Simultaneous Diophantine approximation of the orbits of the dynamical systems $x_2$ and $x_3$
Plan

1. Diophantine approximation and shrinking target problem
2. Simultaneous Dioph. approx. of dynamical systems x2 and x3
3. Ideas and proofs
4. Shrinking target problem for matrix transformations of tori
Diophantine approximation
and shrinking target problem
I. Diophantine approximation

Denote by $\| \cdot \|$ the distance to the nearest integer.

- **Corollary of Dirichlet Theorem 1842** or
  a property of continued fraction (**Legendre 1808**):
  $$\{ \theta \in \mathbb{R} : \|n\theta\| < n^{-1} \text{ for infinitely many } n \} = \mathbb{R}.$$

- **Khintchine 1924**: Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be such that $n \mapsto n \psi(n)$ is decreasing. Then,
  $$\text{Leb}\left(\{ \theta : \|n\theta\| < \psi(n) \text{ i.m. } n \}\right) = \begin{cases} 0 & \text{if } \sum \psi(n) < \infty, \\ \text{full} & \text{if } \sum \psi(n) = \infty. \end{cases}$$

- **Duffin–Schaefer 1941** conjecture, and
  the proof of **Koukoulopoulos–Maynard 2020**.

- **Jarník 1929, Besicovith 1934**: For $v > 1$,
  $$\dim_H \{ \theta : \|n\theta\| < n^{-v} \text{ i.m. } n \} = 2/(1 + v).$$
II. Recall of Hausdorff dimension

$s$-dimensional Hausdorff measure: for $E \subset \mathbb{R}^d$, $s > 0$,

$$
\mathcal{H}^s(E) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subset \bigcup_{i=1}^{\infty} U_i, |U_i| < \delta \right\}.
$$

Hausdorff dimension:

$$
\dim_H(E) := \inf \{ s > 0 : \mathcal{H}^s(E) = 0 \} = \sup \{ s > 0 : \mathcal{H}^s(E) = \infty \}.
$$

**Upper/lower bound estimation**

1. If we can find $\delta$-coverings $(U_i)_{i \geq 1}$ such that $\sum_{i=1}^{\infty} |U_i|^s \leq M$ then $\dim_H(E) \leq s$,

2. If we can find a measure $\mu$ supported on $E$ such that $\mu(U) \leq |U|^s$ for all balls $U$, then $\dim_H(E) \geq s$. 

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III. Dynamics and shrinking target problem

- **Borosh–Fraenkel 1972**:
  \[ \dim_H \{ x \in \mathbb{R} : \|2^n x\| < (2^n)^{-v} \text{ i.m. } n \} = 1/(1 + v). \]

- **Shen–Wang 2013**:
  For a \( \beta \)-transformation \( T_\beta : [0, 1) \to [0, 1) \) defined by \( T_\beta(x) := \beta x - \lfloor \beta x \rfloor \),
  \[ \dim_H \{ x : T_\beta^n(x) < (\beta^n)^{-v} \text{ i.m. } n \} = 1/(1 + v). \]

In general, let \((X, d)\) be a metric space and \( T : X \to X \) be a transformation. Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a decreasing function.

**Shrinking target problem (Hill–Velani 1995)**: for a fixed \( y \in X \), what is the size (measure, dimension) of
\[
W(\psi, y) := \{ x \in X : d(T^n x, y) < \psi(n) \text{ for infinitely many } n \}
= \{ x \in X : T^n x \in B(y, \psi(n)) \text{ for infinitely many } n \}
= \limsup_{n \to \infty} T^{-n} B(y, \psi(n)).
\]
IV. Some measure results

Let $\mu$ be an invariant measure. Then $\mu(T^{-n}B(y,\psi(n))) = \mu(B(y,\psi(n)))$.

