Dumont–Thomas numeration systems for $\mathbb{Z}$

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Outline

1. Introduction to numeration systems
2. Motivation: description of substitutive Wang tilings
3. Numeration systems for $\mathbb{N}$ based on fixed points
4. Numeration systems for $\mathbb{Z}$ based on periodic points
5. Properties of Dumont–Thomas numeration systems for $\mathbb{Z}$

Outline

Binary numeration system for $\mathbb{N}$

n	$\mathrm{rep}_2(n)$
7	111
6	110
5	101
4	100
3	11
2	10
1	1
0	ε

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$\boxed{a}$

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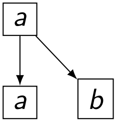

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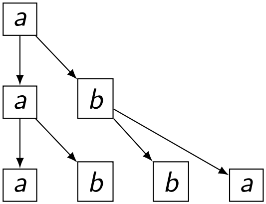

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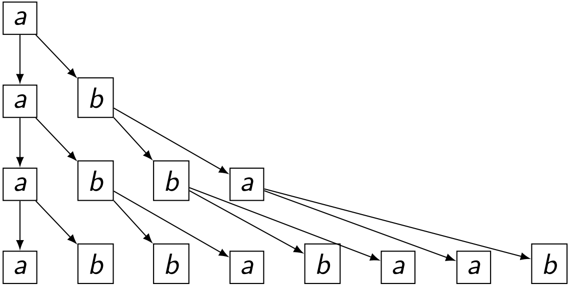

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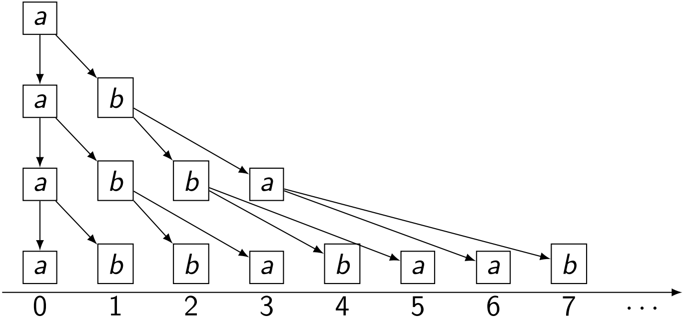

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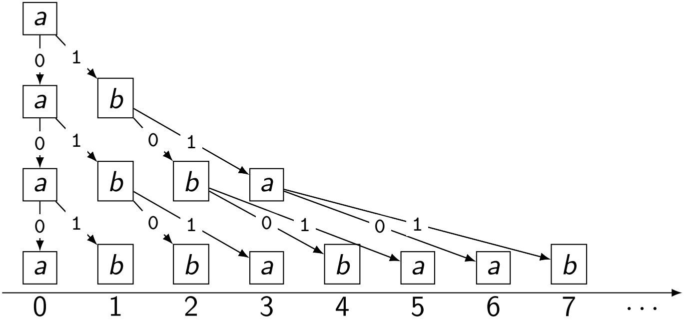

Labelling $\alpha \xrightarrow{j} \beta$ if and only if β is at position j in the image of α.

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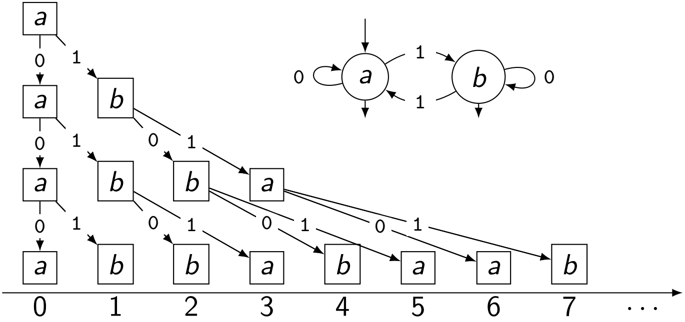

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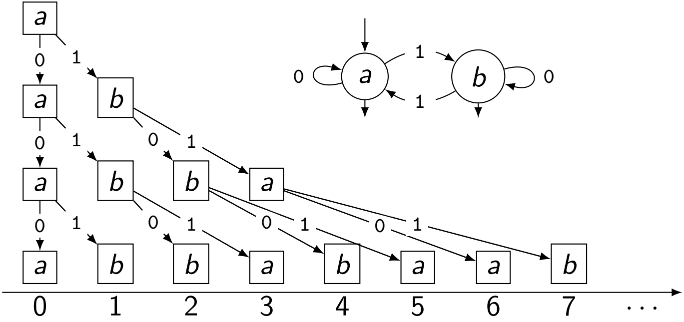


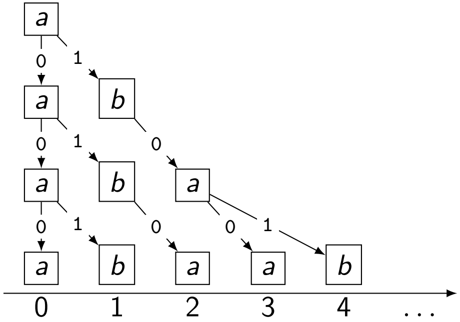
Theorem (Cobham, 1972)

Let $k \geq 2$. A right-infinite word $\mathbf{x} = x_0x_1x_2\cdots$ is a fixed point of a k-uniform morphism if and only if there exists deterministic finite automaton with output $\mathcal{A}$ such that $x_n = \mathcal{A}(\mathrm{rep}_k(n))$, for all $n \in \mathbb{N}$.

Fibonacci numeration system for $\mathbb{N}$

n	$\text{rep}_{\mathcal{F}}(n)$
7	1010
6	1001
5	1000
4	101
3	100
2	10
1	1
0	ε

- Let φ be the substitution $\varphi\colon a \mapsto ab, b \mapsto a$ and denote its fixed point $\mathbf{f} = \varphi(\mathbf{f}) = abaababaabaabab\cdots$



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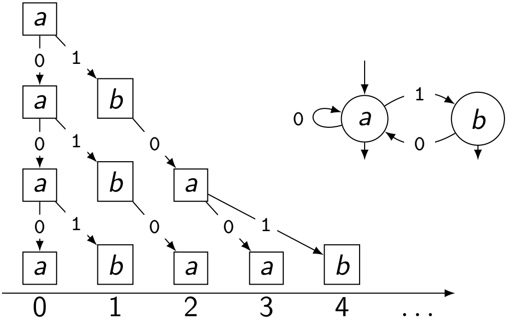

- It holds that $f_n = \mathcal{A}_{\mathbf{f}}(\mathrm{rep}_{\mathcal{F}}(n))$, for all $n \in \mathbb{N}$.

