On a question of Douglass and Ono

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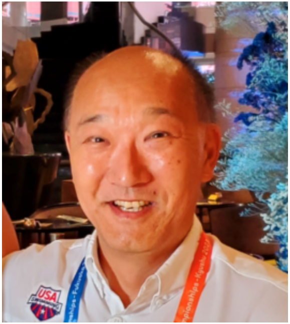
One World Numeration Seminar, November 12, 2024

Kate Douglass



Ken Ono

Uniform distribution

Definition

A sequence $(a_n)_{n\geq 1} \subset [0,1]$ is said to be *uniformly distributed* (UD) if

$$\lim_{X\to\infty} \frac{\#\{n \leq X : a_n \in (\alpha,\beta)\}}{X} = \beta - \alpha \quad \text{for all} \quad 0 \leq \alpha < \beta \leq 1.$$

For a real number x write

$$x = \lfloor x \rfloor + \{x\}, \qquad \text{where} \qquad \lfloor x \rfloor \in \mathbb{Z} \quad \text{and} \quad \{x\} \in [0,1).$$

Definition

A sequence of real numbers is uniformly distributed modulo 1 (UD mod 1), if the sequence of fractional parts $\{a_n\} \in [0,1)$ is UD.

Example

If $\tau \in \mathbb{R}\backslash\mathbb{Q}$, then $n\tau$ is UD mod 1.

Example

Question

What is the proportion of Fibonacci numbers that start with the digit 2?

Recall that the Fibonacci numbers $(F_n)_{n\geq 0}$ form the sequence starting with $F_0 = 0$, $F_1 = 1$ and of recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all} \quad n \geq 0.$$

We write the Binet formula

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n), \quad \text{where} \quad (\phi, \psi) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

The condition is

$$2 \cdot 10^m \le F_n < 3 \cdot 10^m.$$

We take logs and use the Binet formula to get

$$\log(2 \cdot 10^m) \le \log\left(\frac{\phi^n}{\sqrt{5}}\left(1 - \left(\frac{\psi}{\phi}\right)^n\right)\right) < \log(3 \cdot 10^m).$$

This is

$$\log 2 + m \log 10 \le n \log \phi - \log\sqrt{5} + \log\left(1 - \left(\frac{-1}{\phi^2}\right)^n\right) < \log 3 + m \log 10.$$

Dividing by $\log 10$ and rearranging we get

$$\frac{\log 2}{\log 10} + \frac{\log\sqrt{5}}{\log 10} + o(1) \le n\left(\frac{\log\phi}{\log 10}\right) - m \le \frac{\log 3}{\log 10} + \frac{\log\sqrt{5}}{\log 10} + o(1).$$

$$\frac{\log 2}{\log 10} := 0.30103\ldots, \qquad \frac{\log 3}{\log 10} := 0.477121\ldots, \qquad \frac{\log\sqrt{5}}{\log 10} := 0.34948$$

The above inequality means that

$$\alpha + o(1) \leq \{n\tau\} \leq \beta + o(1)$$

where

$$\alpha = \frac{\log 2}{\log 10} + \frac{\log \sqrt{5}}{\log 10}, \quad \beta = \frac{\log 3}{\log 10} + \frac{\log \sqrt{5}}{\log 10}, \quad \tau = \frac{\log \phi}{\log 10}.$$

Since our τ is irrational, the example shows that the proportion is

$$\beta - \alpha = \frac{\log 3}{\log 10} - \frac{\log 2}{\log 10} = \log_{10}\left(1 + \frac{1}{2}\right).$$

Benford's law

Definition

We say that a sequence of integers $(a_n)_{n\geq 1}$ satisfies the Benford law in base $b \geq 2$ if for every string of digits f written in base b, we have

$$\lim_{X\to\infty} \frac{\#\{n \leq X : a_n \text{ in base } b \text{ starts with } f\}}{X} = \log_b(f+1) - \log_b f \mod 1,$$

where on the right we interpret f as an integer in base b (that is, if $f = f_0 f_1 \ldots f_{t-1}$ has t digits in $\{0, 1 \ldots, b-1\}$ with $f_0 \neq 0$, then

$$f = f_0 b^{t-1} + f_1 b^{t-2} + \cdots + b_{t-1}.$$

A criterion of Anderson, Rolen, Stoehr

The following criterion appears in a paper of **2011**.

