Regularity Properties of the Brjuno Functions Associated with By-excess, Odd and Even Continued Fractions

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OWNS

Joint work with Stefano Marmi.

Regular Continued Fraction

Regular continued fraction of $x \in \mathbb{R}$

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}}}, \qquad a_0 \in \mathbb{Z},\ a_i \in \mathbb{N},\ i \geq 1.$$

Principal convergents

$$\frac{P_n(x)}{Q_n(x)} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}, \qquad n \in \mathbb{N}_0$$

$P_n/Q_n \to x$ as $n \to \infty$,

P_n/Q_n's are the best approximation of x.

Analytic Small Divisor Problem

Question: When is a holomorphic germ an analytic perturbation of its linear part?

$f(z) = \lambda z + O(z^2)$, where $\lambda = e^{2\pi i x}$, $x \in \mathbb{R}$.

Definition (linearizability)

f is *linearizable* if $\exists$ a holomorphic germ $H_f(z) = z + O(z^2)$ such that $H_f(0) = 0$, $H_f'(0) = 1$ and $H_f^{-1} \circ f \circ H_f(z) = \lambda z$. We call H_f a linearization of f.

[Siegel, 1942]
$\log Q_{n+1}(x) = O(\log Q_n(x)) \Longrightarrow f(z) = e^{2\pi i x} z + O(z^2)$ is linearizable.

[Brjuno 1965, Yoccoz 1988, 1995]

$$\sum_{n=1}^{\infty} \frac{\log Q_{n+1}(x)}{Q_n(x)} < \infty \Leftrightarrow$$ Every holomorphic germs $f(z) = e^{2\pi i x} z + O(z^2)$ is linearizable.

x is said to be a Brjuno number if $\sum_{n=1}^{\infty} \frac{\log Q_{n+1}(x)}{Q_n(x)} < \infty$

Brjuno Function

$\Phi : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ is an even and $\mathbb{Z}$-periodic function satisfying

$$\Phi(x) = \log \frac{1}{x} + x\Phi\left(\frac{1}{x}\right), \quad 0 < x < \frac{1}{2}.$$

$\Phi(x) < \infty \iff x$ is a Brjuno number.

$R_f :=$ the radius of convergence of the linearization H_f

$R(x) := \inf_{f \in S_x} R_f$,
$S_x = \{f(x) = e^{2\pi i x} z + O(z^2),$ univalent on the unit disk $\}$

[Yoccoz, 1988, 1995]

If $\Phi(x) = +\infty$, then $\exists$ non-linearizable germ $f \in S_x$.
If $\Phi(x) < \infty$, then $R(x) > 0$, and

$$|\log R(x) + \Phi(x)| < C, \text{ uniformly.}$$

Nearest-Integer CF

$x \in \mathbb{R}$

$$x = b_0 + \cfrac{\varepsilon_0}{b_1 + \cfrac{\varepsilon_1}{\ddots + \cfrac{\varepsilon_{n-1}}{b_n + \ddots}}},\ b_0 \in \mathbb{Z}, \varepsilon_i = \pm 1, b_i \in \mathbb{N}_{\geq 2},\ b_i + \varepsilon_i \geq 2$$

NICF Principal convergent $\frac{p_n(x)}{q_n(x)} = b_0 + \cfrac{\varepsilon_0}{b_1 + \cfrac{\varepsilon_1}{\ddots + \cfrac{\varepsilon_{n-1}}{b_n}}}$

NICF Gauss map $A_{1/2}(x) = \left\|\frac{1}{x}\right\|_{\mathbb{Z}} = \inf_{n\in\mathbb{Z}} \left|\frac{1}{x} - n\right|$

$x_0 = \|x\|_{\mathbb{Z}}$, $x_i = A^i_{1/2}(x_0)$

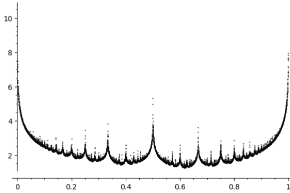
$\Phi(x) = \Phi(x_0) = \log(1/x_0) + x_0\Phi(1/x_0) = \log(1/x_0) + x_0\Phi(x_1) = \cdots$

$= \sum_{n\geq 0} x_0 x_1 \cdots x_{n-1} \log(1/x_n) = \sum_{n\geq 0} |q_{n-1}x_0 - p_{n-1}| \log \frac{|q_{n-1}x_0 - p_{n-1}|}{|q_n x_0 - p_n|}$

$\left|\Phi(x) - \sum_{n\geq 0} \frac{\log q_{n+1}(x)}{q_n(x)}\right| < C$, uniformly.