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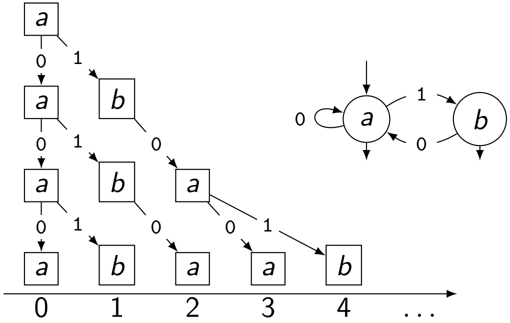

- It holds that $f_n = \mathcal{A}_{\mathbf{f}}(\mathrm{rep}_{\mathcal{F}}(n))$, for all $n \in \mathbb{N}$.

$\mathcal{F}$ may be viewed as a NS for $\mathbb{N}$ described by Dumont and Thomas, 1989.

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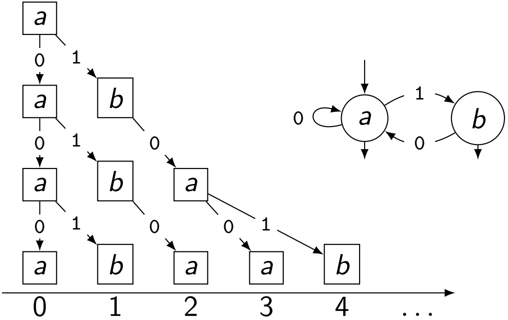
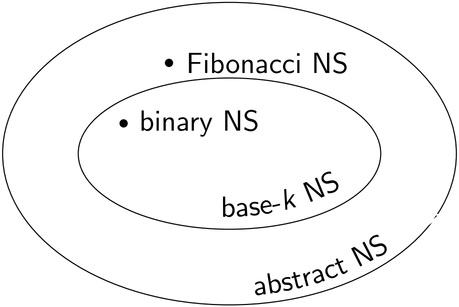


Theorem (P. Lecomte, M. Rigo, 2001)

An infinite word $\mathbf{x} = x_0x_1x_2\ldots$ is an image under a coding of a fixed point of a morphism if and only if there exists deterministic finite automaton with output $\mathcal{A}$ and an abstract numeration system $\mathcal{S}$ s.t. $x_n = \mathcal{A}(\mathrm{rep}_{\mathcal{S}}(n))$, for all $n \in \mathbb{N}$.

Numeration systems for $\mathbb{N}$



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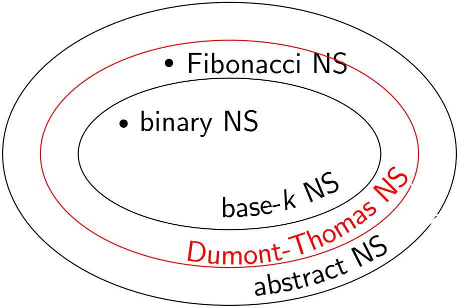

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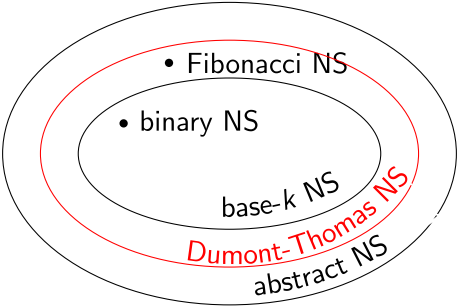
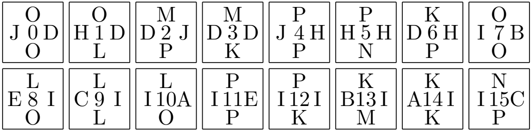


Goal: numeration systems based on ideas of Dumont and Thomas for $\mathbb{Z}$.

Outline

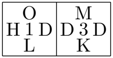
Wang tiles

A Wang tile is a square tile with 4 edge labels and an index.

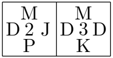


A particular set of 16 Wang tiles $\mathcal{Z}$.

Adjacent tiles with distinct label on common edge are forbidden:



Allowed.



Forbidden.

A configuration $x : \mathbb{Z}^2 \to \mathcal{Z}$ without forbidden pattern is said valid.
The set $\Omega_{\mathcal{Z}}$ of all valid configurations $x : \mathbb{Z}^2 \to \{0, \dots, 15\}$ is called the Wang shift $\Omega_{\mathcal{Z}}$ associated to the set $\mathcal{Z}$ of Wang tiles.

Configuration in a particular Wang shift $\Omega_{\mathcal{Z}}$

	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
5	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	O I 7 B O	K B 13 I M	O I 7 B O	K B 13 I M
4	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	M D 2 J P	O J 0 D O	M D 3 D K
3	P I 11 E P	L E 8 I O	P I 12 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M
2	P J 4 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
1	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 12 I K	L I 10 A O	K A 14 I K
0	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	L I 10 A O	K A 14 I K	O I 7 B O	K B 13 I M
-1	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K

A finite part of a particular configuration $y \in \Omega_{\mathcal{Z}}$.

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3	P I 11 E P	L E 8 I O	P I 12 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M
2	P J 4 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
1	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 12 I K	L I 10 A O	K A 14 I K
0	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	L I 10 A O	K A 14 I K	O I 7 B O	K B 13 I M
-1	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K

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Question

Does there exist a deterministic finite automaton with output $\mathcal{A}$ and a numeration system rep *for $\mathbb{Z}^2$ so that, for all $\mathbf{n} \in \mathbb{Z}^2$, $y_{\mathbf{n}} = \mathcal{A}(\text{rep}(\mathbf{n}))$?*

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1	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 12 I K	L I 10 A O	K A 14 I K
0	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	L I 10 A O	K A 14 I K	O I 7 B O	K B 13 I M
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Necessary step: extend Dumont–Thomas numeration systems to $\mathbb{Z}$ and $\mathbb{Z}^2$.

