_2}{2k_3 + \cdots}}} \mapsto \cfrac{1}{2k_2 + \cfrac{\varepsilon_2}{2k_3 + \cdots}}$$

Romik $\to$ EICF

T_{EICF} is the jump transformation associated to R w.r.t. $E = \left[\frac{1}{3}, 1\right]$.



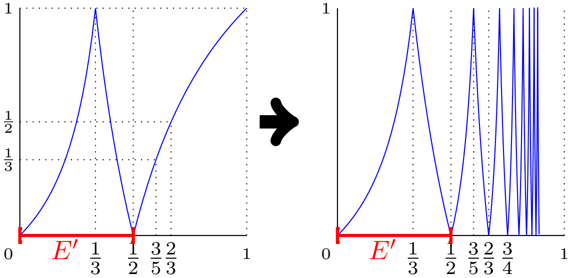
3. Odd-Odd Continued Fraction

Odd-Odd Continued Fraction (OOCF)

Let $E' = [0, \frac{1}{2}]$.

The jump transformation associated to R w.r.t. E':

$$T_{\mathsf{OOCF}}(x) = \begin{cases} \frac{kx-(k-1)}{k-(k+1)x}, & x \in [\frac{k-1}{k}, \frac{2k-1}{2k+1}], \\ \frac{k-(k+1)x}{kx-(k-1)}, & x \in [\frac{2k-1}{2k+1}, \frac{k}{k+1}], \end{cases} \quad \text{and} \quad T_{\mathsf{OOCF}}(1) = 1.$$



Invariant measure of T_{OOCF}

Define $f : [0, 1] \to [0, 1]$ by $f(x) := \frac{1-x}{1+x}$.

$$f \circ T_{\mathsf{OOCF}} \circ f^{-1} = T_{\mathsf{EICF}}.$$

Theorem [Schweiger, 1982]

T_{EICF} admits an ergodic absolutely continuous invariant measure $\dfrac{dx}{1-x^2}$.

Denote by $y = f(x)$. We have

$$\frac{dx}{1-x^2} = \frac{(1+x)^2 dy}{2(1-x^2)} = \frac{(1+x)dy}{2(1-x)} = \frac{dy}{2y}.$$

Thus, T_{OOCF} preserves an infinite ergodic absolutely continuous invariant measure $\dfrac{dx}{x}$.

OOCF

Let $T := T_{\mathsf{OOCF}}$. $T(x) = \begin{cases} \frac{kx-(k-1)}{k-(k+1)x}, & x \in [\frac{k-1}{k}, \frac{2k-1}{2k+1}], \\ \frac{k-(k+1)x}{kx-(k-1)}, & x \in [\frac{2k-1}{2k+1}, \frac{k}{k+1}], \end{cases}$ and $T(1) = 1$.

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$$\frac{1}{1-x} = \begin{cases} (k+1) - \frac{1}{2-(1-T(x))}, & x \in [\frac{k-1}{k}, \frac{2k-1}{2k+1}] \\ k + \frac{1}{2-(1-T(x))}, & x \in [\frac{2k-1}{2k+1}, \frac{k}{k+1}] \end{cases}$$

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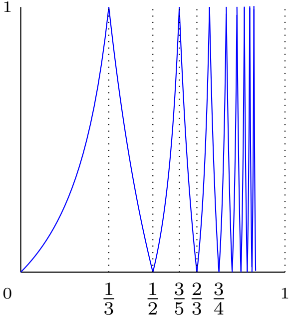
A corresponding continued fraction form is

$$x = 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{2 - \cfrac{1}{\boldsymbol{a_2} + \cfrac{\boldsymbol{\varepsilon_2}}{2 - \ddots}}}} = [\![(a_1, \varepsilon_1), (a_2, \varepsilon_2), \cdots\cdots]\!],$$

where $a_n \in \mathbb{N}$, $\varepsilon_n = \pm 1$,$(a_n, \varepsilon_n) \neq (1, -1)$. ($(a_n, \varepsilon_n)$; partial quotient)

Partial quotients

Define $B(a, \varepsilon)$ by $B(k+1, \text{-}1) := [\frac{k-1}{k}, \frac{2k-1}{2k+1}]$ and $B(k, 1) := [\frac{2k-1}{2k+1}, \frac{k}{k+1}]$.



$$(a_n, \varepsilon_n) = \begin{cases} (k+1, \text{-}1), & \text{if } T^{n-1}(x) \in B(k+1, \text{-}1), \\ (k, 1), & \text{if } T^{n-1}(x) \in B(k, 1). \end{cases}$$

The procedure terminates when $T^n(x) = 1$.

