Some fractal problems in beta-expansions

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Outline

1. Greedy beta-expansions
2. The distribution of orbits and some fractal problems
3. Intermediate β-expansions

Greedy beta-expansions

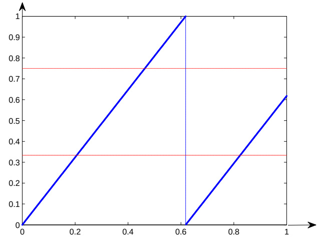
Greedy β-transformations

- $\beta > 1$
- β-transformation $T_\beta : [0,1] \to [0,1]$

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \beta x \rfloor$ denotes the integer part of βx.

- Example: $\beta = \frac{1+\sqrt{5}}{2}$



- β-transformation dynamical system: $([0,1), T_\beta)$

Greedy β-expansion

- digit set

$$\mathcal{A} = \begin{cases} \{0, 1, \dots, \beta - 1\} & \text{when } \beta \text{ is an integer} \\ \{0, 1, \dots, \lfloor \beta \rfloor\} & \text{otherwise.} \end{cases}$$

- digit function

$$\varepsilon_1(\cdot, \beta) : [0, 1] \to \mathcal{A} \text{ as } x \mapsto \lfloor \beta x \rfloor$$

- $\varepsilon_n(x, \beta) := \varepsilon_1(T_\beta^{n-1} x, \beta)$
- greedy β-expansion (Rényi, 1957)

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots$$

- notation:

$$\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \dots, \varepsilon_n(x, \beta), \dots)$$

Convergence Rate Problem

- convergents of the β-expansion

$$\omega_n(x) := \omega_n(x,\beta) = \frac{\varepsilon_1(x,\beta)}{\beta} + \frac{\varepsilon_2(x,\beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x,\beta)}{\beta^n}$$

- For any $x \in [0,1)$,

$$\omega_n(x) \to x \quad \text{as} \quad n \to \infty.$$

- Natural question: What is the rate of convergence of $\omega_n(x)$?

Convergence Rate Problem

- convergents of the β-expansion

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- Natural question: What is the rate of convergence of $\omega_n(x)$?
- Metric result(Fang, Wu and L., 2020, Math. Z.)

  For $\mathcal{L}$-almost all $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} \log_\beta (x - \omega_n(x)) = -1.$$

  That is, $x - \omega_n(x) \approx \beta^{-n}$ for Lebesgue-almost all $x \in [0,1)$.

invariant measure

- (i) If $\beta \in \mathbb{N}$, then the Lebesgue measure $\mathcal{L}$ is T_β-invariant.
  (ii) If $\beta \notin \mathbb{N}$, then $\mathcal{L}$ is not T_β-invariant.
- (Rényi, 1957) There exists a unique invariant measure μ_β which is equivalent to the Lebesgue measure. Moreover, for any $x \in [0,1)$,

$$1 - \frac{1}{\beta} \leq \frac{d\mu_\beta}{d\mathcal{L}}(x) \leq \frac{1}{1-\frac{1}{\beta}}.$$

- (Gelfond, 1959 & Parry, 1960) For any $\beta > 1$, the density function formula

$$h_\beta(x) := \frac{d\mu_\beta}{d\mathcal{L}}(x) = \frac{1}{F(\beta)} \sum_{n \geq 0 : x < T_\beta^n 1} \frac{1}{\beta^n} \quad \mathcal{L}-\text{a.e. } x \in [0,1), \tag{1}$$

where $F(\beta) = \int_0^1 \sum_{n \geq 0 : x < T_\beta^n 1} \frac{1}{\beta^n} dx$ is the normalising function.