Definition

We say that an integer sequence $(a_n)_{n\geq 1}$ is good if

$$a(n) \sim b(n)e^{c(n)} \qquad \text{as} \qquad n \to \infty,$$

where the following conditions are satisfied:

(i) There exists some integer $h \geq 1$ such that $c(n)$ is h-differentiable and $c^{(h)}(n) \to 0$ monotonically for large n.

(ii) We have that

$$\lim_{n\to+\infty} n|c^{(h)}(n)| = +\infty.$$

(iii) We have that

$$\lim_{n\to+\infty} \frac{D^{(h)} \log b(n)}{c^{(h)}(n)} = 0 \quad (D^{(h)} \quad \text{is the } h \text{ derivative}).$$

Their main result is the following:

Theorem

Good integer sequences abide by Benford's law in any integer base $b \geq 2$.

Example

Let

$$p(n) = \#\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1 : \lambda_1 + \cdots + \lambda_k = n\}$$

be the partition function of n. For example, $p(3) = 3$ since

$$3, \quad 2+1, \quad 1+1+1.$$

It is known that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as} \quad n \to +\infty.$$

Then

$$b(n) := \frac{1}{4n\sqrt{3}} \quad \text{and} \quad c(n) := \pi\sqrt{2n/3}.$$

We take $h = 1$ and observe that

$$c'(n) = \frac{\pi}{\sqrt{6n}}$$

tends to zero monotonically, so (i). Also,

$$nc'(n) = \pi\sqrt{n/6} \to +\infty \quad \text{as} \quad n \to +\infty,$$

so (ii). Furthermore,

$$\frac{D' \log b(n)}{c'(n)} = \frac{-1/n}{\pi/\sqrt{6n}} = -\frac{1}{\pi\sqrt{n/6}} \to 0 \quad (n \to +\infty) \quad \text{so, (iii).}$$

So, we proved the following result.

Theorem

The partition function $p(n)$ abides Benford's law in any integer base $b \geq 2$.

There are many other partition functions. One of them is the plane partition function.

Example

We take

$$\mathrm{PL}(n) = \#\{(\pi_{i,j}) : \pi_{i,j} \in \mathbb{N},\ \pi_{i,j} \geq \pi_{i,j+1},\ \pi_{i+1,j} \geq \pi_{i,j},\ \sum_{i,j} \pi_{i,j} = n\}.$$

For $n = 3$, we have

$$3,\ 2+1,\ 1+1+1,\ \begin{matrix} 2 \\ 1 \end{matrix},\ \begin{matrix} 1 \\ 1 \end{matrix}\ +1,\ \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}, \quad \text{so} \quad \mathrm{PL}(3) = 6.$$

It is known that

$$\mathrm{PL}(n) \sim \frac{(2^{25}A^7)^{1/36}e^c}{\sqrt{12\pi}n^{25/36}} \exp\left(\sqrt[3]{\frac{27An^2}{4}}\right) \quad \text{as} \quad n \to +\infty,$$

where

$$A = \zeta(3) \sim 1.202056\ldots, \quad c = \zeta'(-1) = -0.16542\ldots$$

Douglass, Ono proved in **2024** that $(\mathrm{PL}(n))_{n\geq 1}$ abides Benford's law in any integer base $b \geq 2$ and set forward the following effectivity question.

Question

Find $N(b,f)$ such that there is $n \leq N(b,f)$ with $PL(n)$ starting with the string f in base b.

The rest of the talk is devoted to this task.

The case of the partition function

Let $t := \lfloor \log f / \log b \rfloor + 1$ be the number of digits of the string f in base b.

Theorem

For the partition function $p(n)$ one can take

$$N(b,f) = \exp\left(2 \cdot 10^{25}(t+12)(\log b)^2\right).$$

What do we need?

We need three ingredients:

(i) An estimate with explicit error term for the partition function.

(ii) The Erdős, Koksma, Turán inequality which gives an upper bound for the discrepancy of a sequence of real numbers modulo 1.

(iii) A linear in logarithms of algebraic numbers with algebraic coefficients due to Philippon, Waldschmidt.

For (i) we use the following result of Lehmer.

Lemma

Let $\mu(n) := \frac{\pi}{6}\sqrt{24n-1}$. *Then the inequality*

$$\left| p(n) - \frac{\sqrt{3}}{12n} e^{\mu(n)} \right| < \frac{\sqrt{3}}{12n^{3/2}} e^{\mu(n)}$$

holds for all positive integers n.

