Simultaneous Diophantine approximation of the orbits of the dynamical systems x2 and x3

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Plan

Diophantine approximation and shrinking target problem

I. Diophantine approximation

Denote by $\|\cdot\|$ the distance to the nearest integer.

- Corollary of **Dirichlet Theorem 1842** or
  a property of continued fraction (**Legendre 1808**) :

$$\left\{\theta \in \mathbb{R} : \ \|n\theta\| < n^{-1} \text{ for infinitely many } n\right\} = \mathbb{R}.$$

- **Khintchine 1924** : Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be such that $n \mapsto n\psi(n)$ is decreasing. Then,

$$Leb\Big(\{\theta : \|n\theta\| < \psi(n) \text{ i.m. } n\}\Big) = \begin{cases} 0 & \text{if } \sum \psi(n) < \infty, \\ \text{full} & \text{if } \sum \psi(n) = \infty. \end{cases}$$

- **Duffin–Schaefer 1941** conjecture, and
  the proof of **Koukoulopoulos–Maynard 2020**.

- **Jarník 1929, Besicovith 1934** : For $v > 1$,

$$\dim_H \left\{\theta : \ \|n\theta\| < n^{-v} \text{ i.m. } n\right\} = 2/(1+v).$$

II. Recall of Hausdorff dimension

s-dimensional Hausdorff measure : for $E \subset \mathbb{R}^d$, $s > 0$,

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subset \bigcup_{i=1}^{\infty} U_i, \ |U_i| < \delta \right\}.$$

Hausdorff dimension :

$$\dim_H(E) := \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\}.$$

Upper/lower bound estimation

(1) If we can find δ-**coverings** $(U_i)_{i\geq 1}$ such that $\sum_{i=1}^{\infty} |U_i|^s \leq M$ then $\dim_H(E) \leq s$,

(2) If we can find a **measure** μ supported on E such that $\mu(U) \leq |U|^s$ for all balls U, then $\dim_H(E) \geq s$.

III. Dynamics and shrinking target problem

- **Borosh–Fraenkel 1972** :

$$\dim_H \left\{x \in \mathbb{R} : \ \|2^n x\| < (2^n)^{-v} \text{ i.m. } n\right\} = 1/(1+v).$$

- **Shen–Wang 2013** : for β-transformation $T_\beta : [0,1) \to [0,1)$ defined by $T_\beta(x) := \beta x - \lfloor \beta x \rfloor$,

$$\dim_H \{x : T_\beta^n(x) < (\beta^n)^{-v} \text{ i.m. } n\} = 1/(1+v).$$

In general, let (X, d) be a metric space and $T : X \to X$ be a transformation. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function.

Shrinking target problem (**Hill–Velani 1995**) : for a fixed $y \in X$, what is the size (measure, dimension) of

$$\begin{aligned} W(\psi, y) &:= \{x \in X : d(T^n x, y) < \psi(n) \text{ for infinitely many } n\} \\ &= \{x \in X : T^n x \in B(y, \psi(n)) \text{ for infinitely many } n\} \\ &= \limsup_{n\to\infty} T^{-n} B(y, \psi(n)). \end{aligned}$$

IV. Some measure results

Let μ be an invariant measure. Then $\mu(T^{-n}B(y,\psi(n))) = \mu(B(y,\psi(n)))$.

Borel-Cantelli Lemma

$$\sum_{n\geq 1} \mu(B(y,\psi(n))) < \infty \Longrightarrow \mu(W(\psi,y)) = 0.$$

Dynamical Borel-Cantelli Lemma

$$\sum_{n\geq 1} \mu(B(y,\psi(n))) = \infty \ + \ \text{some condition} \Longrightarrow \mu(W(\psi,y)) = \text{full}.$$

- **Kuraweil 1955** : bounded type irrational rotation.
- **Philipp 1967** : β-transformation, Gauss transformation.
- **Kleinbock–Margulis 1999** : dynamics on homogeneous spaces.
- **Chernov–Kleinbock 2001** : Anosov diffeomorphisms and topological Markov chains, Gibbs measure.
- **Kim 2007** : expanding maps on the interval whose derivative has bounded variation.