admissible sequence

- admissible sequence/word

$$\Sigma_\beta = \{\omega \in \mathcal{A}^{\mathbb{N}} : \ \exists\, x \in [0,1) \ \text{such that} \ \varepsilon(x,\beta) = \omega\}$$

$$\Sigma_\beta^n = \{\omega \in \mathcal{A}^{n} : \ \exists\, x \in [0,1) \ \text{s.t.} \ \varepsilon_i(x,\beta) = \omega_i \ \text{for all} \ i = 1, \cdots, n\}$$

- β is an integer: $\Sigma_\beta = \mathcal{A}^{\mathbb{N}}$ (except countable points)
- Example: $\beta_0 = \frac{\sqrt{5}+1}{2}$

$$\Sigma_{\beta_0} = \{\omega \in \{0,1\}^{\mathbb{N}} : \ \text{the word } 11 \text{ dosen't appear in } \omega\}$$

- number of admissible words of length n

$$\beta^n \le \sharp\Sigma_\beta^n \le \frac{\beta^{n+1}}{\beta - 1}$$

- $\beta_1 < \beta_2 \Longrightarrow \Sigma_{\beta_1} \subset \Sigma_{\beta_2}, \quad \Sigma_{\beta_1}^n \subset \Sigma_{\beta_2}^n$

admissible sequence

the infinite expansion of the number 1

$$\varepsilon^*(1,\beta) = \begin{cases} \varepsilon(1,\beta) & \text{if there are infinite many } \varepsilon_n(1,\beta) \neq 0 \text{ in } \varepsilon(1,\beta) \\ \big(\varepsilon_1(1,\beta), \cdots, (\varepsilon_n(1,\beta) - 1)\big)^\infty & \text{otherwise, where } \varepsilon_n(1,\beta) \text{ is the last non-zero element in } \varepsilon(1,\beta). \end{cases}$$

- (Parry, 1960) $\omega \in \Sigma_\beta$ if and only if

$$\sigma^k(\omega) \prec \varepsilon^*(1,\beta) \quad \text{for all} \quad k \geq 0,$$

where $\prec$ means the lexicographical order.

Blanchard's classification (1989)

- C_1: $\varepsilon(1,\beta)$ is finite (simple beta-number). That is, the orbit of 1 under T_β eventually goes to 0.
- (Parry, 1960) The set of simple beta-numbers is dense in $(1,+\infty)$.
- S_β is an SFT iff β is a simple beta-number.
- C_2: $\varepsilon(1,\beta)$ is ultimately periodic (beta-number) but not finite, that is, $\varepsilon^*(1,\beta)$ is ultimately periodic but not purely periodic.
- C_3: all string of 0's in $\varepsilon^*(1,\beta)$ are bounded but $\varepsilon(1,\beta)$ is not ultimately periodic.
- C_4: $\varepsilon(1,\beta)$ does not contain some admissible word, but contains string of 0's with unbounded length.
- C_5: $\varepsilon(1,\beta)$ contains all admissible words, that is, $O_\beta(1)$ is dense in $[0,1]$.

a kind of classification of $\beta > 1$

- $\ell_n(1,\beta) := \max\{\, k \geq 0 : \varepsilon^*_{n+1}(1,\beta) = \cdots = \varepsilon^*_{n+k}(1,\beta) = 0 \,\}$
- $\ell(1,\beta) = \limsup\limits_{n\to\infty} \frac{\ell_n(1,\beta)}{n}$
- A kind of classification of $\beta > 1$:

$$A_0 = \Big\{\, \beta > 1 : \{\ell_n(1,\beta)\} \text{ is bounded } \Big\};$$
$$A_1 = \Big\{\, \beta > 1 : \{\ell_n(1,\beta)\} \text{ is unbounded and } \ \ell(1,\beta) = 0 \,\Big\};$$
$$A_2 = \Big\{\, \beta > 1 : \ell(1,\beta) > 0 \,\Big\}.$$

Theorem (L. and Wu, 2008, JMAA)

(1) $\beta \in A_0 \Longleftrightarrow C\beta^{-n} \leq |I_n(x)| \leq \beta^{-n}$ *for any* $x \in [0,1]$ *and* $n \geq 1$, *where* C *is a constant.*

(2) $\beta \in A_0 \cup A_1 \Longleftrightarrow \lim\limits_{n\to\infty} -\frac{\log|I_n(x)|}{n} = \log\beta$ *for any* $x \in [0,1]$.