Discrepancy

For (ii), we need some further notation and terminology. The discrepancy of a sequence $(b_m)_{m=1}^N$ of real numbers (not necessarily distinct) is defined as

$$D_N := \sup_{0\le\gamma\le1} \left| \frac{\#\{m \le N : \{b_m\} < \gamma\}}{N} - \gamma \right|.$$

From the above definition, we see that the inequality

$$\#\{m \le N : \alpha \le \{b_m\} < \beta\} \ge (\beta - \alpha)N - 2ND_N \qquad (1)$$

holds for all $0 \le \alpha < \beta \le 1$.

Lemma

We have

$$D_N \le \frac{6}{H} + \frac{4}{\pi N} \sum_{h \le H} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h b_n} \right|,$$

where $\|x\|$ is the distance to the nearest integer and $H \le N$ is an arbitrary positive integer.

The proof of the theorem

We write

$$f := b_0 b^{t-1} + b_1 b^{t-2} + \cdots + b_{t-1},$$

where $b_0, \ldots, b_{t-1} \in \{0, 1, \ldots, b-1\},\ b_0 \neq 0$. We set

$$\lambda := \frac{f + 0.5}{b^t} = b_0 + \frac{b_1}{t} + \cdots + \frac{b_{t-2}}{b^{t-2}} + \frac{b_{t-1} + 0.5}{b^{t-1}}.$$

Any real number M such that for some integer $m > t$ we have

$$|M - b^m \lambda| < \frac{b^m}{b^{t+2}} \tag{2}$$

has the property that its first t digits in base b are $b_0, b_1, \ldots, b_{t-1}$. So, we want to infer that we can take $M = p(n)$ for an n at most as large as $N(b, f)$.

By Lehmer's lemma, the equation holds for $M = p(n)$ provided that the following two inequalities

$$\left| p(n) - \frac{\sqrt{3}}{12n} e^{\mu(n)} \right| < \frac{\sqrt{3}}{12nb^{t+5}} e^{\mu(n)}, \tag{3}$$

and

$$\left| \frac{\sqrt{3}}{12n} e^{\mu(n)} - b^m \lambda \right| < \frac{b^m}{b^{t+5}} \tag{4}$$

hold. For the first one it suffices that

$$n > b^{2t+10}. \tag{5}$$

We look at the second one.

This can be rewritten as

$$\left|\left(\frac{\sqrt{3}}{12n}e^{\mu(n)}\right)(b^m\lambda)^{-1}-1\right|<\frac{1}{\lambda b^{t+5}}. \qquad (6)$$

It is implied by

$$\left|-m\log b+\frac{\pi}{6}\sqrt{24n-1}+\log\left(\frac{\sqrt{3}}{12n\lambda}\right)\right|<\frac{1}{2b^{t+6}}. \qquad (7)$$

We evaluate the above in n's of a certain form, namely

$$n := n_j^2, \quad n_j = b^T + j, \quad j = 0, 1, \ldots, N, \tag{8}$$

where

$$T \text{ is an integer such that } b^T > N^2.$$

We can take

$$T := \lfloor 2 \log N / \log b \rfloor + 1.$$

We also assume that (5) is satisfied for $n := N$.

We put

$$f(x) := \sqrt{1-x}.$$

By Taylor's formula one gets

$$\sqrt{24n_j^2-1} = \sqrt{24}n_j + \eta_j, \quad \text{where} \quad |\eta_j| < \frac{1}{2\sqrt{12}N^2}.$$

By the triangular inequality, we then have

$$\begin{aligned} & \left| -m_j \log b + \frac{\pi}{6}\sqrt{24n_j^2-1} + \log\left(\frac{\sqrt{3}}{12n_j^2\lambda}\right)\right| \\ < & \left| -m_j \log b + \sqrt{\frac{2}{3}}\pi n_j + \log\left(\frac{\sqrt{3}}{12n_j^2\lambda}\right)\right| \\ + & \frac{\pi}{12\sqrt{12}N^2}. \end{aligned}$$

One calculates that it is enough that

$$\left| -m_j + \frac{\sqrt{\frac{2}{3}}\pi n_j + \log\left(\frac{\sqrt{3}}{12n_j\lambda}\right)}{\log b}\right| < \frac{1}{b^{t+7}}.$$

So, we take the sequence of general term

$$b_{n_j} := c_b n_j + \frac{\log\left(\frac{\sqrt{3}}{12 n_j^2 \lambda}\right)}{\log b} \quad \text{where} \quad c_b := \frac{\sqrt{\frac{2}{3}}\pi}{\log b},$$

and we need to show that there is

$$n_j \in (b^{2t+10}, N(b, f)),$$

such that $\{b_n\} \in I$, where

$$I := \left(0, \frac{1}{b^{t+7}}\right).$$

It turns out that the conditions

$$D_N \leq 1/(4b^{t+7}) \quad \text{and} \quad N > 2b^{3t+17}$$

are sufficient.