Borel-Cantelli Lemma

$$\sum_{n \geq 1} \mu(B(y,\psi(n))) < \infty \implies \mu(W(\psi,y)) = 0.$$  

Dynamical Borel-Cantelli Lemma

$$\sum_{n \geq 1} \mu(B(y,\psi(n))) = \infty + \text{some condition} \implies \mu(W(\psi,y)) = \text{full}.$$  

- **Kuraweil 1955**: bounded type irrational rotation.
- **Philipp 1967**: $\beta$-transformation, Gauss transformation.
- **Kleinbock–Margulis 1999**: dynamics on homogeneous spaces.
- **Chernov–Kleinbock 2001**: Anosov diffeomorphisms and topological Markov chains, Gibbs measure.
- **Kim 2007**: expanding maps on the interval whose derivative has bounded variation.
V. Some dimension results

- **Hill–Velani 1999**: linear maps on $n$-dimensional torus.
- **Urbański 2002**: conformal IFS.
- **Allen–Bárány 2021**: Hausdorff measure, conformal IFS.
- **Shen–Wang 2013**: $\beta$-transformation.
- **Coons–Hussain–Wang 2016**: Hausdorff measure, $\beta$-transfor.
Simultaneous Diophantine approximation of dynamical systems $x_2$ and $x_3$
I. Simultaneous Diophantine approximation

Let $T_2$ and $T_3$ be two transformations defined on $[0, 1]$ by

$$T_2 x = 2x \mod 1, \quad \text{and} \quad T_3 x = 3x \mod 1.$$ 

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function such that $\psi(n) \to 0$ ($n \to \infty$).

We are interested in the set

$$E(\psi) := \{ x \in [0, 1] : T_2^n x < \psi(n), \quad T_3^n x < \psi(n) \text{ for infinitely many } n \}.$$ 

Attention, it is not the intersection

$$\left\{ x \in [0, 1] : T_2^n x < \psi(n) \text{ i.m. } n \right\} \cap \left\{ x \in [0, 1] : T_3^n x < \psi(n) \text{ i.m. } n \right\}.$$ 

Related things:

- **Furstenberg’s conjecture 1969**: for all $x \in [0, 1] \setminus \mathbb{Q}$,

$$\dim_H \{ T_2^n x : n \geq 0 \} + \dim_H \{ T_3^n x : n \geq 0 \} \geq 1.$$ 

- **Wu 2019**: the exceptional set of Furstenberg’s conjecture is of upper box dimension 0.
II. Our results

Recall

\[ E(\psi) := \{ x \in [0, 1] : T_2^nx < \psi(n), \ T_3^nx < \psi(n) \text{ for infinitely many } n \}. \]

**Theorem 1 (Li–L–Velani-Zorin, in preparation)**

\[
\text{Leb} (E(\psi)) = \begin{cases} 
0 & \text{if } \sum \psi(n)^2 < \infty, \\
\text{full} & \text{if } \sum \psi(n)^2 = \infty. 
\end{cases}
\]

**Theorem 2 (Li–L–Velani-Zorin, in preparation)**

Let \( \psi(n) = 3^{-n\tau} \) with \( 0 < \tau < 1 \).

1. If \( \tau < 1 - \frac{\log 2}{\log 3} \), \( \dim_H E(\psi) = \frac{1-\tau}{1+\tau} \).

2. If \( \tau \geq 1 - \frac{\log 2}{\log 3} \), admitting the abc conjecture, we have

\[
\dim_H E(\psi) = \frac{1 - \tau}{1 + \tau}.
\]
III. Generalizations -intersection with a curve -setting

Remark that the set

\[ E(\psi) := \{x \in [0, 1] : T_2^n x < \psi(n), \ T_3^n x < \psi(n) \text{ i.m. } n \} \]

is the intersection of the diagonal \( L := \{(x, x) : x \in [0, 1]\} \) of \([0, 1]^2\) with

\[ W(A, \psi) := \left\{ x \in [0, 1]^2 : \|A^n x\|_\infty < \psi(n) \text{ i.m. } n \right\}, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \]

→ In general we can study the intersection of a curve \( C \) with \( W(A, \psi) \).

