Dynamics of Ostrowski skew-product

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Intro

- For an irrational $x \in [0,1]$, the expression

$$x = [0; a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

will denote the continued fraction expansion of x.

- The integers $a_i \geq 1$ is uniquely given by $a_i = \lfloor \frac{1}{T^{i-1}(x)} \rfloor$ where T is the Gauss map,

$$T : x \longmapsto \left\{ \frac{1}{x} \right\}.$$

- Then continued fraction expansions can be viewed as trajectories of a one-dimensional system $([0,1], T)$.

Intro

We consider a simple skew-product extension of the Gauss map T:

$$S : (x, y) \longmapsto (\{1/x\}, \{y/x\})$$

which is called the Ostrowski map.

- The system $([0,1]^2, S)$ explicitly determines the expression of real numbers in terms of the convergents of a given irrational.
- Study statistical properties of the digits.
- Estimate the size of a bounded-type (digit) fractal set.

Set up: Ostrowski numeration

- Set $(x, y) := (x_0, y_0)$ and $(x_i, y_i) = S^i(x, y)$ for $i \geq 1$. This defines a sequence of integers

$$(a_i, b_i) = \left(\left\lfloor \frac{1}{x_{i-1}} \right\rfloor, \left\lfloor \frac{y_{i-1}}{x_{i-1}} \right\rfloor \right).$$

- Set $\theta_i := q_i x - p_i$, where $p_i/q_i = [0; a_1, \ldots, a_i]$ denotes the i-th convergent of x.
- Then $b_i (= b_i(x, y))$ yields the digits for the Ostrowski expansion for a real y in an irrational base x.

Ostrowski

Let $x \in [0, 1)$ be an irrational. Every real $y \in [0, 1)$ can be written uniquely in the form

$$y = \sum_{i \geq 1} b_i |\theta_{i-1}|$$

with $0 \leq b_i \leq a_i$ and $b_{i+1} = 0$ if $a_i = b_i$ for some i.

Facet: I. Inhomogeneous approximation

- Ostrowski expansion is used to approximate y modulo 1 by numbers of the form Nx with $N \in \mathbb{N}$.
- Indeed, setting

$$M_n := \sum_{i=1}^{n} (-1)^{i-1} b_i q_{i-1},$$

  the sequence (M_n) yields the inhomogeneous approximation of y modulo 1 by all translates $M_n x$ of x.
- This is the best left approximation in the sense that

$$0 < |M_{n+1}x - y| < |M_n x - y|.$$

Facet: II. Real quadratic field

Let $x = \sqrt{D}$ be a quadratic irrational with $D > 0$.

- Arithmetic of number field $\mathbb{Q}(x)$ has been studied via the characterisation of x: there exist $\ell \geq 1$ and $\{a_1, \cdots, a_\ell\}$ such that

$$x = [0; \overline{a_1, \ldots, a_\ell}],$$

  i.e., it admits purely periodic continued fraction expansion.
- For instance, this yields the solvability of Pell's equation $X^2 - DY^2 = 1$ and (partial) progress for class number one problem.
- More recently, McMullen and Duke-Imamoglu-Toth study arithmetic invariants using geometric realisation. Certain Diophantine conditions are strongly involved.

Facet: II. Real quadratic field

Hara–Ito

Let x be a quadratic irrational. Then $y \in \mathbb{Q}(x)$ if and only if

$$y = \sum_{i \geq 1} b_i |\theta_{i-1}|$$

with (b_i) periodic, i.e., $\{b_i\} = \{b_0; b_1, \ldots, b_k, \overline{b_{k+1}, \ldots, b_{k+\ell}}\}$.

- Ostrowski expansion gives a much stronger characterisation for elements of the real quadratic field.
- Thus, we expect deep interactions of fractal analysis arising from Diophantine nature of Ostrowski system in arithmetic.

Recall: Homogeneous setup

Dirichlet

For irrational $x \in [0,1)$, there exist infinitely many rationals p/q with $q > 0$ satisfying
$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

- Homogeneous best rational approximations p/q are given by the convergents of continued fraction expansion of x.
- We have: x is badly approximable if and only if digits of continued fraction expansion of x are bounded.
- This defines a fractal set in the unit interval: For $N \geq 1$,
$$E_N = \{x = [0; a_1, a_2, \ldots] : a_i \leq N\}$$
which has zero Lebesgue measure.
- Hausdorff dimension $dim_H(E_N) \in [0,1]$ quantifies the size of such fractal sets.

Recall: Homogeneous setup

Powerful approach to $dim_H(E_N)$ is provided by the thermodynamic formalism using transfer operator/dynamical determinant.

- Family of linear operators $\mathcal{L}_s$ defined by

$$\mathcal{L}_s f(x) = \sum_{a \leq N} \frac{1}{(a+x)^{2s}} f\left(\frac{1}{a+x}\right).$$

- Under a suitable choice of function space, $\mathcal{L}_s$ is compact operator and of trace class with the leading eigenvalue λ_s.
- Then we have the following implicit characterisation for the Hausdorff dimension.