Sizes of A_0, A_1, A_2

Theorem (L., Persson, Wang and Wu, 2014, Math. Z.)

(1) $\mathcal{L}(A_0) = 0$ *and* $\dim_H(A_0) = 1$ *(already known by Schmeling, 1997).*

(2) The set A_1 *is of full Lebesgue measure.*

(3) $\mathcal{L}(A_2) = 0$ *and* $\dim_H(A_2) = 1$.

The distribution of orbits and some fractal problems

Exceptional set

Recall: For $\mathcal{L}$-almost all $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} \log_\beta(x - \omega_n(x)) = -1.$$

Further question: Is there any point with other approximation orders ? If yes, how large is the set of such points. That is, what is the size of the set

$$\left\{ x \in [0,1) : \lim_{n\to\infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) = -1 \right\},$$

where ϕ is a positive function defined on $\mathbb{N}$ satisfying $\phi(n) \to \infty$ as $n \to \infty$.

Denote $\eta = \liminf_{n\to\infty} \phi(n)/n$.

Proposition (Fang, Wu and L., 2020, Math. Z.)

Let $\beta > 1$ be a real number. If $\eta > 1$, then the set

$$\left\{ x \in [0,1) : \limsup_{n\to\infty} \frac{1}{\phi(n)} \log_\beta (x - \omega_n(x)) = -1 \right\}$$

is at most countable.

Dimensional result

Recall $\eta = \liminf_{n\to\infty} \phi(n)/n$.

Theorem (Fang, Wu and L., 2020, Math. Z.)

Let $\beta > 1$ be a real number. Additionally assume that ϕ is nondecreasing and denote

$$A_\phi := \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) = -1 \right\}.$$

Then

(i) If $0 \le \eta < 1$, then A_ϕ is empty and hence $\dim_H A_\phi = 0$.

(ii) If $\eta \ge 1$, then $\dim_H A_\phi = 1/\eta$.

A special case: level set

Let $\beta > 1$ be a real number. For any $\alpha \in \mathbb{R}$, we define

$$A(\alpha) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{1}{n} \log_\beta (x - \omega_n(x)) = -\alpha \right\}.$$

Corollary

$$\dim_H A(\alpha) = \begin{cases} \alpha^{-1}, & \textit{if } \alpha \geq 1; \\ 0, & \textit{otherwise.} \end{cases}$$

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Corollary

$$\dim_H A(\alpha) = \begin{cases} \alpha^{-1}, & \textit{if } \alpha \geq 1; \\ 0, & \textit{otherwise.} \end{cases}$$

Notice that

$$x - \omega_n(x) = \frac{T_\beta^n x}{\beta^n}$$

Periodic orbits

- the orbit of $x \in [0,1)$ under β-transformation: $O_\beta(x) = \{T_\beta^n(x)\}_{n\geq 0}$

$$x \to T_\beta(x) \to T_\beta^2(x) \to \cdots \to T_\beta^n(x) \to T_\beta^{n+1}(x) \to \cdots$$

- eventually periodic $\mathrm{Per}(\beta) = \{x \in [0,1) : O_\beta(x) \text{ is finite}\}$.

$$\mathrm{Per}(\beta) \subset \mathbb{Q}(\beta) \cap [0,1)$$

- (Schmidt, 1980)

  $\mathbb{Q} \cap [0,1) \subset \mathrm{Per}(\beta) \Longrightarrow \beta$ is either a Pisot number or Salem number
- (Bertrand 1977, Schmidt, 1980)

$$\beta \text{ is a Pisot number } \Longrightarrow \mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$$