V. Some dimension results

- **Hill–Velani 1995, 1997** : expanding rational map on Julia set.
- **Hill–Velani 1999** : linear maps on n-dimensional torus.
- **Urbański 2002** : conformal IFS.
- **Allen–Bárány 2021** : Hausdorff measure, conformal IFS.
- **Shen–Wang 2013** : β-transformation.
- **Coons–Hussain–Wang 2016** : Hausdorff measure, β-transfor.
- **Li–Wang–Wu–Xu 2014** : continued fractions.
- **Bárány–Rams 2017** : diag. action on Bedford-McMullen carpets.

Simultaneous Diophantine approximation of dynamical systems x2 and x3

I. Simultaneous Diophantine approximation

Let T_2 and T_3 be two transformations defined on $[0,1]$ by

$$T_2 x = 2x \mod 1, \quad \text{and} \quad T_3 x = 3x \mod 1.$$

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function such that $\psi(n) \to 0$ $(n \to \infty)$.

We are interested in the set

$$E(\psi) := \Big\{x \in [0,1] : T_2^n x < \psi(n), \quad T_3^n x < \psi(n) \text{ for infinitely many } n\Big\}.$$

Attention, it is not the intersection

$$\Big\{x \in [0,1] : T_2^n x < \psi(n) \text{ i.m. } n\Big\} \bigcap \Big\{x \in [0,1] : T_3^n x < \psi(n) \text{ i.m. } n\Big\}.$$

Related things :

- Furstenberg's conjecture 1969 : for all $x \in [0,1] \setminus \mathbb{Q}$,

$$\dim_{\mathrm{H}} \overline{\{T_2^n x : n \geq 0\}} + \dim_{\mathrm{H}} \overline{\{T_3^n x : n \geq 0\}} \geq 1.$$

- Wu 2019 : the exceptional set of Furstenberg's conjecture is of upper box dimension 0.

II. Our results

Recall

$$E(\psi) := \{x \in [0,1] : T_2^n x < \psi(n), \ \ T_3^n x < \psi(n) \text{ for infinitely many } n\}.$$

Theorem 1 (Li–L–Velani-Zorin, in preparation)

$$Leb\,(E(\psi)) = \begin{cases} 0 & \text{if } \sum \psi(n)^2 < \infty, \\ \text{full} & \text{if } \sum \psi(n)^2 = \infty. \end{cases}$$

Theorem 2 (Li–L–Velani-Zorin, in preparation)

Let $\psi(n) = 3^{-n\tau}$ with $0 < \tau < 1$.

(1) If $\tau < 1 - \frac{\log 2}{\log 3}$, $\dim_H E(\psi) = \frac{1-\tau}{1+\tau}$.

(2) If $\tau \geq 1 - \frac{\log 2}{\log 3}$, admiting the abc conjecture, we have

$$\dim_H E(\psi) = \frac{1-\tau}{1+\tau}.$$

III. Generalizations -intersection with a curve -setting

Remark that the set

$$E(\psi) := \{x \in [0,1] : T_2^n x < \psi(n), \quad T_3^n x < \psi(n) \text{ i.m. } n\}$$

is the intersection of the diagonal $L := \{(x,x) : x \in [0,1]\}$ of $[0,1]^2$ with

$$W(A,\psi) := \left\{\mathbf{x} \in [0,1]^2 : \|A^n \mathbf{x}\|_\infty < \psi(n) \text{ i.m. } n\right\}, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

→ In general we can study the intersection of a curve $\mathcal{C}$ with $W(A,\psi)$.

- Assume $\mathcal{C} = \mathcal{C}_f := \{(x, f(x)) : x \in [0,1]\}$ with $f : [0,1] \to [0,1]$.
- f is bi-Lipschitz on $I \subset [0,1]$ if $\exists\ 0 < \kappa_1 \le \kappa_2 < \infty$ s.t.

$$\kappa_1 \le \frac{|f(u) - f(v)|}{|u - v|} \le \kappa_2 \qquad \forall\ u, v \in I\,.$$

- Let S be a subset of $[0,1]$. We say that $f \in \mathcal{L}(S^c)$ if

$\forall x \in S^c, \exists$ open interval I_x centred at x s.t. f is bi-Lipschitz on I_x.