Brjuno Function

$B : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ is a $\mathbb{Z}$-periodic function satisfying



$$B(x) = \log \frac{1}{x} + xB\left(\frac{1}{x}\right), \quad 0 < x < 1.$$

Gauss map of the regular CF

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right], \; x \in (0, 1]$$

$x_0 = x - [x]$, $x_i = G^i(x_0)$,

$$B(x) = B(x_0) = \sum_{n=0}^{\infty} x_0 x_1 \cdots x_{n-1} \log(1/x_n)$$
$$= \sum_{n\geq 0} |Q_{n-1}x_0 - P_{n-1}| \log \frac{|Q_{n-1}x_0 - P_{n-1}|}{|Q_n x_0 - P_n|}$$

$$\left| B(x) - \sum_{n\geq 0} \frac{\log Q_{n+1}(x)}{Q_n(x)} \right| < C, \quad \text{uniformly}$$

$B(x) - \Phi(x) \in L^{\infty}$

$\Upsilon := \log R(x) + \Phi(x)$

Marmi, Moussa and Yoccoz conjectured that Υ is $1/2$-Hölder continuous.

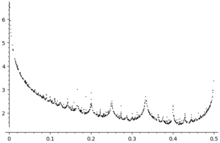
[Buff-Chéritat, 2006] Υ has a continuous extension to $\mathbb{R}$.

[Chéritat, 2008] For $\eta > \frac{1}{2}$, Υ cannot be η-Hölder continuous.

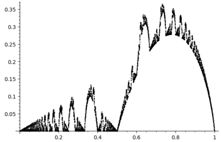
[Cheraghi-Chéritat, 2015] Υ is $\frac{1}{2}$-Hölder continuous on the high type numbers, i.e $\{x : b_n(x) \geq N \text{ for all } n\}$ for some N, where b_n is a partial quotient of NICF.

[Marmi-Moussa-Yoccoz, 1997]

$B - \Phi$ has a $1/2$-Hölder continuous extension to $\mathbb{R}$.



Φ



$\Phi - B$

Operator and Hölder continuity

$\Phi(x) = \log(1/x) + x\Phi(1/x)$ for $0 < x < 1/2$.

For $\nu \geq 0$, $T_\nu f(x) = x^\nu f(1/x)$,
$\{f :$ measurable, even, $\mathbb{Z}$-periodic function, $f|_{(0,\frac{1}{2})} \in L^p((0,\frac{1}{2}), d\mu)\}$

$(1 - T_1)\Phi(x) = \log(1/x) \in L^p((0,\frac{1}{2}), d\mu)$

The spectral radius of T_ν w.r.t. p-norm < 1 for $p \geq 1$.

Φ is a unique solution of the functional equation.

η-Hölder norm: $\|f\|_\eta = \sup|f| + \sup_{0\leq x<y\leq\frac{1}{2}} \frac{|f(x)-f(y)|}{|x-y|^\eta}$

$\mathcal{C}^\eta_{[0,1/2]} = \{f :$ continuous, $\|f\|_\eta < \infty\}$

[Marmi-Moussa-Yoccoz, 1997]

If $f \in \mathcal{C}^\eta_{[0,1/2]}$ for $\nu/2 < \eta \leq 1$, then $\sum_{m\geq 0} T^m_\nu f \in \mathcal{C}^{\nu/2}_{[0,1/2]}$

α-Continued Fractions

α-CF Gauss map, $\alpha \in [0,1]$,

$$A_\alpha(x) = \left| \frac{1}{x} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor \right|, \quad x \in (0, \overline{\alpha}],$$

where $\overline{\alpha} = \max\{\alpha, 1-\alpha\}$.

$$B_\alpha(x) = \sum_{n \geq 0} x_0 A_\alpha(x_0) \cdots A_\alpha^{n-1}(x_0) \log \frac{1}{A_\alpha^n(x_0)},$$

where $x_0 = |x - \lfloor x + 1 - \overline{\alpha} \rfloor|$.