Configuration in a particular Wang shift $\Omega_{\mathcal{Z}}$

	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
5	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M	O / I 7 B / O	K / B 13 I / M
4	K / D 6 H / P	O / H 1 D / L	M / D 2 J / P	P / J 4 H / P	O / H 1 D / L	K / D 6 H / P	O / H 1 D / L	M / D 3 D / K	K / D 6 H / P	O / H 1 D / L	M / D 2 J / P	O / J 0 D / O	M / D 3 D / K
3	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	L / I 10 A / O	K / A 14 I / K	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M
2	P / J 4 H / P	O / H 1 D / L	K / D 6 H / P	P / H 5 H / N	O / H 1 D / L	K / D 6 H / P	O / H 1 D / L	K / D 6 H / P	P / H 5 H / N	O / H 1 D / L	K / D 6 H / P	O / H 1 D / L	M / D 3 D / K
1	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	N / I 15 C / P	L / C 9 I / L	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	N / I 15 C / P	L / C 9 I / L	P / I 12 I / K	L / I 10 A / O	K / A 14 I / K
0	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M	P / I 11 E / P	L / E 8 I / O	P / I 12 I / K	O / I 7 B / O	K / B 13 I / M	P / I 12 I / K	L / I 10 A / O	K / A 14 I / K	O / I 7 B / O	K / B 13 I / M
-1	K / D 6 H / P	O / H 1 D / L	M / D 2 J / P	P / J 4 H / P	O / H 1 D / L	K / D 6 H / P	O / H 1 D / L	M / D 3 D / K	K / D 6 H / P	O / H 1 D / L	K / D 6 H / P	O / H 1 D / L	M / D 3 D / K

A finite part of a particular configuration $y \in \Omega_{\mathcal{Z}}$.

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Does there exist a deterministic finite automaton with output $\mathcal{A}$ and a numeration system rep *for* $\mathbb{Z}^2$ *so that, for all* $\boldsymbol{n} \in \mathbb{Z}^2$, $y_{\boldsymbol{n}} = \mathcal{A}(\text{rep}(\boldsymbol{n}))$?

Necessary step: extend Dumont–Thomas numeration systems to $\mathbb{Z}$ and $\mathbb{Z}^2$.
Moreover, we study two-sided periodic points instead of fixed points.

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5. Properties of Dumont–Thomas numeration systems for $\mathbb{Z}$

Greedy algorithm

- Numeration system based on the sequence $(3^k)_{k=0}^{+\infty}$

k	0	1	2	3	$\cdots$
3^k	1	3	9	27	$\cdots$

$$\begin{aligned} 11 &= 9 + 2 \\ &= 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0 \end{aligned}$$

We obtain $\text{rep}_3(11) = 102$.

Greedy algorithm

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$$11 = 9 + 2 = 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0$$

We obtain $\mathrm{rep}_3(11) = 102$.

- Numeration system based on the fixed point $\mathbf{r} = abcbaacbbbaaab\cdots$ of the substitution $\rho\colon a \mapsto abc, b \mapsto baa, c \mapsto cbb$

k	0	1	2	3	$\cdots$
$\rho^k(a)$	a	abc	$abcbaacbb$	$abcbaacbbbaaabcabccbbbaabaa$	$\cdots$

Greedy algorithm

- Numeration system based on the sequence $(3^k)_{k=0}^{+\infty}$

k	0	1	2	3	$\cdots$
3^k	1	3	9	27	$\cdots$

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k	0	1	2	3	$\cdots$
$\rho^k(a)$	a	abc	$abcbaacbb$	$abcbaacbbbaaabcabccbbbaabaa$	$\cdots$

We aim to represent $n = 11$, thus we take the prefix of $\mathbf{r}$ of length 11:

$$abcbaacbbba = abcbaacbb \cdot \varepsilon \cdot ba = \rho^2(a) \cdot \rho^1(\varepsilon) \cdot \rho^0(ba)$$

We obtain $\mathrm{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$.

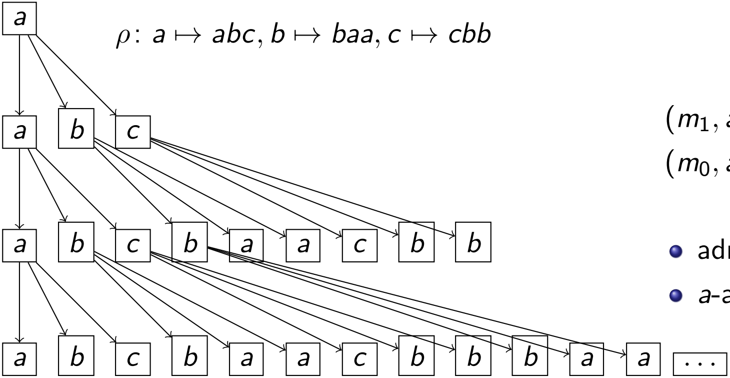
Idea of Dumont and Thomas

Definition (admissible sequence, Dumont, Thomas, 1989)

Let $\eta : A^* \to A^*$ be a substitution. Let $a \in A$, $k \in \mathbb{N}$ and, for each $i \in \mathbb{N}$, $i \leq k$, (m_i, a_i) be an element of $A^* \times A$. The sequence $(m_i, a_i)_{i=0,\dots,k}$ is

- *admissible w.r.t.* η if, for all $i \in \{1, .., k\}$, $m_{i-1}a_{i-1}$ is a prefix of $\eta(a_i)$;
- *a-admissible w.r.t.* η if, moreover, $m_k a_k$ is a prefix of $\eta(a)$.

$\rho\colon a \mapsto abc, b \mapsto baa, c \mapsto cbb$



$(m_1, a_1) = (\varepsilon, b)$,

$(m_0, a_0) = (ca, a)$.

- admissible ×
- a-admissible ×

Idea of Dumont and Thomas

Definition (admissible sequence, Dumont, Thomas, 1989)

Let $\eta : A^* \to A^*$ be a substitution. Let $a \in A$, $k \in \mathbb{N}$ and, for each $i \in \mathbb{N}$, $i \leq k$, (m_i, a_i) be an element of $A^* \times A$. The sequence $(m_i, a_i)_{i=0,\dots,k}$ is

- *admissible w.r.t.* η if, for all $i \in \{1, .., k\}$, $m_{i-1}a_{i-1}$ is a prefix of $\eta(a_i)$;
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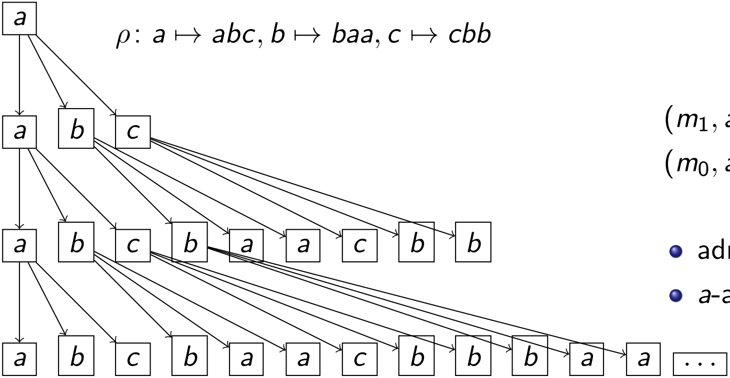