IV. Generalizations -intersection with a curve -results

Theorem 3 (Li–L–Velani-Zorin, in preparation)

Let $f \in \mathcal{L}(S^c)$ with $Leb(S) = 0$. Then

$$Leb\big(\mathcal{C}_f \cap W(A,\psi)\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n)^2 < \infty\,, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n)^2 = \infty\,. \end{cases}$$

Theorem 4 (Li–L–Velani-Zorin, in preparation)

Assume $0 \le \lambda = \liminf_{n\to\infty} \frac{-\log\psi(n)}{n} \le \log 3$ and $\dim_H S \le \frac{\log 3 - \lambda}{\log 3 + \lambda}$. Then

$$\dim_H\big(\mathcal{C}_f \cap W(A,\psi)\big) \le \frac{\log 3 - \lambda}{\log 3 + \lambda}\,, \tag{1}$$

and we have equality in (1) for $0 \le \lambda \le \log 3 - \log 2$.

Moreover, if $\mathcal{C}_f$ is a line with rational slope then we also have equality in (1) for $\log 3 - \log 2 < \lambda \le \log 3$, conditional on the validity of the abc-conjecture.

V. Generalizations -different targets

For $y_1, y_2 \in [0,1]$ and two decreasing positive function φ, ψ, we study

$$G(\varphi,\psi,y_1,y_2) = \{x \in [0,1] : |T_2^n x - y_1| < \varphi(n),\ \ |T_3^n x - y_2| < \psi(n) \text{ i.m. } n\}.$$

Denote

$$\overline{s}(\varphi) := \limsup_{n\to\infty} \frac{1}{1 - \frac{\log\varphi(n)}{n\log 2}}, \quad \text{and} \quad \underline{s}(\varphi) := \liminf_{n\to\infty} \frac{1}{1 - \frac{\log\varphi(n)}{n\log 2}}.$$

$$\overline{\delta}(\varphi,\psi) = \limsup_{n\to\infty} \frac{1 + \frac{\log\varphi(n)}{n\log 3}}{1 - \frac{\log\psi(n)}{n\log 3}}, \quad \text{and} \quad \underline{\delta}(\varphi,\psi) = \liminf_{n\to\infty} \frac{1 + \frac{\log\varphi(n)}{n\log 3}}{1 - \frac{\log\psi(n)}{n\log 3}}.$$

Theorem 5 (Li–L–Velani-Zorin, in preparation)

(1)

$$Leb\,(G(\varphi,\psi,y_1,y_2)) = \begin{cases} 0 & \text{if } \sum \varphi(n)\psi(n) < \infty, \\ 1 & \text{if } \sum \varphi(n)\psi(n) = \infty. \end{cases}$$

(2) Suppose $\limsup_{n\to\infty} \frac{-\log\varphi(n)}{n\log 2} < \frac{\log 3}{\log 2} - 1$. Then

$$\max\left\{\min\{\overline{\delta}(\varphi,\psi),\ \underline{s}(\varphi)\}, \min\{\underline{\delta}(\varphi,\psi),\ \overline{s}(\varphi)\}\right\}$$
$$\le \dim_{\mathrm{H}} G(y_1,y_2,\varphi,\psi) \le \min\left\{\overline{\delta}(\varphi,\psi),\ \overline{s}(\varphi)\right\}.$$

Ideas and proofs

I. Ideas for the measure result

Put

$$A_n := \{x \in [0,1] : \|2^n x\| < \psi(n), \|3^n x\| < \psi(n)\}.$$

Then

$$E(\psi) = \limsup_{n\to\infty} A_n.$$

Suppose $\sum_{n=1}^{\infty} \psi(n)^2 = \infty$. We prove the quasi-independance in average : for all $N \in \mathbb{N}$,

$$\sum_{m=1}^{N} \sum_{n=m+1}^{N} Leb(A_m \cap A_n) \leq C \cdot \left(\sum_{n=1}^{N} \psi(n)^2 \right)^2.$$

Then apply a general version of Borel-Cantelli

$$Leb(\limsup_{n\to\infty} A_n) \geq \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N} Leb(A_n)\right)^2}{\sum_{m=1}^{N}\sum_{n=1}^{N} Leb(A_m \cap A_n)}.$$