Then $\Phi = B_{1/2}$, $B = B_1$.

[Marmi-Moussa-Yoccoz, 1997] $B - B_\alpha \in L^\infty$ for $\alpha \in [1/2, 1]$.

[Luzzi-Marmi-Nakada-Natsui, 2010]

(1) $B - B_\alpha \in L^\infty$ for $\alpha \in (0, 1/2)$.

(2) $B(x) - [B_0(x) + B_0(-x)] \in L^\infty$,

i.e. $B(x) < \infty$ iff $B_0(x) < \infty$ and $B_0(-x) < \infty$.

By-excess CF

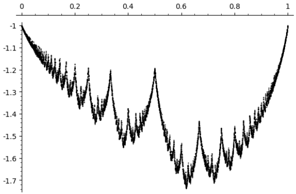


Figure: $B(x) - [B_0(x) + B_0(-x)]$

Theorem 1 [L.-Marmi]

$B(x) - [B_0(x) + B_0(-x)]$ has a $1/2$-Hölder continuous extension to $\mathbb{R}$.

$\Delta := B_{1/2}(x) - [B_0(x) + B_0(-x)]$ even, $\mathbb{Z}$-periodic,

$F(x) := \Delta(x) - x\Delta(1/x) \in \mathcal{C}^{1/2}_{[0,1/2]}$

$$= (1 - \lfloor 1/x \rfloor x) \log(1 - \lfloor 1/x \rfloor x) + x \log x + \sum_{k=1}^{\lfloor 1/x \rfloor - 1} x \log(1 - kx)$$

Even CF

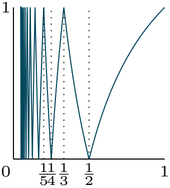
$x \in \mathbb{R}$,

$$x = d_0 + \cfrac{\eta_0}{d_1 + \cfrac{\eta_1}{d_2 + \ddots + \cfrac{\eta_{n-1}}{d_n + \ddots}}},$$ where $d_0 \in 2\mathbb{Z},\ d_i \in 2\mathbb{N},\ \eta_i = \pm 1.$

$A_{even}(x) = \left| \frac{1}{x} - 2\left\lfloor \frac{1}{2x} + \frac{1}{2} \right\rfloor \right|$, $x \in (0,1]$.

$2\left\lfloor \frac{1}{2x} + \frac{1}{2} \right\rfloor$ is the nearest even integer of $1/x$

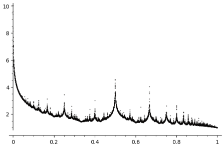
$dm_e(x) = \frac{dx}{1-x^2}$: it has infinite total mass



ECF-Brjuno function

$B_{even} : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$, even, $2\mathbb{Z}$-periodic

$B_{even}(x) = \log \frac{1}{x} + x B_{even}\left(\frac{1}{x}\right)$, $0 < x < 1$.



Even CF

$B_{even}(x) = \sum_{n\geq 0} x_0 A_{even}(x_0)\cdots A_{even}^{n-1}(x_0) \log \frac{1}{A_{even}^n(x_0)}$, where $x_0 = \|x\|_{2\mathbb{Z}}$.

Theorem 2 [L.-Marmi]

$$B(x) - \left[B_{even}(x) + \frac{\|x\|_{2\mathbb{Z}}+1}{2} B_{even}\left(\frac{1-\|x\|_{2\mathbb{Z}}}{1+\|x\|_{2\mathbb{Z}}}\right)\right] \in L^\infty.$$

$B_{even,\nu}(x) := \log \frac{1}{x} + x^{\nu} B_{even,\nu}\left(\frac{1}{x}\right)$

[Rivoal-Seuret '15] $F(x,t) = \sum_{k=1}^{\infty} \frac{\exp(i\pi k^2 x + 2\pi i k t)}{k}$ converges if $B_{even,1/2}(x) < \infty$ and $\sum_{n=0}^{\infty} (x A_e(x)\cdots A_e^{n-1}(x))^{1/2} < \infty$.