Prop. Every irrational has a unique infinite OOCF expansion.

Principal Convergents

A principal convergent: $\left(\frac{p_{-1}}{q_{-1}} = \frac{-1}{1}, \frac{p_0}{q_0} = \frac{1}{1} \text{ and } \frac{p_1}{q_1} = \frac{2a_1+\varepsilon_1-2}{2a_1+\varepsilon_1}\right)$

$$\frac{p_n}{q_n} = [\![(a_1,\varepsilon_1),(a_2,\varepsilon_2),\cdots,(a_n,\varepsilon_n)]\!] := 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{\ddots + \cfrac{\ddots}{2 - \cfrac{1}{\boldsymbol{a_n} + \cfrac{\boldsymbol{\varepsilon_n}}{2}}}}}$$

$$\begin{cases} p_n = (2a_n + \varepsilon_n - 1)p_{n-1} + \varepsilon_{n-1}p_{n-2}, \\ q_n = (2a_n + \varepsilon_n - 1)q_{n-1} + \varepsilon_{n-1}q_{n-2}. \end{cases}$$

Then, we have $p_n/q_n \in \Theta(1)$.

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Then, we have $p_n/q_n \in \Theta(1)$.

A sub-convergent : $\displaystyle \frac{p'_n}{q'_n} := 1 - \cfrac{1}{\boldsymbol{a_1} + \cfrac{\boldsymbol{\varepsilon_1}}{2 - \cfrac{1}{\ddots \cfrac{\boldsymbol{\varepsilon_{n-1}}}{2 - \cfrac{1}{\boldsymbol{a_n}}}}}} \in \Theta(\infty)$

$\frac{p'_0}{q'_0} = \frac{1}{0}, \frac{p'_1}{q'_1} = \frac{a_1-1}{a_1}$

$$\begin{cases} p'_n = a_n p_{n-1} - p'_{n-1}, \\ q'_n = a_n q_{n-1} - q'_{n-1}. \end{cases}$$

$f_{(a_n,\varepsilon_n)} = (T|_{B(a_n,\varepsilon_n)})^{-1}$

$$f_{(a_n,\varepsilon_n)}(t) = 1 - \cfrac{1}{a_n + \cfrac{\varepsilon_n}{1+t}}.$$

$f_{(a_n,\varepsilon_n)}$ is a linear fractional map corresponding to

$$A_{(a_n,\varepsilon_n)} := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_n - 1 & a_n + \varepsilon_n - 1 \\ a_n & a_n + \varepsilon_n \end{pmatrix}$$

$$\mathrm{det} A_{(a_n,\varepsilon_n)} = \pm 1$$

$$A_{(a_1,\varepsilon_1)} A_{(a_2,\varepsilon_2)} \cdots A_{(a_n,\varepsilon_n)} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_n' \\ q_n & q_n' \end{pmatrix}$$

$$A_{(a_1,\varepsilon_1)} A_{(a_2,\varepsilon_2)} \cdots A_{(a_n,\varepsilon_n)} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n' \\ q_{n-1} & q_n' \end{pmatrix}$$

OOCF of 1-rationals

For $\frac{r}{s} \in \Theta(1)$, $\exists n$ s.t. $T^n(\frac{r}{s}) = 1$.