- (Schmidt's conjecture, 1980)

  $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1) \Longrightarrow \beta$ is either a Pisot number or Salem number

Diophantine type problem

- dense orbit (L. and Chen, 2011) $D_\beta = \{x \in [0,1] : \overline{O_\beta(x)} = [0,1]\}$

  D_β is dense, of full Lebesgue measure and second category

  D_β^C is dense, of full Hausdorff dimension and first category
- well approximable set (shrinking target problem)

  $$W_y(T_\beta, \psi) = \{x \in [0,1) : |T_\beta^n(x) - y| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$
- (Coons, Hussain, Wang, 2016) $\mathcal{H}^f(W_y(T_\beta, \psi))$ is either zero or $\mathcal{H}^f([0,1])$ depending on the convergence of certain series
- badly approximable set (avoiding a point)

  $$A_y(T_\beta) = \{x \in [0,1) : y \notin \overline{O_\beta(x)}\}$$
- (Färm, Persson, Schmeling, 2010) $A_y(T_\beta)$ is α-winning $(0 < \alpha < 1/64)$ for Schmidt game, in particular, $\dim_{\mathrm{H}} A_y(T_\beta) = 1$.

Diophantine type problem

- Uniformly Diophantine approximation (Bugeaud and Liao, 2016)

Recurrence problem

- recurrent

$$\mathcal{L}\{x \in [0,1] : \liminf_{n\to\infty} |T_\beta^n(x) - x| = 0\} = 1$$

- quantitative recurrence

$$R(T_\beta, \psi) := \left\{x \in [0,1] : |T_\beta^n x - x| < \psi(n) \ \text{ for i.m. } n \in \mathbb{N}\right\}$$

metric result (Hussain, L., Simmons and Wang, preprint)

$$\mathcal{L}(R(T_\beta, \psi)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases}$$

dimensional result (Tan and Wang, 2011)

$$\dim_{\mathrm{H}} R(T_\beta, \psi) = \frac{1}{1+b} \text{ with } b = \liminf_{n\to\infty} \frac{-\log_\beta \psi(n)}{n}$$

Covering problem

- covering set

$$E(x) = \{y \in [0,1) : |T_\beta^n(x) - y| < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

- (Hu and L., 2020, SPL) $\mathcal{L}$-almost every $x \in [0,1)$,

$$\mathcal{L}(E(x)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases}$$

$$\dim_{\mathrm{H}} E(x) = \inf\{s > 0 : \sum_{n=1}^{\infty} \psi^s(n) < \infty\} = \limsup_{n \to \infty} \frac{\log n}{-\log \psi(n)}$$

Chaotic problem

- Li-Yorke chaos: $([0,1),T_\beta)$, uncountable scrambled set
- scrambled set: $\forall x,y\in S$

$$\liminf_{n\to\infty}|T_\beta^n(x)-T_\beta^n(y)|=0,,\limsup_{n\to\infty}|T_\beta^n(x)-T_\beta^n(y)|>0$$

- (Liu, Huang, Li and Wang, 2018): there exists a scrambled set in $[0,1)$ whose Hausdorff dimension is 1.

Some other fractal problems

- frequency set
- level set for the lengths of cylinders
- multifractal (Birkhoff average)
- first return time/recurrence rate
- parameter space
- $\cdots\cdots$

Key proposition: exponentially mixing

Proposition (Philipp, 1967; Aaronson and Nakada, 2005)

There exist the constants $C_\beta > 0$ (only depending on β) and $\beta^{-1} \leq \rho < 1$ such that

$$\left| \mu_\beta(E \cap T_\beta^{-(n+k)} F) - \mu_\beta(E)\mu_\beta(F) \right| \leq C_\beta \rho^n \mu_\beta(F)$$

holds for any $E \in \mathcal{B}_k$, $F \in \mathcal{B}$ and $k, n \in \mathbb{N}$, where $\mathcal{B}_k$ denotes the σ-algebra generated by the k-th cylinders of β-expansions and $\mathcal{B}$ is the Borel σ-algebra on $[0, 1)$.