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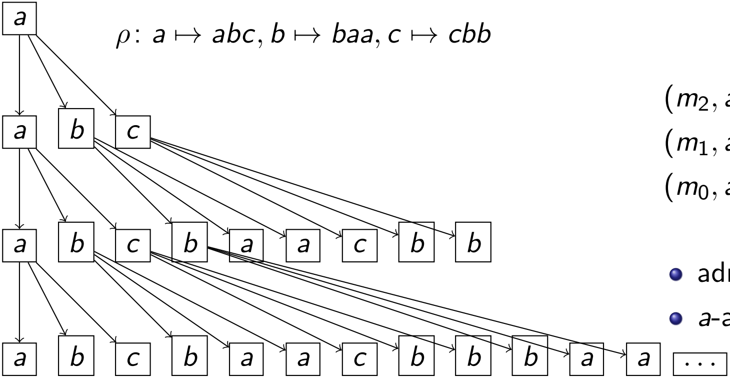
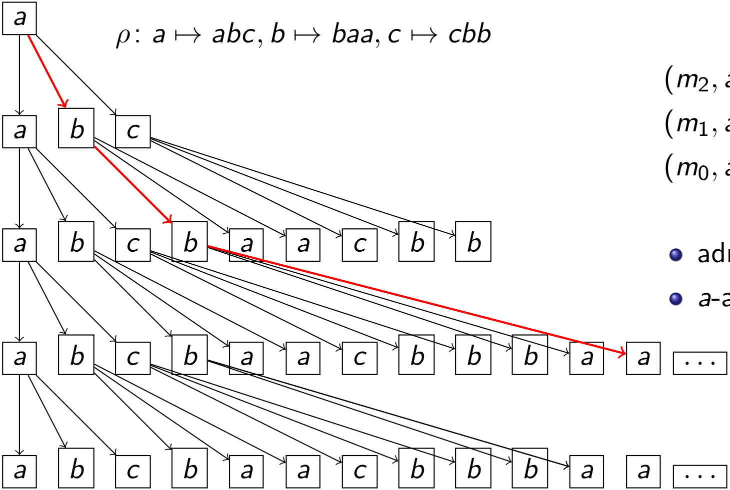


$(m_2, a_2) = (a, b)$,
$(m_1, a_1) = (\varepsilon, b)$,
$(m_0, a_0) = (ba, a)$.

- admissible ✓
- a-admissible ✓

Illustration of Dumont and Thomas theorem

$\rho\colon a \mapsto abc, b \mapsto baa, c \mapsto cbb$



$(m_2, a_2) = (a, b),$
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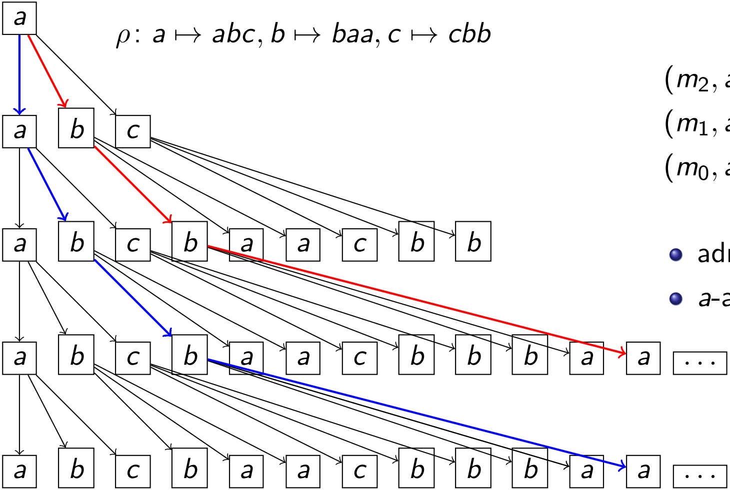
- admissible ✓
- *a*-admissible ✓

Recall: we obtained $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$, where $\mathbf{r} = \rho(\mathbf{r})$.

Illustration of Dumont and Thomas theorem

$\rho \colon a \mapsto abc, b \mapsto baa, c \mapsto cbb$



$(m_2, a_2) = (a, b),$

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- admissible ✓
- a-admissible ✓

Recall: we obtained $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$, where $\mathbf{r} = \rho(\mathbf{r})$.

There exists a unique representation of every $n \geq 1$ with an a-admissible sequence $(m_i, a_i)_{i=0,\dots,k-1}$ such that $m_{k-1} \neq \varepsilon$.

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Let $a \in A$ and let $\eta : A^* \to A^*$ be a substitution. Let $\mathbf{u} = \eta(\mathbf{u})$ be a right-infinite fixed point of η with growing letter $u_0 = a$.
For every $n \geq 1$, there exists a unique sequence $(m_i, a_i)_{i=0,\dots,k-1}$ s. t.

- *this sequence is a-admissible and $m_{k-1} \neq \varepsilon$,*
- $u_0 u_1 \cdots u_{n-1} = \eta^{k-1}(m_{k-1})\eta^{k-2}(m_{k-2}) \cdots \eta^0(m_0)$.

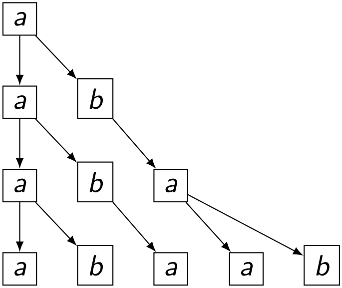
Note: requiring $m_{k-1} \neq \varepsilon$ is analogical to forbidding leading zeroes in base-k numeration systems.
E.g. $\mathrm{rep}_3(11) = 102$, $\mathrm{rep}_3(11) \neq 0102$.

Outline

Fibonacci substitution has no two-sided fixed point

Fibonacci substitution φ: $a \mapsto ab$, $b \mapsto a$ has

- a right-infinite fixed point with growing letter a;



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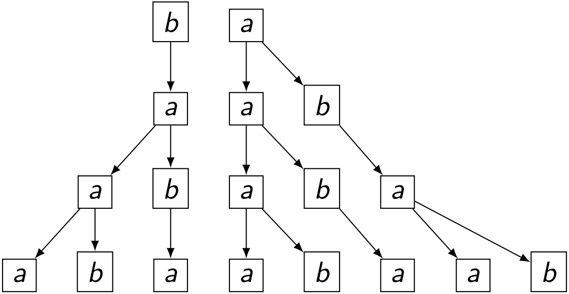
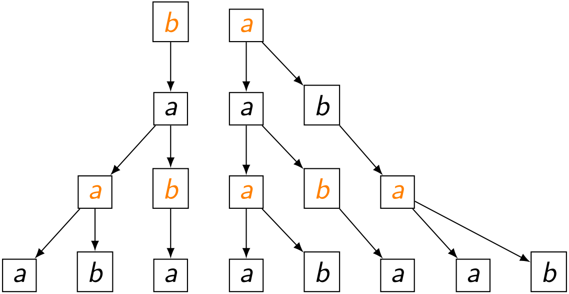