Prop. Each 1-rational has exactly two finite OOCF expansions which differ only in the last partial quotient:

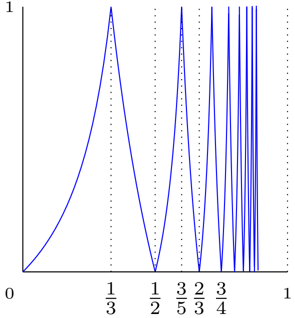
$$\begin{cases} [\![(a_1, \varepsilon_1), \cdots, (a_{n-1}, \varepsilon_{n-1}), (k+1, -1)]\!] \\ [\![(a_1, \varepsilon_1), \cdots, (a_{n-1}, \varepsilon_{n-1}), (k, 1)]\!]. \end{cases}$$

e.g.

$$\frac{1}{3} = 1 - \cfrac{1}{2 + \cfrac{-1}{2}} = 1 - \cfrac{1}{1 + \cfrac{1}{2}}$$

$$T\left(\frac{1}{5}\right) = \frac{1}{3},$$

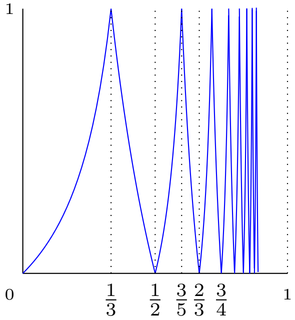
$$\frac{1}{5} = [\![(2,-1),(2,-1)]\!] = [\![(2,-1),(1,1)]\!]$$



OOCF of ∞-rationals

For $\frac{r}{s} \in \Theta(\infty)$, $\exists n$ s.t. $T^n\left(\frac{r}{s}\right) = 0$.

$$0 = 1 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{\ddots}}}}$$



Prop. Any non-zero ∞-rational has exactly two infinite OOCF expansions ending with $(2, -1)^\infty$.

e.g. $$\frac{1}{2} = 1 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - (1 - 0)}} = 1 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{2} + \cfrac{\mathbf{-1}}{2 - \cfrac{1}{\mathbf{2} + \cdots}}}}} = [\![(1,1),(2,\text{-}1)^\infty]\!]$$

$$= 1 - \cfrac{1}{\mathbf{3} + \cfrac{\mathbf{-1}}{2 - (1 - 0)}} = [\![(3,\text{-}1),(2,\text{-}1)^\infty]\!]$$

Finite and Periodic OOCFs

Theorem 1 [Dong Han Kim - L. - Lingmin Liao]

1. finite OOCF $\Longleftrightarrow$ 1-rational
2. periodic OOCF $\Longleftrightarrow$ quad. irr. or ∞-rational

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2. periodic OOCF $\Longleftrightarrow$ quad. irr. or ∞-rational

CF	RCF	EICF	OOCF
finite	rationals	∞-rationals [1] $\left(\frac{\text{even}}{\text{odd}} \text{ or } \frac{\text{odd}}{\text{even}}\right)$	1-rationals
periodic	quad. irr.	1-rationals $\left(\frac{\text{odd}}{\text{odd}}\right)$ [1] or quad. irr [2]	∞-rationals or quad. irr.

[1] [Short-Walker, 2014]

[2] [Boca-Merriman, 2018]

Finite and Periodic OOCFs

Theorem 1 [Dong Han Kim - L. - Lingmin Liao]

1. finite OOCF $\Longleftrightarrow$ 1-rational
2. periodic OOCF $\Longleftrightarrow$ quad. irr. or ∞-rational

$\zeta_i = \zeta_i(x) := T^i(x)$.

$$x = \frac{(p_i - p_i') + p_i'\zeta_{i+1}}{(q_i - q_i') + q_i'\zeta_{i+1}} \quad \rightarrow \quad \zeta_{i+1} = -\frac{(q_i - q_i')x - (p_i - p_i')}{q_i' x - p_i'}$$

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$(\Rightarrow)$ x has a periodic OOCF $\Rightarrow \exists\, i < j$ s.t. $\zeta_i(x) = \zeta_j(x)$.

$(\Leftarrow)$ $a_1x^2 + b_1x + c_1 = 0 \Rightarrow a_i\zeta_i^2 + b_i\zeta_i + c_i = 0$

$\{a_i, b_i, c_i \text{ for } i \geq 1\} \subset \mathbb{Z}$ are bounded. Then, $\exists\, i < j$ s.t. $\zeta_i = \zeta_j$.