- Assume \( C = C_f := \{(x, f(x)) : x \in [0, 1]\} \) with \( f : [0, 1] \to [0, 1] \).
- \( f \) is bi-Lipschitz on \( I \subset [0, 1] \) if \( \exists 0 < \kappa_1 \leq \kappa_2 < \infty \) s.t.

\[
\kappa_1 \leq \frac{|f(u) - f(v)|}{|u - v|} \leq \kappa_2 \quad \forall u, v \in I. 
\]

- Let \( S \) be a subset of \([0, 1]\). We say that \( f \in \mathcal{L}(S^c) \) if

\[ \forall x \in S^c, \exists \text{ open interval } I_x \text{ centred at } x \text{ s.t. } f \text{ is bi-Lipschitz on } I_x. \]
IV. Generalizations -intersection with a curve -results

Theorem 3 (Li–L–Velani-Zorin, in preparation)

Let \( f \in \mathcal{L}(S^c) \) with \( \text{Leb}(S) = 0 \). Then

\[
\text{Leb}(C_f \cap W(A, \psi)) = \begin{cases} 
0 & \text{if } \sum_{n=1}^{\infty} \psi(n)^2 < \infty, \\
1 & \text{if } \sum_{n=1}^{\infty} \psi(n)^2 = \infty.
\end{cases}
\]

Theorem 4 (Li–L–Velani-Zorin, in preparation)

Assume \( 0 \leq \lambda = \liminf_{n \to \infty} -\frac{\log \psi(n)}{n} \leq \log 3 \) and \( \dim_H S \leq \frac{\log 3 - \lambda}{\log 3 + \lambda} \). Then

\[
\dim_H (C_f \cap W(A, \psi)) \leq \frac{\log 3 - \lambda}{\log 3 + \lambda},
\]

and we have equality in (1) for \( 0 \leq \lambda \leq \log 3 - \log 2 \).

Moreover, if \( C_f \) is a line with rational slope then we also have equality in (1) for \( \log 3 - \log 2 < \lambda \leq \log 3 \), conditional on the validity of the abc-conjecture.
V. Generalizations -different targets

For \( y_1, y_2 \in [0, 1] \) and two decreasing positive function \( \varphi, \psi \), we study

\[
G(\varphi, \psi, y_1, y_2) = \{ x \in [0, 1] : |T_2^n x - y_1| < \varphi(n), |T_3^n x - y_2| < \psi(n) \text{ i.m. } n \}.
\]

Denote

\[
\overline{s}(\varphi) := \limsup_{n \to \infty} \frac{1}{1 - \frac{\log \varphi(n)}{n \log 2}}, \quad \text{and} \quad \underline{s}(\varphi) := \liminf_{n \to \infty} \frac{1}{1 - \frac{\log \varphi(n)}{n \log 2}}.
\]

\[
\overline{\delta}(\varphi, \psi) = \limsup_{n \to \infty} \frac{1 + \frac{\log \varphi(n)}{n \log 3}}{1 - \frac{\log \psi(n)}{n \log 3}}, \quad \text{and} \quad \underline{\delta}(\varphi, \psi) = \liminf_{n \to \infty} \frac{1 + \frac{\log \varphi(n)}{n \log 3}}{1 - \frac{\log \psi(n)}{n \log 3}}.
\]

**Theorem 5 (Li–L–Velani-Zorin, in preparation)**

(1)

\[
\text{Leb} \left( G(\varphi, \psi, y_1, y_2) \right) = \begin{cases} 
0 & \text{if } \sum \varphi(n)\psi(n) < \infty, \\
1 & \text{if } \sum \varphi(n)\psi(n) = \infty.
\end{cases}
\]

(2) Suppose \( \limsup_{n \to \infty} -\frac{\log \varphi(n)}{n \log 2} < \frac{\log 3}{\log 2} - 1 \). Then

\[
\max \left\{ \min \{\overline{\delta}(\varphi, \psi), \underline{s}(\varphi)\}, \min \{\underline{\delta}(\varphi, \psi), \overline{s}(\varphi)\} \right\} \leq \dim_H G(y_1, y_2, \varphi, \psi) \leq \min \{\overline{\delta}(\varphi, \psi), \overline{s}(\varphi)\}.
\]
Ideas and proofs
I. Ideas for the measure result

Put
\[ A_n := \{ x \in [0, 1] : \|2^n x\| < \psi(n), \|3^n x\| < \psi(n) \}. \]

Then
\[ E(\psi) = \limsup_{n \to \infty} A_n. \]

Suppose \( \sum_{n=1}^{\infty} \psi(n)^2 = \infty \). We prove the quasi-independance in average: for all \( N \in \mathbb{N} \),
\[ \sum_{m=1}^{N} \sum_{n=m+1}^{N} \text{Leb}(A_m \cap A_n) \leq C \cdot \left( \sum_{n=1}^{N} \psi(n)^2 \right)^2. \]