Fibonacci substitution has no two-sided fixed point

Fibonacci substitution φ: $a \mapsto ab$, $b \mapsto a$ has

- a right-infinite fixed point with growing letter a;
- a two-sided periodic point with growing seed $b|a$ and period $p = 2$:

$$\mathbf{g} = \cdots abaab.abaababaab \cdots$$



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- a two-sided periodic point with growing seed $a|a$ and period $p = 2$:

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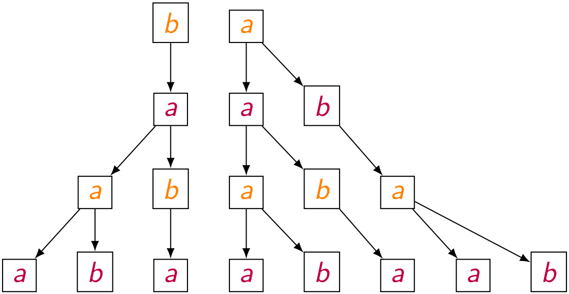

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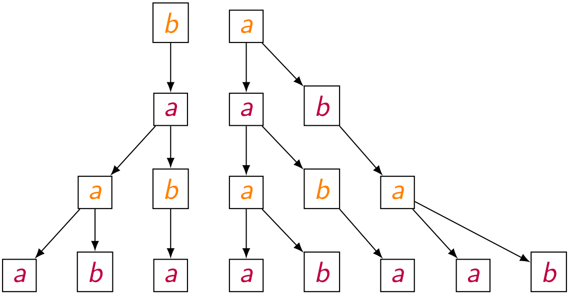
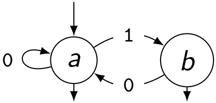


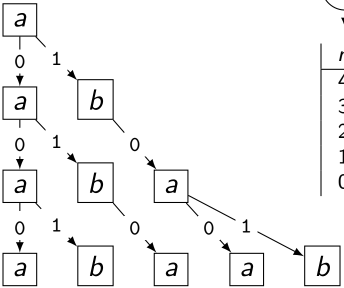
It holds that $\mathbf{g} = \varphi^2(\mathbf{g})$ and $\mathbf{h} = \varphi^2(\mathbf{h})$ are fixed points of φ^2.

Extending Fibonacci numeration system from $\mathbb{N}$ to $\mathbb{Z}$

$\mathbf{f}$ is the fixed point of φ: $a \mapsto ab$, $b \mapsto a$ $\qquad$ $\mathbf{f} = abaababaab\cdots$



n	$\mathrm{rep}_{\mathcal{F}}(n)$
4	101
3	100
2	10
1	1
0	ε



Labelling $\alpha \xrightarrow{j} \beta$ if and only if β is at position j in the image of α.

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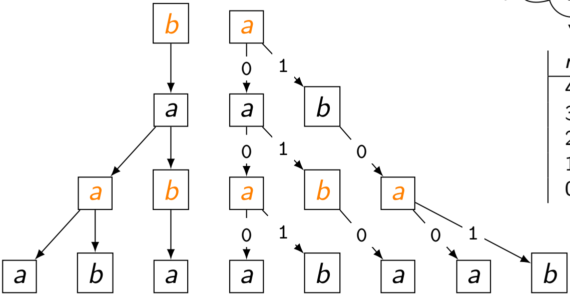

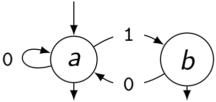

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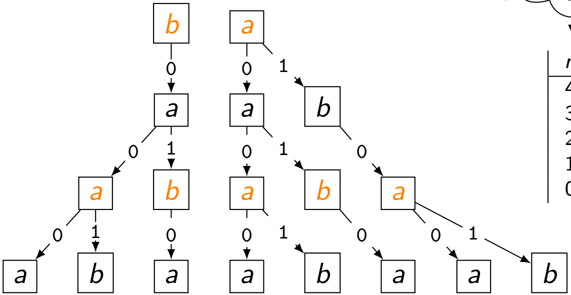

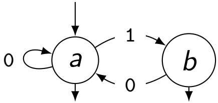

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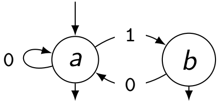
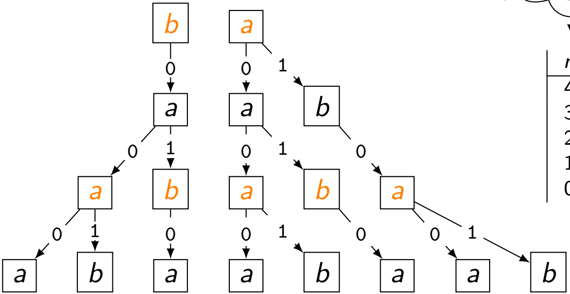


n	$\mathrm{rep}_{\mathcal{F}}(n)$
4	101
3	100
2	10
1	1
0	ε

n	$\mathrm{rep}_{\mathcal{F}c}(n)$
2	
1	
0	
-1	
-2	



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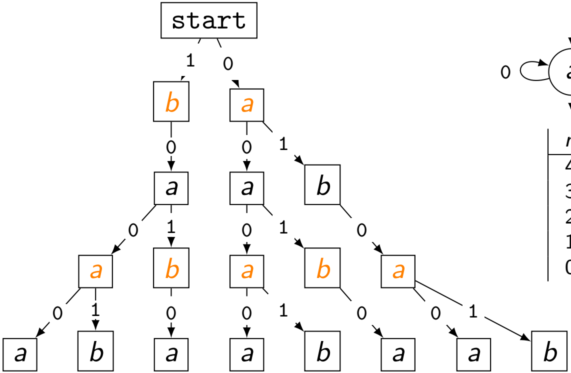
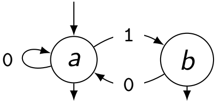
n	$\mathrm{rep}_{\mathcal{F}^c}(n)$
2	
1	
0	0
-1	1
-2	