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e.g. $$\frac{\sqrt{5}-1}{2} = [\![(1,1)^\infty]\!] = 1 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - \cfrac{1}{\mathbf{1} + \cfrac{\mathbf{1}}{2 - \ddots}}}}$$

Best 1-rational approximations

Definition $p/q \in \Theta(1)$ is *a best 1-**rational** approximation of x* if

$$|qx - p| < |bx - a| \quad \text{for any } 1\text{-}\textbf{rational } a/b \neq p/q \text{ s.t. } 0 < b \leq q.$$

Theorem [Dong Han Kim - L. - Lingmin Liao]

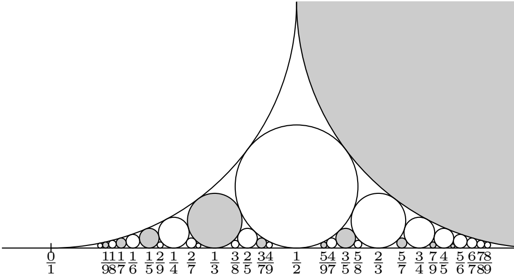
For $x \notin \mathbb{Q}$,

best 1-rational approximation $\Longleftrightarrow$ OOCF principal convergent.

Ford Circles

Def. *A Ford circle $C_{\frac{a}{b}}$:*

A horocycle based at $\frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$ with Euclidean radius $\frac{1}{2b^2}$.



Fact. $C_{p/q}$ is tangent to $C_{p'/q'}$

$$\Leftrightarrow \left| \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \right| = 1$$

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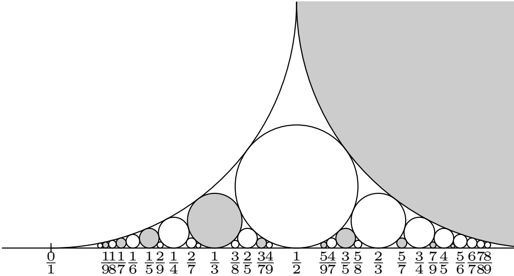
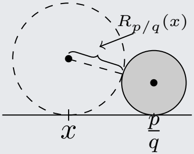


Fact. $C_{p/q}$ is tangent to $C_{p'/q'}$

$$\Leftrightarrow \left|\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\right| = 1$$

$R_{p/q}(x)$: the Euclidean radius of the horocycle based at x tangent to $C_{p/q}$

$$R_{p/q}(x) = \frac{1}{2}|qx - p|^2$$



$$|qx - p| < |bx - a| \iff R_{p/q}(x) < R_{a/b}(x)$$

Theorem 2. [Dong Han Kim - L. - Lingmin Liao]

Best 1-rational approximation $\Longleftrightarrow$ OOCF principal convergent

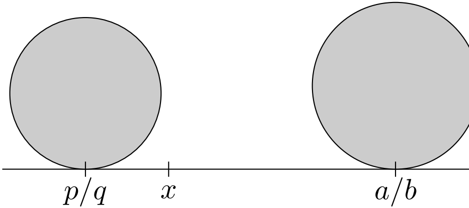
Sketch of proof

$(\Leftarrow)$

- $x \notin \mathbb{Q}$
- p/q : an OOCF principal convergent of x
- a/b : a 1-rational s.t. $b \leq q$
- ETS: $R_{p/q}(x) < R_{a/b}(x)$

$(\Rightarrow)$

- $a/b \in \Theta(1)$: not an OOCF principal convergent of x
- $\exists\ n$ s.t. $q_n \leq b < q_{n+1}$
- C (dashed arc) : a horocycle based at x tangent to $C_{a/b}$
- $R_{a/b}(x) > R_{p_n/q_n}(x)$



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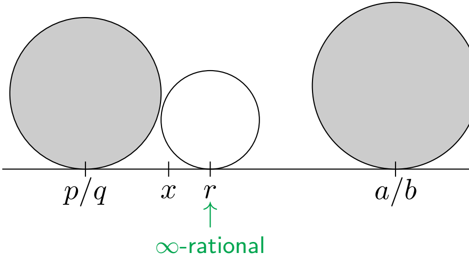

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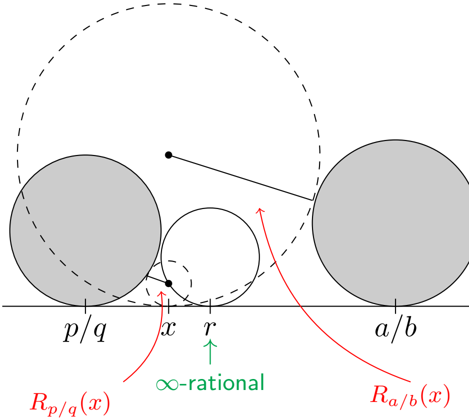