II. Upper bound of the dimension

Natural covering : $E(\psi) \subset \bigcup_{n=N}^{\infty} A_n$, with

$$A_n = \bigcup_{s,t} \left\{ x : |x - \frac{s}{2^n}| < \frac{\psi(n)}{2^n}, |x - \frac{t}{3^n}| < \frac{\psi(n)}{3^n} \right\}.$$

$\rightarrow$ The diameter of each small interval of A_n :

$$\left| \left\{ x : |x - \frac{s}{2^n}| < \frac{\psi(n)}{2^n}, |x - \frac{t}{3^n}| < \frac{\psi(n)}{3^n} \right\} \right| \leq \frac{2\psi(n)}{3^n}.$$

$\rightarrow$ We have at most $2 \cdot 3^n \psi(n)$ such intervals, since

- for each such small interval, we have

$$|\frac{s}{2^n} - \frac{t}{3^n}| < \frac{\psi(n)}{2^n} + \frac{\psi(n)}{3^n} \Rightarrow |3^n s - 2^n t| < 3^n \psi(n) + 2^n \psi(n) < 2 \cdot 3^n \psi(n);$$

- 3^n and 2^n are coprime, $\exists$ unique s_0 et t_0 s.t. $3^n s_0 - 2^n t_0 = 1$, and then unique s_k and t_k s.t. $3^n s_k - 2^n t_k = k$.

So,

$$\mathcal{H}^{\alpha}(E(\psi)) \leq \sum_{n=N}^{\infty} 2 \cdot 3^n \psi(n) \cdot (\frac{2\psi(n)}{3^n})^{\alpha} \rightarrow 0, \text{ if } \alpha > \frac{1-\tau}{1+\tau}.$$

III. Lower bound - construction of a Cantor type set

Method for part (1) : Construct a Cantor type subset and distribute a measure.

- Take $\{n_k\}$, s.t. $n_{k+1} \gg n_k$. Put
$$i_k = \lfloor -\log_2 \psi(n_k) \rfloor, \; j_k = \lfloor -\log_3 \psi(n_k) \rfloor.$$
- In each dyadic interval $I(\varepsilon_1, \dots, \varepsilon_{n_1})$ ($\varepsilon_i \in \{0, 1\}$), find a dyadic subinterval $I(\varepsilon_1, \dots, \varepsilon_{n_1}, 0^{i_1})$.
- For each $I(\varepsilon_1, \dots, \varepsilon_{n_1}, 0^{i_1})$, find triadic subintervals $J(a_1, \dots, a_{n_1})$. (Here, we use the condition $\tau < 1 - \frac{\log 2}{\log 3}$ to guarantee the existence of such triadic subintervals).
- In each $J(a_1, \dots, a_{n_1})$, find a triadic subinterval $J(a_1, \dots, a_{n_1}, 0^{j_1})$.
- Set $J^{(1)}$ the union of these obtained triadic subintervals.
- We continue to find dyadic subintervals $I(\varepsilon_1, \dots, \varepsilon_{n_2})$ in each interval of $J \in J^{(1)}$. Do the same thing, we will have $J^{(2)}$....
- The wanted Cantor type set is $\bigcap_{k=1}^{\infty} J^{(k)}$.

IV. Mass transference principle

We have a useful tool for estimating the lower bound of a limsup set.

Mass transference principle (Beresnevich–Velani 2006)

Let $B(x_n, r_n)$ be a sequence of balls in $I \subset \mathbb{R}$. Let $0 < \alpha < 1$. Then,

$$Leb(\limsup B(x_n, r_n^\alpha)) = \text{full} \Longrightarrow \mathcal{H}^\alpha(\limsup B(x_n, r_n)) = \infty.$$

Consequently, $\dim_H \limsup B(x_n, r_n) \geq \alpha$.

Remark : the lower bound is often the correct one when $\{x_n\}$ are equi-distributed.