Remark: In their paper, the constant ρ just satisfies $0 < \rho < 1$. However, using the result of Keller (i.e., Theorem 1, *Comm. Math. Phys.*, 1984), we can give a lower bound to ρ by following the original proofs of Aaronson and Nakada.

Key proposition: Approximation method for the β-shift (Persson and Schmeling, 2008; Tan and Wang, 2011)

- projection function $\pi_\beta : \mathcal{S}_\beta \longrightarrow [0,1]$

$$\pi_\beta(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{\beta^i}$$

- Let $1 < \beta' < \beta$ such that $S_{\beta'}$ is SFT.
- $H_\beta^{\beta'} := \pi_\beta(\Sigma_{\beta'})$ is a Cantor set of $\pi_\beta(\Sigma_\beta) = [0,1)$.
- Define the function $h : H_\beta^{\beta'} \longrightarrow [0,1)$ as

$$h(x) = \pi_{\beta'}(\varepsilon(x,\beta)).$$

Proposition (Ban and L., 2014)

(i) For any $x \in H_\beta^{\beta'}$, we have $\varepsilon(h(x), \beta') = \varepsilon(x, \beta)$.

(ii) The function h is bijective and strictly increasing on $H_\beta^{\beta'}$.

(iii) The function h is continuous on $H_\beta^{\beta'}$.

(iv) Let $\beta' \in A_0$ with $M = \sup\{\ell_n(1, \beta') : n \geq 1\}$. Then h is Hölder continuous on $H_\beta^{\beta'}$. More precisely,

$$|h(x) - h(y)| \leq \beta'^{M+2} |x - y|^{\frac{\log \beta'}{\log \beta}}$$

for any $x, y \in H_\beta^{\beta'}$.

Key proposition: distribution of full cylinders

- cylinder of order n ($(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \Sigma_\beta^n$)

$$I_n(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = \{x \in [0,1) : \varepsilon_k(x) = \varepsilon_k, 1 \le k \le n\}$$

- full cylinder

$$\big|I_n(w_1, \cdots, w_n)\big| = \beta^{-n}$$

- $I_n(w)$ is a full cylinder if and only if (w, w') is admissible for all $w' \in \Sigma_\beta^*$
- (Bugeaud and Wang, 2014) Every $n+1$ consecutive cylinders of order n contains at least a full cylinder.
- (Li and L., 2018, JNT) finer structure of the distribution of full cylinders

Intermediate β-expansions

Intermediate β-transformations

Let $1 < \beta \leq 2$ and $0 \leq \alpha \leq 2-\beta$. $T_{\beta,\alpha} : [0,1] \circlearrowleft$

$$T_{\beta,\alpha}(x) = \beta x + \alpha \pmod{1}$$

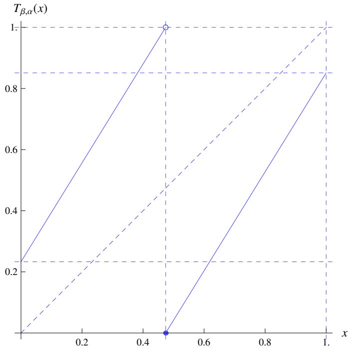

Figure: Plot of $T^+_{\beta,\alpha}$ for $\beta = (\sqrt{5}+1)/2$ and $\alpha = 1 - 0.474\beta$

We refer to the set

$$\Delta := \left\{(\beta, \alpha) \in \mathbb{R}^2 \colon \beta \in (1,2) \text{ and } 0 \leq \alpha \leq 2-\beta\right\}$$

as the parameter space for the set of intermediate β-transformations.