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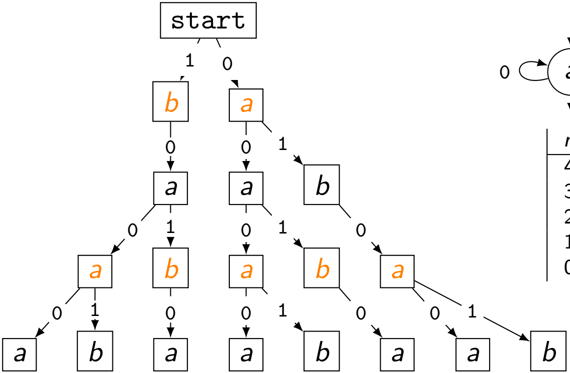
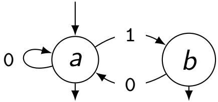
n	$\text{rep}_{\mathcal{F}^c}(n)$
2	010
1	001
0	0
-1	1
-2	





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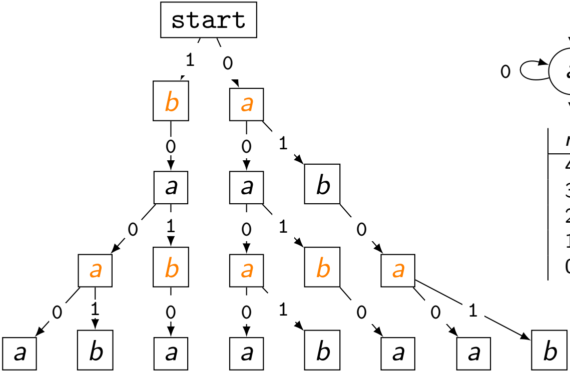

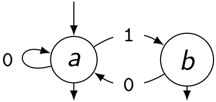

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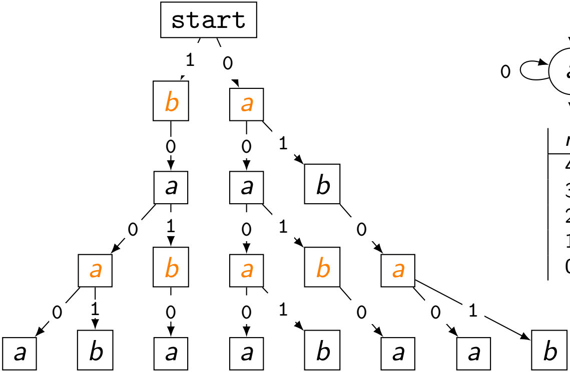

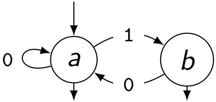

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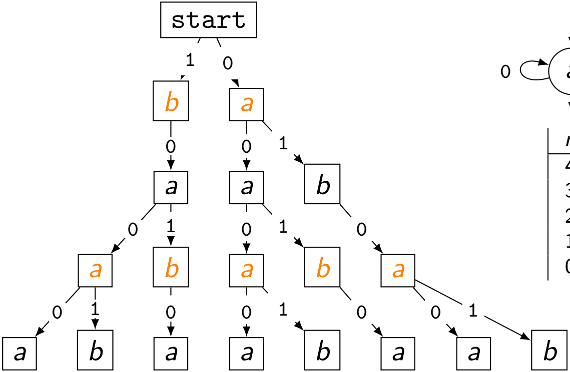

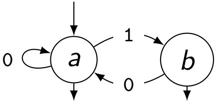

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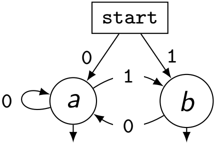
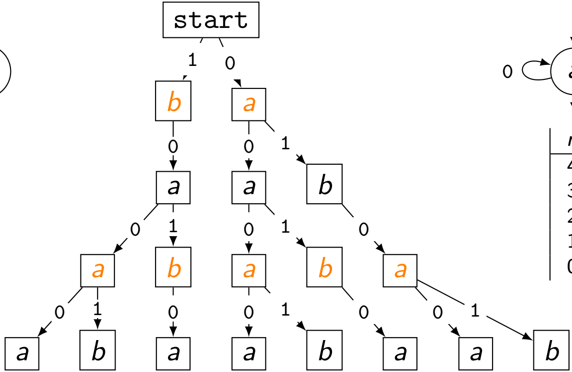
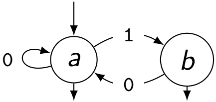


n	$\mathrm{rep}_{\mathcal{F}c}(n)$
2	010
1	001
0	0
-1	1
-2	100





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4	101
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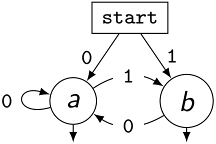
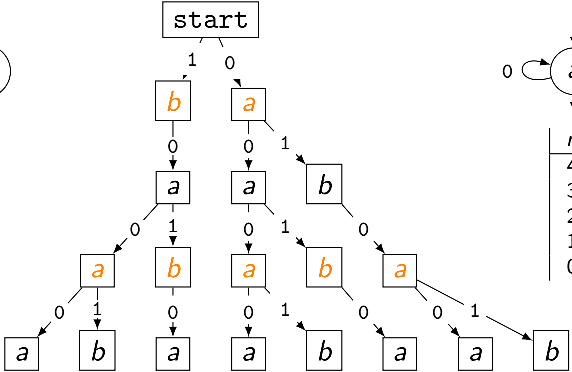
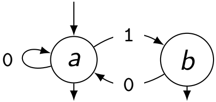


n	$\mathrm{rep}_{\mathcal{F}^c}(n)$
2	010
1	001
0	0
-1	1
-2	100





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4	101
3	100
2	10
1	1
0	ε

There exists a unique representation of every $n \leq -2$ with a b-admissible sequence $(m_i, a_i)_{i=0,..,2k-1}$ such that $\varphi(m_{2k-1})m_{2k-2}a_{2k-2} \neq \varphi^2(b)$.

Numeration systems for $\mathbb{Z}$ based on periodic points

Theorem (Dumont, Thomas, 1989)

Let $a \in A$ and let $\eta : A^* \to A^*$ be a substitution. Let $\mathbf{u} = \eta(\mathbf{u})$ be a right-infinite fixed point of η with growing letter $u_0 = a$.
For every $n \geq 1$, there exists a unique sequence $(m_i, a_i)_{i=0,\ldots,k-1}$ s. t.

- *this sequence is a-admissible and $m_{k-1} \neq \varepsilon$,*
- $u_0 u_1 \cdots u_{n-1} = \eta^{k-1}(m_{k-1})\eta^{k-2}(m_{k-2}) \cdots \eta^0(m_0)$.

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- Our Theorem A is an analogy of Dumont–Thomas theorem for right-infinite periodic points with $u_0 = a$ and period $p \geq 1$, where
  - p divides k,
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- Our Theorem A is an analogy of Dumont–Thomas theorem for right-infinite periodic points with $u_0 = a$ and period $p \geq 1$, where
  - p divides k,
  - $m_{k-1} m_{k-2} \cdots m_{k-p} \neq \varepsilon$.
- Our Theorem B is an analogy of Dumont–Thomas theorem for left-infinite periodic points with $u_{-1} = b$ and period $p \geq 1$, where
  - p divides k,
  - $\eta^{p-1}(m_{k-1})\eta^{p-2}(m_{k-2}) \cdots \eta^0(m_{k-p}) a_{k-p} \neq \eta^p(b)$.