Then apply a general version of Borel-Cantelli
\[ \text{Leb}(\limsup_{n \to \infty} A_n) \geq \limsup_{N \to \infty} \frac{\left( \sum_{n=1}^{N} \text{Leb}(A_n) \right)^2}{\sum_{m=1}^{N} \sum_{n=1}^{N} \text{Leb}(A_m \cap A_n)}. \]
II. Upper bound of the dimension

Natural covering: \( E(\psi) \subset \bigcup_{n=N}^{\infty} A_n \), with

\[
A_n = \bigcup_{s,t} \left\{ x : \left| x - \frac{s}{2^n} \right| < \frac{\psi(n)}{2^n}, \left| x - \frac{t}{3^n} \right| < \frac{\psi(n)}{3^n} \right\}.
\]

→ The diameter of each small interval of \( A_n \):

\[
\left| \left\{ x : \left| x - \frac{s}{2^n} \right| < \frac{\psi(n)}{2^n}, \left| x - \frac{t}{3^n} \right| < \frac{\psi(n)}{3^n} \right\} \right| \leq \frac{2\psi(n)}{3^n}.
\]

→ We have at most \( 2 \cdot 3^n \psi(n) \) such intervals, since

- for each such small interval, we have

\[
\left| \frac{s}{2^n} - \frac{t}{3^n} \right| < \frac{\psi(n)}{2^n} + \frac{\psi(n)}{3^n} \Rightarrow \left| 3^n s - 2^n t \right| < 3^n \psi(n) + 2^n \psi(n) < 2 \cdot 3^n \psi(n);
\]

- \( 3^n \) and \( 2^n \) are coprime, \( \exists \) unique \( s_0 \) et \( t_0 \) s.t. \( 3^n s_0 - 2^n t_0 = 1 \), and then unique \( s_k \) and \( t_k \) s.t. \( 3^n s_k - 2^n t_k = k \).

So,

\[
\mathcal{H}^{\alpha}(E(\psi)) \leq \sum_{n=N}^{\infty} 2 \cdot 3^n \psi(n) \cdot \left( \frac{2\psi(n)}{3^n} \right)^\alpha \rightarrow 0, \text{ if } \alpha > \frac{1 - \tau}{1 + \tau}.
\]
III. Lower bound - construction of a Cantor type set

Method for part (1): Construct a Cantor type subset and distribute a measure.

- Take \( \{n_k\} \), s.t. \( n_{k+1} \gg n_k \). Put
  \[
i_k = \lfloor -\log_2 \psi(n_k) \rfloor, \quad j_k = \lfloor -\log_3 \psi(n_k) \rfloor.
\]

- In each dyadic interval \( I(\varepsilon_1, \ldots, \varepsilon_{n_1}) \) (\( \varepsilon_i \in \{0, 1\} \)), find a dyadic subinterval \( I(\varepsilon_1, \ldots, \varepsilon_{n_1}, 0^{i_1}) \).

- For each \( I(\varepsilon_1, \ldots, \varepsilon_{n_1}, 0^{i_1}) \), find triadic subintervals \( J(a_1, \ldots, a_{n_1}) \).
  (Here, we use the condition \( \tau < 1 - \frac{\log 2}{\log 3} \) to guarantee the existence of such triadic subintervals).

- In each \( J(a_1, \ldots, a_{n_1}) \), find a triadic subinterval \( J(a_1, \ldots, a_{n_1}, 0^{j_1}) \).

- Set \( J^{(1)} \) the union of these obtained triadic subintervals.

- We continue to find dyadic subintervals \( I(\varepsilon_1, \ldots, \varepsilon_{n_2}) \) in each interval of \( J \in J^{(1)} \). Do the same thing, we will have \( J^{(2)} \)....

- The wanted Cantor type set is \( \bigcap_{k=1}^{\infty} J^{(k)} \).
IV. Mass transference principle

We have a useful tool for estimating the lower bound of a limsup set.

**Mass transference principle (Beresnevich–Velani 2006)**

Let $B(x_n, r_n)$ be a sequence of balls in $I \subset \mathbb{R}$. Let $0 < \alpha < 1$. Then,

$$Leb(\limsup B(x_n, r_n^\alpha)) = \text{full} \implies \mathcal{H}^\alpha(\limsup B(x_n, r_n)) = \infty.$$ 

Consequently, $\dim_H \limsup B(x_n, r_n) \geq \alpha$.