Best Approximation

p/q is a best approximation of x if

$$|qx - p| < |bx - a| \text{ for all } \frac{a}{b} \neq \frac{p}{q},\ b \leq q.$$

circle rotation:

[Lagrange]
$\frac{P_n(x)}{Q_n(x)} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$ is a best approximation of x.

Any best approximation is one of $\frac{P_n(x)}{Q_n(x)}$.

∞-,1-rationals

$$\Theta = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle < \mathrm{SL}_2(\mathbb{Z})$$

$$\widetilde{\Theta} = \Theta \sqcup \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Theta < \mathrm{GL}_2(\mathbb{Z})$$

Each branch of A_{even} is an element of $P\widetilde{\Theta}$ as a Möbius transformation.

$\mathbb{H}/\Theta$ has two cusps corresponding to $\Theta.\infty$ and $\Theta.1$

$\partial\mathbb{H} = \mathbb{R} \cup \{\infty\} = \Theta.\infty \sqcup \Theta.1$

$\Theta.\infty = \{\frac{even}{odd}, \frac{odd}{even}\}, \ \Theta.1 = \{\frac{odd}{odd}\}$

$$\sum_{\frac{P_n(x)}{Q_n(x)} \in \Theta.\infty} \frac{\log Q_{n+1}(x)}{Q_n(x)} + \sum_{\frac{P_n(x)}{Q_n(x)} \in \Theta.1} \frac{\log Q_{n+1}(x)}{Q_n(x)}$$

Best ∞-rational Approximation

Definition. $p/q \in \Theta.\infty$ is a best ∞-rational approximation if

$$|qx - p| < |bx - a| \text{ for all } a/b \in \Theta.\infty,\ a/b \neq p/q,\ b \leq q.$$

[Short-Walker '14]
ECF principal convergent $\frac{p_{e,n}(x)}{q_{e,n}(x)} = b_0 + \cfrac{\varepsilon_0}{b_1 + \ddots + \frac{\varepsilon_{n-1}}{b_n}}$ is a best ∞-rational approximation.
Every best ∞-rational approximation is $\frac{p_{e,n}}{q_{e,n}}$

Lemma [L.-Marmi]

$$\left| B_{even}(x) - \sum_{n\geq 0} \frac{\log q_{e,n+1}(x)}{q_{e,n}(x)} \right| < C, \text{ uniformly}$$

$$\left| \sum_{\frac{P_n(x)}{Q_n(x)} \in \Theta.\infty} \frac{\log Q_{n+1}(x)}{Q_n(x)} - \sum_{n\geq 0} \frac{\log q_{e,n+1}(x)}{q_{e,n}(x)} \right| < C, \text{ uniformly}$$

Odd-Odd CF

Definition. $p/q \in \Theta.1$ is a best 1-rational approximation if

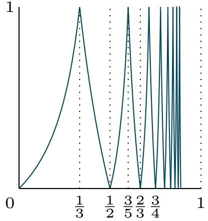
$$|qx - p| < |bx - a| \text{ for all } a/b \in \Theta.1,\ a/b \neq p/q,\ b \leq q.$$

$A_{oo}(x) = \iota^{-1} \circ A_{even} \circ \iota(x)$, $\iota(x) = \frac{1-x}{1+x}$

$x \in (0,1)$,

$$x = 1 - \cfrac{1}{c_1 + \cfrac{e_1}{2 - \cfrac{1}{c_2 + \cfrac{e_2}{2 - \ddots}}}},$$

$c_i \in \mathbb{N}$, $e_i = \pm 1$, $c_i + e_i \geq 2$.



[Kim-L.-Liao '22]

$\frac{p}{q}$ is a best 1-rational approximation if and only if

$\frac{p}{q} = \frac{p_{oo,n}}{q_{oo,n}} = 1 - \cfrac{1}{c_1 + \cfrac{e_1}{2 - \ddots + \frac{e_n}{2}}}$ for some n.

Odd-Odd CF

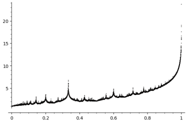
$B_{oo}(x) = \sum_{n\geq 0} |q_{oo,n-1}x_0 - p_{oo,n-1}| \log\left(\frac{|q_{oo,n-1}x_0 - p_{oo,n-1}|}{|q_{oo,n}x_0 - p_{oo,n}|}\right)$, $x_0 = \|x\|_{2\mathbb{Z}}$.