Bowen-Ruelle

We have $dim_H(E_N) = s$ if and only if s is the unique solution to $\lambda_s = 1$. Equivalently, the unique zero of $det(I - \mathcal{L}_s)$.

Recall: Homogeneous setup

- Hensley: $dim_H(E_N) = 1 - \frac{6}{\pi^2}\frac{1}{N} - \frac{72}{\pi^4}\frac{\log N}{N^2} + O\left(\frac{1}{N^2}\right)$, later improved with the better error rates.
- Jenkinson-Pollicott: Rigorous estimate $dim_H(E_N)$ accurate over 100 decimal places using trace formula/resonances.
- In particular, they have *$dim_H(E_5) > \frac{5}{6}$ which implies

$$\lim_{M\to\infty} \frac{1}{M} \# \left\{ 1 \leq q \leq M : \exists p;\ \frac{p}{q} = [0; a_1, \ldots, a_\ell] \text{ with } a_i \leq 5 \right\} = 1.$$

  This is well-known density one result for Zaremba conjecture due to Bourgain-Kontorovich, Huang. Their argument was conditional on the fact (*).
- However, no analogous progress for the corresponding problems in inhomogeneous setup.

Result

- Recall that $M_n = \sum_{i=1}^{n} (-1)^{i-1} b_i q_{i-1}$ defines a sequence $M_n x$ of approximants converges to y.
- To study badly approximable numbers with respect to the Ostrowski expansion, consider a bounded-type fractal set: For $N \geq 1$

$$F_N = \{(x, y) \in [0, 1)^2 : \ ^* a_i \leq N \text{ and } a_i - 1 \leq b_i < a_i\}.$$

- Further we associate the quantities, e.g.,

$$D_n = \#\{1 \leq i \leq n : (a_i, b_i) \text{ satisfying } (*)\}$$

which can be viewed as a Birkhoff sum with an observable.

Berthé-L

There exists an absolutely continuous invariant measure for $([0, 1]^2, S)$ that mixes exponentially fast. Accordingly, we have

- Central Limit Theorem for Birkhoff sums.
- Bowen-Ruelle type formula for Hausdorff dimensions.

Transfer operator: two partitions

- For $(x, y) \in [0, 1)^2$, there are unique integers a, b such that

$$\lfloor 1/x \rfloor = a \ \text{ and } \ \lfloor y/x \rfloor = b.$$

- Let $A = \{(a, b) : 1 \leq a,\ 0 \leq b \leq a\}$. For $(a, b) \in A$,

$$I_{a,b} = \left\{(x, y) : \frac{1}{a+1} \leq x < \frac{1}{a},\ bx \leq y < (b+1)x\right\}.$$

- Note that S is piecewise smooth, expanding map, i.e., $\{I_{a,b}\}$ forms a partition for $[0, 1]^2$ and the restriction $S|_{I_{a,b}}$ extends to an invertible map. Inverse branches are of the form:

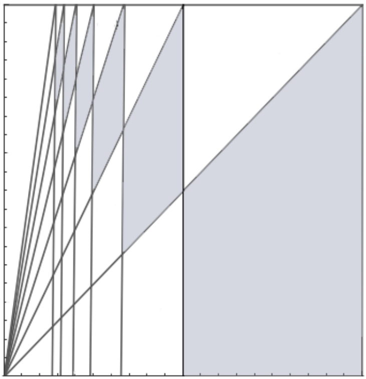
$$S|_{I_{a,b}}^{-1}(x, y) := h_{a,b}(x, y) = \left(\frac{1}{a+x}, \frac{b+y}{a+x}\right)$$

- Also consider the coarse partition Δ_0 and Δ_1 for unit square

$$\Delta_0 = \{y < x\} \text{ and } \Delta_1 = \{x \leq y\}.$$

Transfer operator: two partitions

More precisely, this explains the digit conditions of the Ostrowski expansion in an alternative way:



We have $S(I_{a,b}) = \Delta_0$ if $a = b$, and
$S(I_{a,b}) = [0,1]^2$ if $a \neq b$.

Transfer operator

Let ϕ be an observable and s, t be real parameters.

- Define the weighted transfer operator by

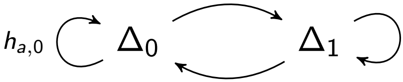
$$\mathcal{L}_{s,t} f(x,y) = \sum_{S(u,v)=(x,y)} |J_S(u,v)|^s e^{it\phi(u,v)} \cdot f(u,v)$$
$$= \sum_{(a,b)\in A} \frac{e^{it\phi(\frac{1}{a+x}, \frac{b+y}{a+x})}}{(a+x)^{3s}} \cdot f\left(\frac{1}{a+x}, \frac{b+y}{a+x}\right) \cdot 1_{SI_{a,b}}(x,y).$$

  Here, Jacobian determinant is uniform in regard to skew part.
- Point is to take a suitable function space on which the transfer operator acts compactly, thus has an interesting spectrum.
- Problem is that we have discontinuities of $\mathcal{L}_{s,t}$ are at boundaries of partition Δ_0, Δ_1.