Numeration systems for $\mathbb{Z}$ based on periodic points

Definition (Dumont–Thomas numeration systems for $\mathbb{Z}$)

Let $\eta : A^* \to A^*$ be a substitution and $\mathbf{u}$ be its two-sided periodic point with growing seed s and the period $p \geq 1$. We define

$$\mathrm{rep}_{\mathbf{u}} : \quad n \quad \mapsto \quad \begin{cases} 0 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdot \ldots \cdot |m_0|, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0; \\ 1, & \text{if } n = -1; \\ 1 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdot \ldots \cdot |m_0|, & \text{if } n \leq -2, \end{cases}$$

where

- $(m_i, a_i)_{i=0,..,k-1}$ is the unique sequence from Theorem A if $n \geq 1$;
- $(m_i, a_i)_{i=0,..,k-1}$ is the unique sequence from Theorem B if $n \leq -2$.

Note that, for every $n \in \mathbb{Z}$, we have

$$|\mathrm{rep}_{\mathbf{u}}(n)| = \ell p + 1, \text{ for some } \ell \in \mathbb{N}.$$

Outline

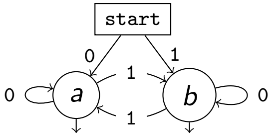
Periodic points as automatic sequences

Theorem (Labbé, L.)

Let $\eta : A^* \to A^*$ be a substitution and $\mathbf{u}$ be its two-sided periodic point with growing seed $s = u_{-1}|u_0$. Then

$$u_n = \mathcal{A}_{\eta,s}(\mathrm{rep}_{\mathbf{u}}(n)), \qquad \text{for every } n \in \mathbb{Z}.$$

The Thue–Morse substitution $\psi\colon a \mapsto ab, b \mapsto ba$ has periodic point $\mathbf{t} = \cdots abbabaab.abbabaab \cdots$ with $s = b|a$ and period $p = 2$.



$\mathcal{A}_{\psi,s}$

n	$\mathrm{rep}_{\mathbf{t}}(n)$
4	00100
3	011
2	010
1	001
0	0

n	$\mathrm{rep}_{\mathbf{t}}(n)$
-1	1
-2	110
-3	101
-4	100
-5	11011

Note that Thue–Morse substitution has no two-sided fixed point.

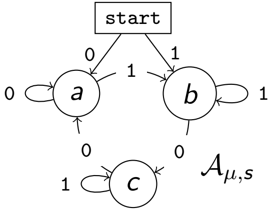
Periodic points as automatic sequences

Theorem (Labbé, L.)

Let $\eta : A^* \to A^*$ be a substitution and $\mathbf{u}$ be its two-sided periodic point with growing seed $s = u_{-1}|u_0$. Then

$$u_n = \mathcal{A}_{\eta,s}(\mathrm{rep}_{\mathbf{u}}(n)), \qquad \text{for every } n \in \mathbb{Z}.$$

A 2-uniform substitution $\mu\colon a \mapsto ab, b \mapsto cb, c \mapsto ac$ has fixed point $\mathbf{c} = \cdots abacaccb.abcbaccb \cdots$ with $s = b|a$ (and period $p = 1$).



n	$\mathrm{rep}_{\mathbf{c}}(n)$
4	0100
3	011
2	010
1	01
0	0

n	$\mathrm{rep}_{\mathbf{c}}(n)$
-1	1
-2	10
-3	101
-4	100
-5	1011

This numeration system is known as the two's complement notation.

Characterization by total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
$\text{rep}_{\mathbf{c}}(n)$	1011	100	101	10	1	0	01	010	011	0100

Radix order: $u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$

Reversed order: $u <_{rev} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$

Characterization by total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
$\mathrm{rep}_{\mathbf{c}}(n)$	1011	100	101	10	1	0	01	010	011	0100

Radix order: $u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$

Reversed order: $u <_{rev} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$

Definition (total order $\prec$)

For every $u, v \in \{0,1\}\mathcal{D}^*$, we define $u \prec v$ if and only if

- $u \in 1\mathcal{D}^*$ and $v \in 0\mathcal{D}^*$, or
- $u, v \in 0\mathcal{D}^*$ and $u <_{rad} v$, or
- $u, v \in 1\mathcal{D}^*$ and $u <_{rev} v$.

Characterization by total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
$\mathrm{rep}_{\mathbf{c}}(n)$	1011	100	101	10	1	0	01	010	011	0100

Radix order: $u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$

Reversed order: $u <_{rev} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$

Definition (total order $\prec$)

For every $u, v \in \{0,1\}\mathcal{D}^*$, we define $u \prec v$ if and only if

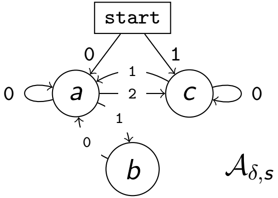
- $u \in 1\mathcal{D}^*$ and $v \in 0\mathcal{D}^*$, or
- $u, v \in 0\mathcal{D}^*$ and $u <_{rad} v$, or
- $u, v \in 1\mathcal{D}^*$ and $u <_{rev} v$.

Theorem (Labbé, L.)

Let $\eta : A^* \to A^*$ be a substitution and $\mathbf{u}$ be its two-sided periodic point with growing seed s and period $p \geq 1$. Let $f : \mathbb{Z} \to \{0,1\}\mathcal{D}^*$ be a map. Then $f = \mathrm{rep}_{\mathbf{u}}$ if and only if f is increasing w.r.t. $\prec$, its image is $f(\mathbb{Z}) = \bigcup_{\ell\in\mathbb{N}} \mathcal{L}_{\ell p+1}(\mathcal{A}_{\eta,s}) \setminus \{0\mathtt{W}_{\min}, 1\mathtt{W}_{\max}\}\mathcal{D}^*$ and $f(0) = 0$.