**Remark**: the lower bound is often the correct one when $\{x_n\}$ are equi-distributed.

Recall that $E(\psi) = \limsup A_n$, with

$$A_n = \bigcup_{s,t} \left\{ x : \left| x - \frac{s}{2^n} \right| < \frac{\psi(n)}{2^n}, \left| x - \frac{t}{3^n} \right| < \frac{\psi(n)}{3^n} \right\}.$$

We need to study the distribution of the points $\frac{s}{2^n}$ with

$$\left| 3^n s - 2^n t \right| < 2 \cdot 3^n \psi(n).$$

They are $\frac{s_k}{2^n}$, with $s_k = k s_0 \mod 2^n$, where $s_0$ satisfies

$$3^n s_0 - 2^n t_0 = 1,$$

and

$$1 \leq k \leq 2 \cdot 3^n \psi(n) = 2 \cdot 3^n(1-\tau).$$
V. Application of the abc conjecture

**Abc conjecture**

\[ \forall \varepsilon > 0, \exists K_\varepsilon \text{ s.t. } \forall (a, b, c) \in \mathbb{Z} \text{ being coprime and satisfying } a + b = c, \]

\[ \max(|a|, |b|, |c|) \leq K_\varepsilon \cdot (\text{rad}(abc))^{1+\varepsilon} \]

where \( \text{rad}(n) \) is the product of the prime numbers dividing \( n \).

- By **Three Distance Theorem**, the \( 2 \cdot 3^{n(1-\tau)} \) points \( \frac{s_k}{2^n} \) are distributed in blocks with distance \( \ell_1 < 3^{-n(1-\tau)} \) in block and distance \( \ell_2 \) or \( \ell_3 = \ell_1 + \ell_2 \) between blocks.

- Let \( B \) be the number of blocks. Then the two points \( \frac{s_0}{2^n} \) and \( \frac{(B+1)s_0-2^n u}{2^n} \) (for certain integer \( u \)) are neighbours, separated by \( \ell_1 \).

- Applying the abc conjecture for

\[ a = \pm B, \quad b = 2^n (Bt_0 - 3^n u), \quad c = 3^n (Bs_0 - 2^n u), \]

we have \( \varepsilon > 0, \quad B > 3^{n(1-\tau)(1-\varepsilon)}. \) (\( \tau \geq 1 - \frac{\log 2}{\log 3} \) is needed.)

- Taking balls of radius \( 3^{-n(1-\tau)(1-\varepsilon)} \) for our \( 2 \cdot 3^{n(1-\tau)} \) points, we can cover \([0, 1]\). Then apply the mass transference principle to conclude: enlarge the radius \( 3^{-n(1+\tau)} \) to radius \( 3^{-n(1-\tau)(1-\varepsilon)} \).
Shrinking target problem
for matrix transformations of tori
I. Setting

Recall that the set

$$E(\psi) := \{x \in [0, 1] : T_2^nx < \psi(n), \ T_3^nx < \psi(n) \ i.m. \ n\}$$

is the intersection of the diagonal of $[0, 1]^2$ with

$$\{x \in [0, 1]^2 : \|A^nx\|_\infty < \psi(n) \ i.m. \ n\}, \ A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$  

→ The shrinking target problems on $\mathbb{T}^d$ are not totally solved:

- $\mathbb{T}^d : d$-dimensional torus, with Lebesgue measure $m_d$.
- $T : d \times d$ non-singular matrix with real coefficients.
- $T$ defines a transformation: $x \mapsto Tx \ (mod \ 1)$.
- $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of subsets (as targets) in of $\mathbb{T}^d$.