Recall that $E(\psi) = \limsup A_n$, with

$$A_n = \bigcup_{s,t} \left\{ x : |x - \tfrac{s}{2^n}| < \tfrac{\psi(n)}{2^n}, |x - \tfrac{t}{3^n}| < \tfrac{\psi(n)}{3^n} \right\}.$$

We need to study the distribution of the points $\frac{s}{2^n}$ with

$$|3^n s - 2^n t| < 2 \cdot 3^n \psi(n).$$

They are $\frac{s_k}{2^n}$, with $s_k = k s_0 \mod 2^n$, where s_0 satisfies

$$3^n s_0 - 2^n t_0 = 1, \quad \text{and} \quad 1 \leq k \leq 2 \cdot 3^n \psi(n) = 2 \cdot 3^{n(1-\tau)}.$$

V. Application of the abc conjecture

Abc conjecture

$\forall \varepsilon > 0$, $\exists K_\varepsilon$ s.t. $\forall (a,b,c) \in \mathbb{Z}$ being coprime and satisfying $a+b=c$,

$$\max(|a|,|b|,|c|) \le K_\varepsilon \cdot (\mathrm{rad}(abc))^{1+\varepsilon}$$

where $\mathrm{rad}(n)$ is the product of the prime numbers dividing n.

- By Three Distance Theorem, the $2 \cdot 3^{n(1-\tau)}$ points $\frac{s_k}{2^n}$ are distributed in blocks with distance $\ell_1 < 3^{-n(1-\tau)}$ in block and distance ℓ_2 or $\ell_3 = \ell_1 + \ell_2$ between blocks.
- Let B be the number of blocks. Then the two points $\frac{s_0}{2^n}$ and $\frac{(B+1)s_0 - 2^n u}{2^n}$ (for certain integer u) are neighbours, separated by ℓ_1.
- Applying the abc conjecture for

$$a = \pm B, \quad b = 2^n(Bt_0 - 3^n u), \quad c = 3^n(Bs_0 - 2^n u),$$

  we have $\epsilon > 0$, $\underline{B > 3^{n(1-\tau)(1-\epsilon)}}$. ($\tau \ge 1 - \frac{\log 2}{\log 3}$ is needed.)
- Taking balls of radius $3^{-n(1-\tau)(1-\epsilon)}$ for our $2 \cdot 3^{n(1-\tau)}$ points, we can cover $[0,1]$. Then apply the mass transference principle to conclude : enlarge the radius $3^{-n(1+\tau)}$ to radius $3^{-n(1-\tau)(1-\epsilon)}$.

Shrinking target problem for matrix transformations of tori

I. Setting

Recall that the set

$$E(\psi) := \{x \in [0,1] : T_2^n x < \psi(n), \quad T_3^n x < \psi(n) \text{ i.m. } n\}$$

is the intersection of the diagonal of $[0,1]^2$ with

$$\{\mathbf{x} \in [0,1]^2 : \|A^n \mathbf{x}\|_\infty < \psi(n) \text{ i.m. } n\}, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

→ The shrinking target problems on $\mathbb{T}^d$ are not totally solved :

- $\mathbb{T}^d$: d-dimensional torus, with Lebesgue measure m_d.
- T : $d \times d$ non-singular matrix with real coefficients.
- T defines a transformation : $\mathbf{x} \mapsto T\mathbf{x} \pmod 1$.
- $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of subsets (as targets) in of $\mathbb{T}^d$.

We are interested in

$$\begin{aligned} W\big(T, \{E_n\}\big) &:= \limsup_{n\to\infty} T^{-n}(E_n) \\ &= \{x \in \mathbb{T}^d : T^n(x) \in E_n \text{ for infinitely many } n \in \mathbb{N}\}. \end{aligned}$$

II. Results

Let $\mathcal{C}$ be a collection of subsets E of $\mathbb{T}^d$ satisfying the bounded property

$$(\mathbf{B}): \quad \sup_{E\in\mathcal{C}} M^{*(d-1)}(\partial E) < \infty,$$

where ∂E is the boundary of E and $M^{*s}(\cdot)$ is the upper Minkowski content defined by

$$M^{*s}(A) := \limsup_{\epsilon\to 0^+} \frac{m_d(\{x\in\mathbb{T}^d : \mathrm{dist}(x,A)<\epsilon\})}{\epsilon^{d-s}}.$$