So, it suffices that

$$D_N < \frac{1}{b^{t+9}} \le \frac{1}{4b^{t+7}}.$$

But

$$D_N \le \frac{6}{H} + \frac{2}{N} \sum_{m=1}^{H} \sum_{h \le H} \frac{1}{h} \left| \sum_{j=1}^{N} e^{2\pi i h b_{n_j}} \right|. \tag{9}$$

Since

$$b_{n_j} - b_{n_0} = c_b(n_j - n_0) - \frac{2\log(n_j/n_0)}{\log b} = c_b j - \left(\frac{2}{\log b}\right) \log\left(1 + \frac{j}{b^T}\right).$$

and the second term is small, so one estimates that

$$2\pi i h(b_{n_j} - b_{n_0}) = 2\pi i h c_b j + \zeta_{h,j}, \quad \text{where} \quad |\zeta_{h,j}| < \frac{4\pi H}{(\log b)N} < \frac{6\pi H}{N}.$$

Hence,

$$e^{2\pi i h(b_{n_j}-b_{n_0})} = e^{2\pi i h c_b j} e^{\zeta_{h,j}} = e^{2\pi i h c_b j}(1+\eta_{h,j}) = e^{2\pi i h c_b j} + \eta'_{h,j},$$

where

$$|\eta'_{h,j}| = |\eta_{h,j}| < \frac{12\pi H}{N}.$$

So, in the exponential sum we get

$$\begin{aligned}\left|\sum_{j=1}^{N} e^{2\pi i h b_{n_j}}\right| &= \left|\sum_{j=1}^{N} e^{2\pi i h (b_{n_j}-b_{n_0})}\right| \\ &= \left|\sum_{j=1}^{N} \left(e^{2\pi i h c_b j} + \eta'_{h,j}\right)\right| \\ &\le \left|\sum_{j=1}^{N} e^{2\pi i h c_b j}\right| + \left|\sum_{j=1}^{N} \eta'_{h,j}\right| \\ &\le \left|\frac{e^{2\pi i N h c_b}-1}{e^{2\pi i h c_b}-1}\right| + 12\pi H. \qquad (10)\end{aligned}$$

The first term on the right–hand side in the last line of (22) is in absolute value at most

$$\frac{1}{|\sin(\pi h c_b)|} = \frac{1}{|\sin(\pi\|hc_b\|)} \le \frac{1}{2\|hc_b\|}. \qquad (11)$$

We get

$$D_N \le \frac{6}{H} + \frac{\log H + 1}{\min_{1\le h\le H} \|hc_b\|} + \frac{48H(1+\log H)}{N}.$$

Denoting by ℓ_h the closest integer to hc_b, we get that

$$\|hc_b\| = \left|\ell_h - \frac{\sqrt{\frac{2}{3}}\pi h}{\log b}\right| = \frac{1}{\log b}\left|\ell_h \log b - \sqrt{\frac{2}{3}}\pi h\right|.$$

Writing $\pi = (-i^2)\pi = -i\log(-1)$, where $i^2 = -1$, we get that the right–hand side above is

$$\left|-\ell_h \log b + \left(-i\sqrt{\frac{2}{3}}h\right)\log(-1)\right|. \qquad (12)$$

The above expression is non-zero by the Gelfond–Schneider theorem. It is a linear form in logarithms of algebraic numbers with algebraic coefficients and we need a lower estimate on it.

We use a linear form in logarithms of Phillipon, Waldschmidt of **1988**.