Modifications

Extend and adapt combinatorial ideas due to D. Mayer:

- Two partitions $I_{a,b}$ and Δ_i for $([0,1]^2, S)$ satisfy certain admissibilities under the action of $h_{a,b}$.
- More precisely, we have a strongly connected graph



- Set a transition matrix: For $(a,b) \in A$ and $i,j \in \{0,1\}$,

$$A_{i,j}^{(a,b)} = \begin{cases} 1 & \text{if } h_{a,b}(\Delta_i) \subseteq \Delta_j \cap I_{a,b} \\ 0 & \text{if } h_{a,b}(\Delta_i) \subseteq \mathbb{R}^2 \backslash \Delta_j. \end{cases}$$

- Finally define a generalised operator: for each $i \in \{0,1\}$,

$$\widetilde{\mathcal{L}}_{s,t} F(i,x,y) = \sum_{(a,b)\in A} A_{i,\tau_{a,b}(i)}^{(a,b)} \cdot - \cdot F(\tau_{a,b}(i), h_{a,b}(x,y))$$

with $\tau_{a,b}(i) = j$ if $A_{i,j}^{(a,b)} = 1$ for some j, and i otherwise.

Modifications

- Say we have $\widetilde{B}$ on which $\widetilde{\mathcal{L}}$ has spectral gap with leading eigenvalue λ. Write Φ for the corresponding eigenfunction.
- Specialisation $(\kappa F)(x, y) := F(i, x, y)$ if $(x, y) \in \Delta_i$. Then the diagram commutes, and $\kappa\Phi$ is an eigenfunction of $\mathcal{L}$ with the same eigenvalue λ.

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{\widetilde{\mathcal{L}}} & \widetilde{B} \\ \kappa \downarrow & & \downarrow \kappa \\ L^1 & \xrightarrow{\mathcal{L}} & L^1 \end{array}$$

- Hence take a suitable function space on which $\widetilde{\mathcal{L}}_{s,t}$ admits good spectral properties.

Spectrum: Compactness

- Contraction: There is a bounded domain $\Omega \subseteq \mathbb{C}^2$ that contains $[0,1]^2$ and $h_{a,b}(\Omega) \subseteq \Omega$.
- Then we have the Banach space $\widetilde{B}(\Omega)$ of locally holomorphic functions (partition Δ_i) on which $\widetilde{\mathcal{L}}_{s,t}$ acts boundedly.

Spectral gap

For s, t close to 1,0

- The operator $\widetilde{\mathcal{L}}_{s,t}$ on $\widetilde{B}(\Omega)$ is compact and of trace class.
- There is a simple eigenvalue $\lambda_{s,t}$ whose modulus is strictly larger than all other eigenvalues.
- The corresponding eigenfunction $\Phi_{s,t}$ is positive.

- This yields an invariant measure for $([0,1]^2, S)$ that mixes exponentially fast, hence CLT basically via the identity:

$$\mathbb{E}[e^{itS_n\phi}] = \int \mathcal{L}^n_{1,it}1 = \int \kappa(\widetilde{\mathcal{L}}^n_{1,it}1).$$

Dimension estimate

- Problem: Ostrowski system $([0,1]^2, S)$ is non-conformal.
- However, we have: Jacobian determinant of $h_{a,b}$

$$|J_{h_{a,b}}(x,y)| = \left(\frac{1}{q_n + xq_{n-1}}\right)^3$$

  is uniform with respect to the skew coordinate.
  Thus it further satisfies bounded distortion property.
- This allows us to have explicit info on fundamental cylinders: For $n \geq 1$ and $(a,b) \in A^n$, the images $I_{a,b} = h_{a,b}([0,1]^2)$ of depth n, which form a (canonical) cover for F_N.
- Diameter and height in $1/q_n$, and measure in $1/{q_n}^3$.

Dimension estimate

- Let $A_N \subseteq A$ be set of (a, b) satisfying $a \leq N$ and $b = a - 1$.
- Then consider the constrained operator$\mathcal{L}_{N,s} := \mathcal{L}_{N,s,0}$ whose sum is over A_N. Write $\lambda_{N,s}$ for the dominant eigenvalue.
- We can now control the diameter of $I_{a,b}$ with Jacobian:

$$\inf |J_{h_{a,b}}(x, y)| \leq diam(I_{a,b}) \leq P_{2n}(N) \cdot \sup |J_{h_{a,b}}(x, y)|$$

  where $P_{2n}(N)$ is a monomial in N of degree $2n$.

Bowen-Ruelle type formula

We have $s_1 \leq dim_H(E_N) \leq s_2$ for real variables s_1, s_2 satisfying $\lambda_{N,s_1} = 1$ and $N^2 \lambda_{N,s_2} = 1$.