Theorem 6 (Li–L–Velani-Zorin, in preparation)

Suppose that all eigenvalues of T are of modulus strictly larger than 1. Let μ be an absolute continuous invariant measure (acim) with support $\mathbb{T}^d$ and mixing. Then for any sequence $\{E_n\}_{n\in\mathbb{N}}$ of subsets in $\mathcal{C}$,

$$m_d\big(W(T,\{E_n\})\big) = \mu\big(W(T,\{E_n\})\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty}\mu\big(E_n\big) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty}\mu\big(E_n\big) = \infty. \end{cases}$$

III. Results - continued

Let $\beta \in \mathbb{R}$, $|\beta| > 1$. Study the (negative) β-transformation on $[0,1]$:

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor \ (\beta > 1) \quad \text{or} \quad T_\beta(x) = \beta x + \lfloor -\beta x \rfloor + 1 \ (\beta < -1).$$

- μ_β : acim (Parry measure, or Yrrap measure).
- $K(\beta)$: support of μ_β.

L–Steiner : let g be the golden mean, then
$K(\beta) = [0,1]$ if $\beta \in (-\infty, -g] \cup (1, +\infty)$,
$K(\beta)$ is a finite union of closed intervals if $\beta \in (-g, -1)$.

- $T = \mathrm{diag}(\beta_1, \ldots, \beta_d)$ with $|\beta_i| > 1$.
- $\mu = \mu_{\beta_1} \times \mu_{\beta_2} \times \cdots \times \mu_{\beta_d}$, $K = \prod_{i=1}^d K(\beta_i)$.

Theorem 7 (Li–L–Velani-Zorin, in preparation)

For any sequence $\{E_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$, we have

$$m_d|_K\big(W(T,\{E_n\})\big) = \mu\big(W(T,\{E_n\})\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty \mu(E_n) < \infty \\ 1 & \text{if } \sum_{n=1}^\infty \mu(E_n) = \infty. \end{cases}$$

IV. Corollary

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be real positive function and $\mathbf{a} \in \mathbb{T}^d$. Consider the targets

$$R_n := R(\mathbf{a}, \psi(n)) = \Big\{\mathbf{x} \in \mathbb{T}^d : \max_{1 \le i \le d} |a_i - x_i| \le \psi(n)\Big\}.$$

We are interested in

$$W(T, \psi, \mathbf{a}) = \{\mathbf{x} \in \mathbb{T}^d : T^n(\mathbf{x}) \in R(\mathbf{a}, \psi(n)) \ \text{ for infinitely many } n \in \mathbb{N}\}.$$

Corollary (Li–L–Velani-Zorin, in preparation)

Let T be a real, non-singular matrix transformation of the torus $\mathbb{T}^d$. Suppose that T is diagonal with all eigenvalues in $(-\infty, -g] \cup (1, +\infty)$. Then

$$m_d\big(W(T, \psi, \mathbf{a})\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n)^d < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n)^d = \infty. \end{cases}$$

V. A dimension result

Let $T = \mathrm{diag}(\beta_1, \dots, \beta_d)$ with $\beta_i > 1$.

For $\mathbf{a} \in \mathbb{T}^d$ and function ψ, let $W(T, \psi, \mathbf{a})$ be as before :

$W(T, \psi, \mathbf{a}) = \{\mathbf{x} \in \mathbb{T}^d : T^n(\mathbf{x}) \in R(\mathbf{a}, \psi(n))$ for infinitely many $n \in \mathbb{N}\}$.

Theorem 8 (Li–L–Velani-Zorin, in preparation)

Let $1 < e_1 \leq e_2 \leq \cdots \leq e_d$ be a reordering of $\beta_1, \dots, \beta_d$. Then

$$\dim_{\mathrm{H}} W(T, \psi, \mathbf{a}) = \min_{i=1,\cdots,d} \left\{ \frac{i \log e_i - \sum_{j: e_j > e_i e^{\lambda}} (\log e_j - \log e_i - \lambda) + \sum_{j>i} \log e_j}{\lambda + \log e_i} \right\},$$

where

$$\lambda = \lambda(\psi) = \liminf_{n \to \infty} \frac{-\log \psi(n)}{n}.$$