After calculations, we choose

$$H := \left\lfloor \exp\left(\frac{\log N}{4.7 \cdot 10^{24} \log b}\right)\right\rfloor.$$

Then

$$D_N \leq \frac{6N + H\exp(4.6 \cdot 10^{24} \log b \log H)}{HN} < \frac{7}{H}.$$

We need that $D_N \leq 1/b^{t+9}$. So, we need that

$$\frac{7}{H} \leq \frac{1}{b^{t+9}},$$

which gives

$$7b^{t+9} \leq H.$$

which implies that one has to choose N to be at least

$$\exp(4.7 \cdot 10^{24}(t+12)(\log b)^2)$$

Next our n_j were of the form $b^T + j$ for $j \leq N$ and $b^T > N^2$. So, it suffices to take

$$T := \left\lfloor \frac{2 \log N}{\log b} \right\rfloor + 1 = 1 + 9.4 \cdot 10^{24}(t + 12) \log b.$$

With this choice for T, and since our n's were

$$n_j^2 \leq (2b^T)^2 \leq b^{2T+2},$$

we can take

$$N(b, f) := \exp(2 \cdot 10^{25}(t + 12)(\log b)^2),$$

as promised.

The plane partition function

One of the key elements from the previous proof fails so we get a better result.

Theorem

For the plane partition function $PL(n)$ one can take

$$N_{PL}(b, f) = b^{51t+688}.$$

This part will appear in *Annals of Combinatorics*.

Let's follow along the previous proof. The following result is from a paper of Ono, Pujahari, Rollen of **2022**.

Lemma

Let $A := \zeta(3) \approx 1.202056\ldots$, $c := \zeta'(-1) \approx -0.16542\ldots$,

$$\mu(n) := 3 \cdot (A/4)^{1/3} n^{2/3}.$$

Then the inequality

$$\left| \mathrm{PL}(n) - \frac{B}{n^{25/36}} e^{\mu(n)} \right| < \frac{100B}{n^{25/36+2/3}} e^{\mu(n)}$$

holds for all $n \geq 105$*, where*

$$B := \frac{2^{25/26} e^{c} A^{7/26}}{\sqrt{12\pi}}.$$

We set up the same machine. We write

$$f = b_0 b^{t-1} + b_1 b^{t-2} + \cdots + b_{t-1},$$

where $b_0, \ldots, b_{t-1} \in \{0, 1, \ldots, b-1\}$, $b_0 \neq 0$. We set

$$\lambda := \frac{f + 0.5}{b^{t-1}} = b_0 + \frac{b_1}{b} + \cdots + \frac{b_{t-2}}{b^{t-2}} + \frac{b_{t-1} + 0.5}{b^{t-1}}.$$

Any real number M such that for some integer $m > t$ we have

$$|M - b^m \lambda| < \frac{b^m}{b^{t+2}} \tag{13}$$

has the property that its first t digits in base b are $b_0, b_1, \ldots, b_{t-1}$. So, we want to infer that we can take $M = \mathrm{PL}(n)$ for an n at most as large as $N_{\mathrm{PL}}(b, f)$.

For us, this happens if

$$\left|\frac{B}{n^{25/36}}e^{\mu(n)} - b^m\lambda\right| < \frac{b^m}{b^{t+5}} \tag{14}$$

and

$$n > b^{2t+20}. \tag{15}$$

The first relation above is implied by

$$\left|-m + \frac{3\cdot 2^{-2/3}A^{1/3}n^{2/3} - (25/36)\log n + \log(B/\lambda)}{\log b}\right| < \frac{1}{b^{t+7}}. \tag{16}$$

Again we specialize in numbers of a certain form.
Namely, we take $1 \le H < N$ with H an integer depending on N to be determined later,

$$p \in [2H, 3H]$$

to be prime, $q := q(p)$ to be the smallest prime $q > p$ and

$$n_{p,j} := q^3 b^{3T} + pjb^T \qquad \text{for} \quad j = 0, \ldots, N, \tag{17}$$

where T is an integer such that $b^T > N^2$.

We can take again $T := \lfloor 2 \log N / \log b \rfloor + 1$.

Notice that $n_{p,j}$ are distinct for distinct pairs (p, j). Indeed, if

$$n_{p,j} = n_{p',j'},$$

then

$$q^3 b^{3T} + pjb^T = q'^3 b^{3T} + p'j'b^T.$$

If $q = q'$, then also $p = p'$ and the above relation gives us $j = j'$. So, we may assume that $q \neq q'$ and to fix ideas that $q > q'$. We then get

$$\begin{aligned} 2 \cdot (2H)^2 b^{3T} &\leq (p^2 + p'^2) b^{3T} < (q^2 + qq' + q'^2) b^{3T} \\ &< (q^3 - q'^3) b^{3T} = (p'j' - pj) b^T < p'j'b^T < 3HNb^T, \end{aligned}$$

which is of course absurd since $b^T > N^2$.