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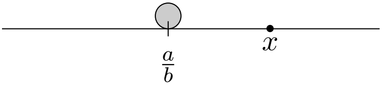

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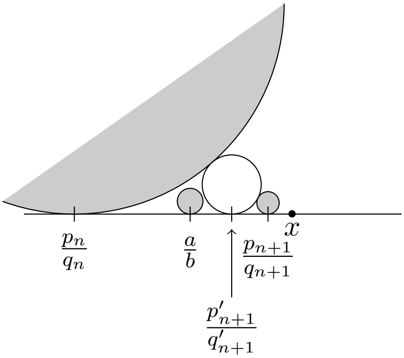

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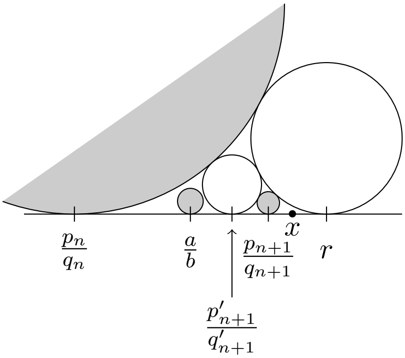

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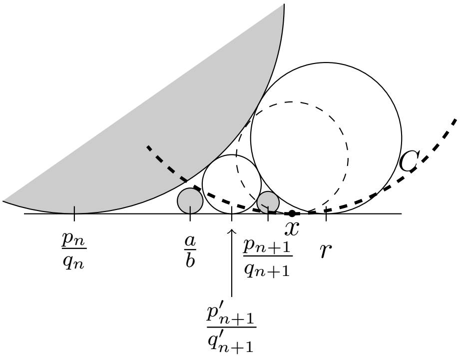

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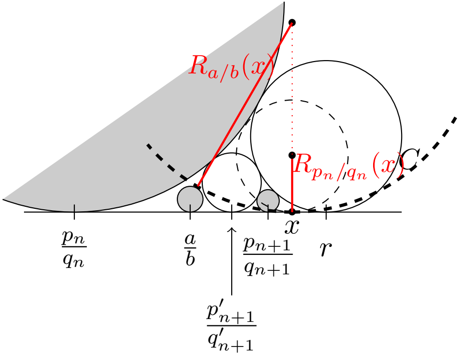

4. Relation between RCF and OOCF

Relation between RCF and OOCF

$$(T|_{B(a_n,\varepsilon_n)})^{-1}(x) = f_{(a_n,\varepsilon_n)}(x) = 1 - \cfrac{1}{a_n + \cfrac{\varepsilon_n}{1+x}}$$

$$x = [0; d_1, d_2, \cdots, d_j, \cdots] \in [0,1]$$

Lemma

The RCF expansion of $f_{(a,\varepsilon)}(x)$ is as follows:

$$f_{(a,\varepsilon)}(x) = \begin{cases} [0; 2, d_1, d_2, \cdots] & \text{if } \varepsilon = 1,\ a = 1, \\ [0; 1, (a-1), 1, d_1, d_2, \cdots] & \text{if } \varepsilon = 1,\ a \geq 2, \\ [0; (d_1+2), d_2, \cdots] & \text{if } \varepsilon = -1,\ a = 2, \\ [0; 1, (a-1), (d_1+1), d_2, \cdots] & \text{if } \varepsilon = -1,\ a \geq 3. \end{cases}$$

Relation between RCF and OOCF

The intermediate convergents are the fractions

$$\frac{p^R_{n,j}}{q^R_{n,j}} := \frac{p^R_{n-2} + j p^R_{n-1}}{q^R_{n-2} + j q^R_{n-1}} \quad \text{for } 0 \le j \le d_n,\ n \ge 1,$$

where p^R_n/q^R_n is a principal convergent.

Theorem [Kim-L.-Liao]

The OOCF principal convergents of x are intermediate convergents of x.

Let $x = [0; d_1, d_2, \cdots] = [\![(a_1, \varepsilon_1), (a_2, \varepsilon_2), \cdots]\!]$.