$\left| B_{oo}(x) - \frac{1}{2}\sum_{n\geq 1} \frac{\log q_{oo,n+1}(x)}{q_{oo,n}(x)} \right| < C$, uniformly.

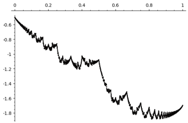
Lemma [L.-Marmi] $\left| \sum_{\frac{P_n(x)}{Q_n(x)} \in \Theta.1} \frac{\log Q_{n+1}(x)}{Q_n(x)} - \frac{1}{2}\sum_{n\geq 1} \frac{\log q_{oo,n+1}(x)}{q_{oo,n}(x)} \right| < C$, unif.

$\iota A_{oo} = A_{even}\iota$, thus

$B_{oo}(x) = (1+\|x\|_{2\mathbb{Z}})B_{even}(\iota(\|x\|_{2\mathbb{Z}})$



B_{oo}



$B - (B_{even} + \frac{1}{2}B_{oo})$

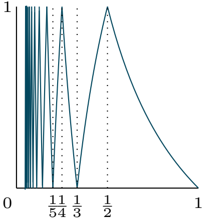
Theorem 2 [L.-Marmi]

$$B(x) - \left[B_{even}(x) + \frac{1+\|x\|_{2\mathbb{Z}}}{2} B_{even}\left(\frac{1-\|x\|_{2\mathbb{Z}}}{1+\|x\|_{2\mathbb{Z}}}\right)\right] \in L^\infty.$$

Odd CF

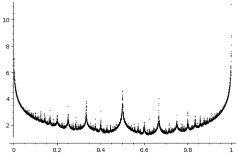
$x \in (0,1)$

$$x = \cfrac{1}{m_1 + \cfrac{f_1}{m_2 + \ddots + \cfrac{f_{n-1}}{m_n + \ddots}}}, \; m_i \in 2\mathbb{Z} - 1, \; f_i = \pm 1, \; m_i + f_i \geq 2$$



$$A_{odd}(x) = \left| \frac{1}{x} - \left(2 \left\lfloor \frac{1}{2x} \right\rfloor + 1 \right) \right|, \; x \in (0,1]$$

$$dm_{odd} = \frac{1}{3 \log G} \left(\frac{1}{G - 1 + x} + \frac{1}{G + 1 - x} \right) dx$$



$B_{odd} : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$, even, $2\mathbb{Z}$-periodic,

$$B_{odd}(x) = \log\left(\frac{1}{x}\right) + x B_{odd}\left(1 - \frac{1}{x}\right), \; 0 < x < 1.$$

Odd CF

Theorem 3 [L.-Marmi]

$B_{odd} - B \in L^\infty$ and $B_{odd} - B$ has an $1/2$-Hölder continuous extension to $\mathbb{R}$.

$\frac{p_{odd,n}}{q_{odd,n}} = \frac{1}{m_1 + \frac{f_1}{\ddots + \frac{f_{n-1}}{m_n}}}$, $n \geq 0$ is a subsequence of

$$\frac{P_0}{Q_0}, \frac{P_0+P_1}{Q_0+Q_1}, \frac{P_1}{Q_1}, \frac{P_1+P_2}{Q_1+Q_2}, \dots, \frac{P_n}{Q_n}, \frac{P_n+P_{n+1}}{Q_n+Q_{n+1}}, \frac{P_{n+1}}{Q_{n+1}}, \dots$$

containing all P_n/Q_n.

$\left| B_{odd}(x) - \sum_{n\geq 0} \frac{\log q_{odd,n+1}(x)}{q_{odd,n}(x)} \right| < C$, uniformly

$T_{odd}f(x) = xf(1-1/x)$, f: even, $2\mathbb{Z}$-periodic, $L^p([0,1], dm_{odd})$.

Proposition. [L.-Marmi]

- The spectral radius of $T_{odd} < 1$.

- If $f \in \mathcal{C}^\eta_{[0,1]}$ for $\eta > 1/2$, then $\sum_{n\geq 0} {T_{odd}}^n f \in \mathcal{C}^{1/2}_{[0,1]}$.

Thank you for your attention!