We take

$$a_{n_{p,j}} := \frac{3 \cdot 2^{-2/3} A^{1/3} n_{p,j}^{2/3} - (25/36) \log n_{p,j} + \log(B/\lambda)}{\log b}. \quad (18)$$

By (16), we need to find N such as to guarantee that there is $j \in [1, N]$ such that

$$n_j > b^{2t+20}$$

and

$$\{a_{n_{p,j}}\} < \frac{1}{b^{t+7}}. \quad (19)$$

Lemma

The estimate

$$a_{n_{p,j}} - a_{n_{p,0}} = \frac{2^{1/3} A^{1/3} jp}{q \log b} + \zeta_{p,j}, \qquad \text{where} \qquad |\zeta_{p,j}| < \frac{2}{N^2}$$

holds for all $j = 1, \ldots, N$ *and* $p \in [2H, 3H]$ *provided* $N > \max\{H, 10\}$.

The proof follows from Taylor's formula, as in the case of the partition function.

Lemma

Put $c_b := 2^{1/3}A^{1/3}/\log b$. *We then have*

$$e^{2\pi ih(a_{n_{p,j}}-a_{n_{p,0}})} = e^{2\pi ihc_bjp/q} + \eta_{p,j,h}, \quad \textit{with} \quad |\eta_{p,j,h}| < \frac{8\pi}{N}$$

for all $p \in [2H, 3H]$, $h \in [1, H]$ *and* $j \in [1, N]$ *provided* $N > \max\{H, 30\}$.

So, for fixed $p \in [2H, 3H]$, we take the sequence of general term $a_{n_{p,j}}$ given by (18) for $j = 1, \ldots, N$ and we want to show that for some p and $j \le N$ such that $n_{p,j}$ satisfies (15) we have $\{a_{n_{p,j}}\} \in J$, where

$$J = \left(0, \frac{1}{b^{t+7}}\right).$$

Assume that

$$D_{p,N} \le 1/(4b^{t+7}).$$

Then, by (1), the number of $j \le N$ for which $\{a_{n_{p,j}}\} \in J$ is at least

$$\#JN - 2ND_{p,N} \ge \frac{N}{b^{t+7}} - \frac{N}{2b^{t+7}} \ge \frac{N}{2b^{t+7}}. \tag{20}$$

and this is large enough for

$$N > b^{3t+38}.$$

So, we go again to the estimate:

$$D_{p,N} \leq \frac{6}{H} + \frac{4}{\pi N} \sum_{h=1}^{H} \frac{1}{h} \left| \sum_{j=1}^{N} e^{2\pi i h a_{n_{p,j}}} \right|. \tag{21}$$

With our choices we manipulate the inner sum as

$$\begin{aligned}
\left| \sum_{j=1}^{N} e^{2\pi i h a_{n_{p,j}}} \right| &= \left| \sum_{j=1}^{N} e^{2\pi i h (a_{n_{p,j}} - a_{n_{p,0}})} \right| \\
&= \left| \sum_{j=1}^{N} \left(e^{2\pi i h c_b j p/q} + \eta_{p,j,h} \right) \right| \\
&\leq \left| \sum_{j=1}^{N} e^{2\pi h c_b j p/q} \right| + \left| \sum_{j=1}^{N} \eta_{p,j,h} \right| \\
&\leq \left| \frac{e^{2\pi N h c_b p/q} - 1}{e^{2\pi h c_b p/q} - 1} \right| + 8\pi. \qquad (22)
\end{aligned}$$

The above argument assumes that $c_b hp/q$ is not an integer. But if

$$c_b hp/q = m_{b,h,p,q}$$

is an integer, then $m_{b,h,p,q}$ is smaller that

$$c_b h < (1.34/\log b)p < 6H.$$

Thus,

$$c_b = m_{b,h,p,q} q/(hp)$$

is a rational number whose numerator is a divisor of

$$m_{b,p,q,h} q < 6Hq < 6H(2p) \leq 36H^2 < (2H)^3$$

for $H \geq 5$ and a multiple of q since $q > p \geq 2H > h$. In the above, we used Bertrand's postulate to conclude that $q < 2p \leq 6H$. Thus, the unlikely case when $c_b hp/q$ ends up being an integer, which in particular implies that c_b is rational, can happen for at most two values of q which must be prime factors larger than $2H$ of the numerator of c_b in reduced form.