$g = f_{(a_1,\varepsilon_1)} \circ f_{(a_2,\varepsilon_2)} \circ \cdots \circ f_{(a_k,\varepsilon_k)}$

$x = g(T^k(x))$ and $p_k/q_k = g(1)$, $(1 = [0; 1])$

x and p_k/q_k have the same prefix in their RCF expansions, except for the last partial quotient of p_k/q_k (by Lemma)

$\Rightarrow$ p_k/q_k is an intermediate convergent of x.

Relation between RCF and OOCF

If $p_n^R/q_n^R \in \Theta(1)$, then p_n^R/q_n^R is a best 1-rational approximation.

$\Rightarrow p_n^R/q_n^R$ is an OOCF convergent.

[Keita, 2017] (1) $q_{n,0}^R = q_{n-2}^R < q_{n-1}^R \le q_{n,1}^R < \cdots < q_{n,d_n}^R = q_n^R$,

(2) $$|q_{n,d_n}^R x - p_{n,d_n}^R| = |q_n^R x - p_n^R| < |q_{n-1}^R x - p_{n-1}^R| \le |q_{n,d_n-1}^R x - p_{n,d_n-1}^R| < \cdots |q_{n,1}^R x - p_{n,1}^R| < |q_{n,0}^R x - p_{n,0}^R| = |q_{n-2}^R x - p_{n-2}^R|.$$

- If $p_{n-1}^R/q_{n-1}^R \in \Theta(1)$, then $p_{n,j}^R/q_{n,j}^R$ is not an OOCF principal convergent for any $j = 1, \cdots, d_n - 1$.
- If $p_{n-1}^R/q_{n-1}^R \in \Theta(\infty)$, then $p_{n,j}^R/q_{n,j}^R$ is an OOCF principal convergent whenever $p_{n,j}^R/q_{n,j}^R \in \Theta(1)$

e.g.

$$x = [0; 1, 2, 3, 4, 3, 7, 15, 1, 292, \cdots]$$
$$= [\![(0,\text{-}1), (4,\text{-}1), (1,1), (2,\text{-}1), (1,1), (2,\text{-}1), (9,\text{-}1), (2,\text{-}1), (2,\text{-}1), (2,\text{-}1), \cdots]\!]$$

RCF Principal convergents (red: 1-rationals, green: ∞-rationals)

$\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{30}{43}, \frac{97}{139}, \frac{709}{1016}, \frac{10732}{15379}, \frac{11441}{16395}$

Intermediate convergents

$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \frac{16}{23}, \frac{23}{33}, \frac{30}{43}, \frac{37}{53}, \frac{67}{96}, \frac{97}{139}, \frac{127}{182}, \frac{224}{321}, \frac{321}{460}, \frac{418}{599}, \frac{515}{738}, \frac{61}{87}$

EICF

$\frac{1}{2}, \frac{2}{3}, \frac{7}{10}, \frac{30}{43}, \frac{127}{182}, \frac{224}{321}, \frac{321}{460}, \frac{418}{599}, \frac{515}{738}, \frac{6}{8}$

OOCF

$\frac{1}{1}, \frac{5}{7}, \frac{9}{13}, \frac{23}{33}, \frac{37}{53}, \frac{97}{139},$

$\frac{1515}{2171}, \frac{2933}{4203}, \frac{4351}{6235}, \frac{5769}{8267}, \frac{7187}{10299}, \frac{8605}{12331}, \frac{10023}{14363}, \frac{11441}{16395}, \ldots$

Thank you for your attention!

$$[0; d_1, d_2, \tau] = \begin{cases} [\![(2,-1)^{\frac{d_1-1}{2}}, (d_2+1, 1), F(\tau)]\!] & \text{if } d_1 \text{ is odd and } \tau \in [\frac{1}{2}, 1), \\ [\![(2,-1)^{\frac{d_1-1}{2}}, (d_2+2, -1), F(\tau)]\!] & \text{if } d_1 \text{ is odd and } \tau \in [0, \frac{1}{2}), \\ [\![(2,-1)^{\frac{d_1}{2}-1}, (1, 1), G(x)]\!] & \text{if } d_1 \text{ is even.} \end{cases}$$