Fix $(\beta, \alpha) \in \Delta$ and let $p = p_{\beta,\alpha} := (1-\alpha)/\beta$. The maps $T^{\mp}_{\beta,\alpha} \colon [0,1] \circlearrowleft$ defined by

$$T^{-}_{\beta,\alpha}(p) := 1 \quad \text{and} \quad T^{-}_{\beta,\alpha}(x) := \beta x + \alpha \bmod 1,$$

and

$$T^{+}_{\beta,\alpha}(p) := 0 \quad \text{and} \quad T^{+}_{\beta,\alpha}(x) := \beta x + \alpha \bmod 1,$$

for all $x \in [0,1] \setminus \{p\}$, are, respective, called the lower and upper intermediate β-transformations.

Intermediate β-shifts

The $T^-_{\beta,\alpha}$-expansion $\tau^-_{\beta,\alpha}(x)$, of a point x with respect to $T^-_{\beta,\alpha}$ is defined to be the infinite word $\omega^- = (\omega^-_1, \omega^-_2, \ldots,) \in \{0,1\}^{\mathbb{N}}$, where

$$\omega^-_n := \begin{cases} 0, & \text{if } (T^-_{\beta,\alpha})^{n-1}(x) \leq p, \\ 1, & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, where we recall that $p = (1-\alpha)/\beta$.

Intermediate β-shifts

The $T^+_{\beta,\alpha}$-expansion $\tau^+_{\beta,\alpha}(x)$, of a point x with respect to $T^+_{\beta,\alpha}$ is defined to be the infinite word $\omega^+ = (\omega^+_1, \omega^+_2, \ldots,) \in \{0,1\}^{\mathbb{N}}$, where

$$\omega^+_n := \begin{cases} 0, & \text{if } (T^+_{\beta,\alpha})^{n-1}(x) < p, \\ 1, & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, where we recall that $p = (1-\alpha)/\beta$.

Intermediate β-shifts

The $T^+_{\beta,\alpha}$-expansion $\tau^+_{\beta,\alpha}(x)$, of a point x with respect to $T^+_{\beta,\alpha}$ is defined to be the infinite word $\omega^+ = (\omega^+_1, \omega^+_2, \ldots,) \in \{0,1\}^{\mathbb{N}}$, where

$$\omega^+_n := \begin{cases} 0, & \text{if } (T^+_{\beta,\alpha})^{n-1}(x) < p, \\ 1, & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$, where we recall that $p = (1-\alpha)/\beta$.

The intermediate β-shift associated to a point $(\beta, \alpha) \in \Delta$ is defined to be the space $\Omega_{\beta,\alpha}$ given by

$$\Omega_{\beta,\alpha} = \tau^-_{\beta,\alpha}([0,1]) \cup \tau^+_{\beta,\alpha}([0,1]).$$

The upper and lower kneading invariants of $T^\pm_{\beta,\alpha}$ are defined to be the infinite words $\tau^\pm_{\beta,\alpha}(p)$, respectively.

Theorem (Parry, 1960)

For $\beta \in (1,2)$, we have that

1. *the greedy β-shift is a subshift of finite type if and only if $\tau^{-}_{\beta,0}(p)$ is periodic and*
2. *the lazy β-shift is a subshift of finite type if and only if $\tau^{+}_{\beta,2-\beta}(p)$ is periodic.*

Theorem (L., Sahlsten and Samuel, 2016, DCDS)

Let $(\beta,\alpha) \in \Delta$ with $\alpha \in (0, 2-\beta)$ and let $p = (1-\alpha)/\beta$. The intermediate β-shift $\Omega_{\beta,\alpha}$ is SFT if and only if both $\tau^{\pm}_{\beta,\alpha}(p)$ are periodic.

Denseness

Theorem (L., Sahlsten, Samuel and Steiner, 2019, PAMS)

The set of $(\beta, \alpha) \in \Delta$ such that the corresponding $\Omega_{\beta,\alpha}$ is SFT is dense in Δ.

Thanks for your attention!