Lemma

The inequality

$$D_{p,N} \le \frac{6}{H} + \frac{32(\log H + 1)}{N} + \sum_{h=1}^{H} \frac{1}{h\|hc_b p/q\|}. \qquad (23)$$

holds for all $p \in [2H, 3H]$ *with at most two exceptions provided* $5 \le H < N$ *and* $N > 30$.

Lemma

There are at most two values of $p \in [2H, 3H]$ *such that there exists* $h \in [1, H]$ *with the property that*

$$\|hc_b p/q\| < \frac{1}{162H^6}. \qquad (24)$$

Assume $m_h \neq 0$ and $m_h - c_b hp/q \neq 0$. We then have

$$m_h < c_b hp/q + 1/(162H^6) < 1.94H + 1/162 < 2H$$

(since in fact $c_b < 1.94$). Further,

$$0 < \left| \frac{m_h q}{hp} - c_b \right| < \frac{q}{162hpH^6} < \frac{1}{81H^6} = \frac{1}{3(3H^2)^3}. \qquad (25)$$

Note that $hp < 3H^2$ so writing $m_h q/(hp) = P/Q$ the inequality

$$0 < \left| \frac{P}{Q} - c_b \right| < \frac{1}{3Q^3}$$

is satisfied. We show that there is at most one p for which the above inequality is satisfied for some h. Indeed, assume that $p' \neq p \in [2H, 3H]$ is such that the above inequality is also satisfied with $P'/Q' = m_{h'} q'/(h' p')$ for some $h' \in \{1, \ldots, H\}$. Noting that $Q,\ Q' \in [2H, 3H^2]$, it follows that $Q < Q'^2$ and $Q' < Q^2$. In particular, $2QQ' < \min\{2Q^3, 2Q'^3\}$. Thus, we get

$$\left| \frac{PQ' - P'Q}{QQ'} \right| = \left| \frac{P'}{Q'} - \frac{P}{Q} \right| \leq \left| \frac{P'}{Q'} - c_b \right| + \left| \frac{P}{Q} - c_b \right| < \frac{2}{3QQ'}.$$

So, the last two lemmas show that with at most four exceptions in $p \in [2H, 3H]$ such that inequality (24) holds for some $h \in \{1, \ldots, H\}$.
Next, note that there are $\pi(3H) - \pi(2H)$ primes in $[2H, 3H]$. Here, $\pi(x)$ is the number of primes $p \le x$. Using estimates from Rosser, Schoenfeld **1962** we have that

$$\pi(3H) - \pi(2H) > \frac{3H}{\log(3H)} - \frac{5H}{2\log(2H)} \qquad (H > 60).$$

The above function exceeds 4 for all $H \ge 73$. Thus, if $H \ge 73$, there are at least five primes in $[2H, 3H]$ and in particular there exists a prime $p \in [2H, 3H]$ such that inequality (23) holds and additionally

$$\|c_b hp/q\| \ge \frac{1}{162H^6}$$

holds for all $h = 1, \ldots, H$.

This gives

$$D_{p,N} < \frac{6}{H} + \frac{12H^7}{N}$$

for $H \geq 73$. We choose $H := \lfloor N^{1/8} \rfloor$, and we get

$$D_{p,N} = \frac{6N + 12H^8}{NH} \leq \frac{18}{H} < \frac{19}{H+1} < \frac{19}{N^{1/8}}.$$

Of course, we want $N \geq \max\{b^8, 73^8\}$. Since $b \geq 2$, it suffices that $N \geq b^{50}$. We need that

$$\frac{19}{N^{1/8}} \leq \frac{1}{b^{t+9}}.$$

This works when $N \geq b^{8t+106}$.

Recall that our n's are of the form

$$n_j = q^3 b^{3T} + pjb^T \qquad \text{for} \qquad j = 1, \ldots, N.$$

We want $b^T > N^2$, so we take $T = 16t + 213$. Then

$$q < 2p \leq 6H \leq 6N^{1/8} = 6b^{t+13.25} < b^{t+16},$$

and then

$$n < 2q^3 b^{3T} < 2b^{3t+48+48t+639} \leq b^{51t+688}.$$

MERCI BEAUCOUP!