We are interested in

$$W(T, \{E_n\}) := \limsup_{n \to \infty} T^{-n}(E_n)$$

$$= \{x \in \mathbb{T}^d : T^n(x) \in E_n \ for \ infinitely \ many \ n \in \mathbb{N}\}.$$
II. Results

Let $C$ be a collection of subsets $E$ of $\mathbb{T}^d$ satisfying the bounded property

\[(B) : \sup_{E \in C} M^{* (d-1)}(\partial E) < \infty,\]

where $\partial E$ is the boundary of $E$ and $M^{* s}(\cdot)$ is the upper Minkowski content defined by

$$M^{* s}(A) := \limsup_{\epsilon \to 0^+} \frac{m_d(\{x \in \mathbb{T}^d : \text{dist}(x, A) < \epsilon\})}{\epsilon^{d-s}}.$$ 

**Theorem 6 (Li–L–Velani-Zorin, in preparation)**

Suppose that all eigenvalues of $T$ are of modulus strictly larger than 1. Let $\mu$ be an absolute continuous invariant measure (acim) with support $\mathbb{T}^d$ and mixing. Then for any sequence $\{E_n\}_{n \in \mathbb{N}}$ of subsets in $C$,

$$m_d(W(T, \{E_n\})) = \mu(W(T, \{E_n\})) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(E_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(E_n) = \infty. \end{cases}$$
III. Results - continued

Let $\beta \in \mathbb{R}, \ |\beta| > 1$. Study the (negative) $\beta$-transformation on $[0, 1]$:

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \ (\beta > 1) \quad \text{or} \quad T_\beta(x) = \beta x + \lfloor -\beta x \rfloor + 1 \ (\beta < -1).$$

- $\mu_\beta$ : acim (Parry measure, or Yrrap measure).
- $K(\beta)$ : support of $\mu_\beta$.

**L–Steiner** : let $g$ be the golden mean, then

$K(\beta) = [0, 1] \text{ if } \beta \in (-\infty, -g] \cup (1, +\infty),$

$K(\beta)$ is a finite union of closed intervals if $\beta \in (-g, -1)$.

- $T = \text{diag}(\beta_1, \ldots, \beta_d)$ with $|\beta_i| > 1$.
- $\mu = \mu_{\beta_1} \times \mu_{\beta_2} \times \cdots \times \mu_{\beta_d}, \ K = \prod_{i=1}^d K(\beta_i)$.

**Theorem 7 (Li–L–Velani-Zorin, in preparation)**

For any sequence $\{E_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}$, we have

$$m_d|_K\left(W(T, \{E_n\})\right) = \mu\left(W(T, \{E_n\})\right) = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty \mu(E_n) < \infty \\ 1 & \text{if } \sum_{n=1}^\infty \mu(E_n) = \infty. \end{cases}$$
IV. Corollary

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real positive function and $a \in \mathbb{T}^d$. Consider the targets

$$R_n := R(a, \psi(n)) = \left\{ x \in \mathbb{T}^d : \max_{1 \leq i \leq d} |a_i - x_i| \leq \psi(n) \right\}.$$ 

We are interested in

$$W(T, \psi, a) = \left\{ x \in \mathbb{T}^d : T^n(x) \in R(a, \psi(n)) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$ 

Corollary (Li–L–Velani–Zorin, in preparation)

Let $T$ be a real, non-singular matrix transformation of the torus $\mathbb{T}^d$. Suppose that $T$ is diagonal with all eigenvalues in $(-\infty, -g] \cup (1, +\infty)$. Then

$$m_d(W(T, \psi, a)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n)^d < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n)^d = \infty. \end{cases}$$
V. A dimension result

Let $T = \text{diag}(\beta_1, \ldots, \beta_d)$ with $\beta_i > 1$.

For $a \in \mathbb{T}^d$ and function $\psi$, let $W(T, \psi, a)$ be as before:

$$W(T, \psi, a) = \{x \in \mathbb{T}^d : T^n(x) \in R(a, \psi(n)) \text{ for infinitely many } n \in \mathbb{N}\}.$$

**Theorem 8 (Li–L–Velani-Zorin, in preparation)**

Let $1 < e_1 \leq e_2 \leq \cdots \leq e_d$ be a reordering of $\beta_1, \ldots, \beta_d$. Then

$$\dim_H W(T, \psi, a) = \min_{i=1,\ldots,d} \left\{ \frac{i \log e_i - \sum_{j : e_j > e_i} (\log e_j - \log e_i - \lambda) + \sum_{j > i} \log e_j}{\lambda + \log e_i} \right\},$$

where

$$\lambda = \lambda(\psi) = \liminf_{n \to \infty} -\frac{\log \psi(n)}{n}.$$