Extension to $\mathbb{Z}^2$

A substitution $\delta\colon a \mapsto abc, b \mapsto a, c \mapsto ca$ has a periodic point $\mathbf{d} = \cdots caabc.abcacaabc \cdots$ with growing seed $s = c|a$ and $p = 2$



n	$\mathrm{rep}_{\mathbf{d}}(n)$
6	00100
5	021
4	020
3	010
2	002
1	001
0	0

n	$\mathrm{rep}_{\mathbf{d}}(n)$
-1	1
-2	111
-3	110
-4	101
-5	100
-6	11102
-7	11101

We denote $\mathtt{W}_{\min} = 00$ and $\mathtt{W}_{\max} = 12$. Let $w \in \mathcal{L}(\mathcal{A}_{\delta,s})$. Then

$$\mathcal{A}_{\delta,s}(w) = \begin{cases} \mathcal{A}_{\delta,s}(0(\mathtt{W}_{\min})^* v), & \text{if } w = 0v; \\ \mathcal{A}_{\delta,s}(1(\mathtt{W}_{\max})^* v), & \text{if } w = 1v. \end{cases}$$

This is why $\mathtt{W}_{\min}$ and $\mathtt{W}_{\max}$ are called neutral words.

We represent $\boldsymbol{n} = (6, -5) \in \mathbb{Z}^2$ as

$$\begin{pmatrix} \mathrm{rep}_{\mathbf{d}}(6) \\ \mathrm{rep}_{\mathbf{d}}(-5) \end{pmatrix} = \begin{pmatrix} 00100 \\ 100 \end{pmatrix} = \begin{pmatrix} 00100 \\ 11200 \end{pmatrix}.$$

Extension to $\mathbb{Z}^d$

Definition (Numeration system for $\mathbb{Z}^d$)

Let $\eta : A^* \to A^*$ be a substitution and $\mathbf{u}$ its periodic point with growing seed. For every $\boldsymbol{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, we define

$$\mathrm{rep}_{\mathbf{u}}(\boldsymbol{n}) = \begin{pmatrix} \mathrm{pad}_t(\mathrm{rep}_{\mathbf{u}}(n_1)) \\ \mathrm{pad}_t(\mathrm{rep}_{\mathbf{u}}(n_2)) \\ \ldots \\ \mathrm{pad}_t(\mathrm{rep}_{\mathbf{u}}(n_d)) \end{pmatrix} \in \{0,1\}^d(\mathcal{D}^d)^*,$$

where pad_t inserts corresponding neutral word $\mathtt{W}_{\mathrm{min}}$ or $\mathtt{W}_{\mathrm{max}}$ into representations up to length $t = \max\{|\mathrm{rep}_{\mathbf{u}}(n_i)| : 1 \le i \le d\}$.

Fibonacci numeration system $\mathcal{F}c$ extended to $\mathbb{Z}^2$

7	10000 01010	10001 01010	10010 01010	10100 01010	10101 01010	00000 01010	00001 01010	00010 01010	00100 01010	00101 01010	01000 01010	01001 01010	01010 01010
6	10000 01001	10001 01001	10010 01001	10100 01001	10101 01001	00000 01001	00001 01001	00010 01001	00100 01001	00101 01001	01000 01001	01001 01001	01010 01001
5	10000 01000	10001 01000	10010 01000	10100 01000	10101 01000	00000 01000	00001 01000	00010 01000	00100 01000	00101 01000	01000 01000	01001 01000	01010 01000
4	10000 00101	10001 00101	10010 00101	10100 00101	10101 00101	00000 00101	00001 00101	00010 00101	00100 00101	00101 00101	01000 00101	01001 00101	01010 00101
3	10000 00100	10001 00100	10010 00100	10100 00100	10101 00100	00000 00100	00001 00100	00010 00100	00100 00100	00101 00100	01000 00100	01001 00100	01010 00100
2	10000 00010	10001 00010	10010 00010	100 010	101 010	000 010	001 010	010 010	00100 00010	00101 00010	01000 00010	01001 00010	01010 00010
1	10000 00001	10001 00001	10010 00001	100 001	101 001	000 001	001 001	010 001	00100 00001	00101 00001	01000 00001	01001 00001	01010 00001
0	10000 00000	10001 00000	10010 00000	100 000	1 0	0 0	001 000	010 000	00100 00000	00101 00000	01000 00000	01001 00000	01010 00000
−1	10000 10101	10001 10101	10010 10101	100 101	1 1	0 1	001 101	010 101	00100 10101	00101 10101	01000 10101	01001 10101	01010 10101
−2	10000 10100	10001 10100	10010 10100	100 100	101 100	000 100	001 100	010 100	00100 10100	00101 10100	01000 10100	01001 10100	01010 10100
−3	10000 10010	10001 10010	10010 10010	10100 10010	10101 10010	00000 10010	00001 10010	00010 10010	00100 10010	00101 10010	01000 10010	01001 10010	01010 10010
−4	10000 10001	10001 10001	10010 10001	10100 10001	10101 10001	00000 10001	00001 10001	00010 10001	00100 10001	00101 10001	01000 10001	01001 10001	01010 10001
−5	10000 10000	10001 10000	10010 10000	10100 10000	10101 10000	00000 10000	00001 10000	00010 10000	00100 10000	00101 10000	01000 10000	01001 10000	01010 10000
	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7

Description of a particular Wang configuration

The numeration system $\mathcal{F}c$ extended to $\mathbb{Z}^2$ describes the Wang configuration $y \in \Omega_{\mathcal{Z}}$.

5	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	O I 7 B O	K B 13 I M	O I 7 B O	K B 13 I M
4	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	M D 2 J P	O J 0 D O	M D 3 D K
3	P I 11 E P	L E 8 I O	P I 12 I K	P I 11 E P	L E 8 I O	P I 12 I K	L I 10 A O	K A 14 I K	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M
2	P J 4 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	K D 6 H P	P H 5 H N	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
1	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 11 E P	L E 8 I O	P I 12 I K	N I 15 C P	L C 9 I L	P I 12 I K	L I 10 A O	K A 14 I K
0	P I 12 I K	O I 7 B O	K B 13 I M	P I 11 E P	L E 8 I O	P I 12 I K	O I 7 B O	K B 13 I M	P I 12 I K	L I 10 A O	K A 14 I K	O I 7 B O	K B 13 I M
-1	K D 6 H P	O H 1 D L	M D 2 J P	P J 4 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K	K D 6 H P	O H 1 D L	K D 6 H P	O H 1 D L	M D 3 D K
	-5	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Theorem (Labbé, L.)

There exists a deterministic finite automaton with output $\mathcal{A}$ so that

$$y_{\boldsymbol{n}} = \mathcal{A}(\mathrm{rep}_{\mathcal{F}c}(\boldsymbol{n})), \qquad \text{for every } \boldsymbol{n} \in \mathbb{Z}^2.$$

Open problems

- Explore links between Dumont–Thomas numeration systems for $\mathbb{Z}^d$ and d-dimensional periodic points.
- Find other Wang shifts with an automatic characterization.

Thank you for your attention!