Serendipitous decompositions of higher-dimensional continued fractions

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Abstract:

Complex continued fractions (CFs) represent a complex number using a descending fraction with Gaussian integer coefficients. The associated dynamical system is exact (Nakada 1981) with a piecewise-analytic invariant measure (Hensley 2006). Certain higher-dimensional CFs, including CFs over quaternions, octonions, as well as the non-commutative Heisenberg group can be understood in a unified way using the Iwasawa CF framework (L-Vandehey 2022). Under some natural and robust assumptions, ergodicity of the associated systems can then be derived from a connection to hyperbolic geodesic flow, but stronger mixing results and information about the invariant measure remain elusive. Here, we study Iwasawa CFs under a more delicate serendipity assumption that yields the finite range condition, allowing us to extend the Nakada-Hensley results to certain Iwasawa CFs over the quaternions, octonions, and in R^3.

This is joint work with Joseph Vandehey, University of Texas at Tyler.

Real continued fractions (CFs)

Regular CFs represent $x \in \mathbb{R}_+$ as a descending fraction with coefficients in $\mathbb{N}$:

$$3 + \frac{1|}{|7} + \frac{1|}{|15} + \frac{1|}{|1} + \frac{1|}{|292} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|\cdots}$$

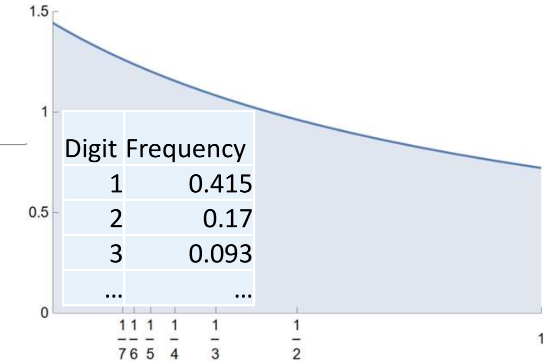
Digits of $x \in [0,1]$ are extracted using the Gauss map $T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$:

- Invariant probability measure $\left(\log 2 \ (1+x)\right)^{-1}$,
- Ergodicity, exactness: straightforward because of full cylinders,
- Factor of geodesic flow on modular surface $\mathbb{H}^2/SL(2,\mathbb{Z})$.

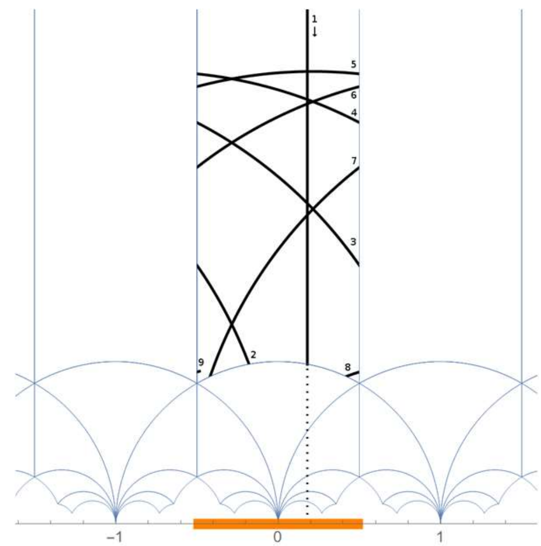
Many real CF variants exist, including:

- Backwards CFs: numerator -1, interval [-1,0], with measure $(1-x)^{-1}$
- Nearest-integer CFs: with interval [- ½, ½]
- α-CFs: interval [-α, $1-\alpha$]

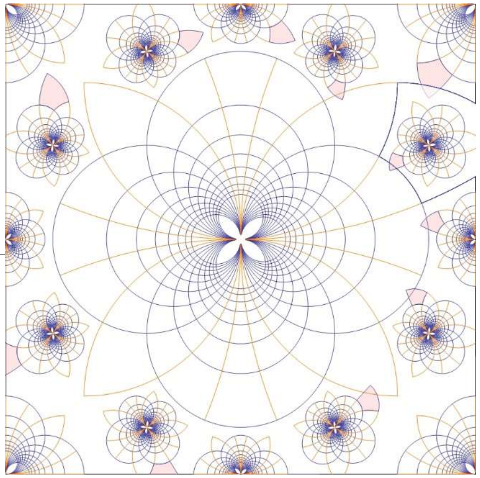
The latter were shown to be exact with positive inversion (Nakada-Steiner 2000) and ergodic with negative inversion (LV 2022).



Digit	Frequency
1	0.415
2	0.17
3	0.093
…	…

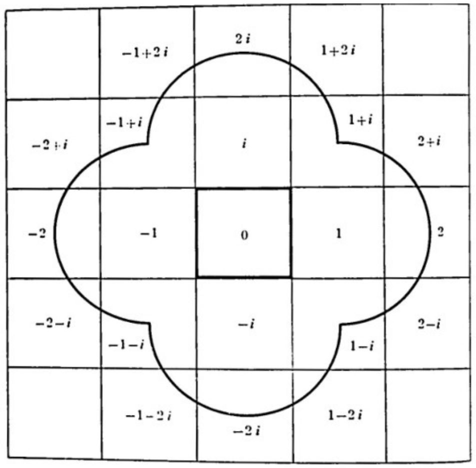


Complex continued fractions



Complex CFs represent $x \in \mathbb{C}$ as a descending fraction with coefficients in $\mathbb{Z}[\mathbb{i}]$:

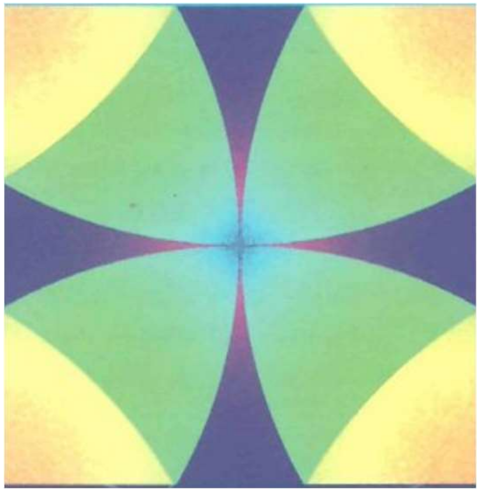
$$2+3\mathbb{i}+\frac{1}{\lfloor -2-\mathbb{i}}+\frac{1}{\lfloor -3-\mathbb{i}}+\frac{1}{\lfloor 2+2\mathbb{i}}+\frac{1}{\lfloor -1+\mathbb{i}}+\frac{1}{\lfloor -1-2\mathbb{i}}+\frac{1}{\lfloor \dots}$$



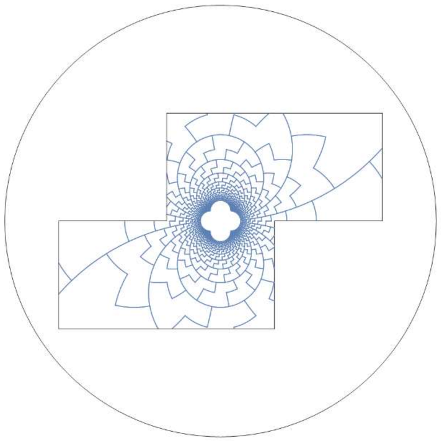
Digits of $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ are extracted using the map

$$T(z) = \frac{1}{z} - \left[\frac{1}{z}\right].$$

- First discovered by A. Hurwitz in 1887,
- Cylinders not full, but has the finite range property,
- (Nakada, 1981) Exact, with a unique piecewise-Lipschitz invariant measure; using direct measure theory and cylinder analysis,
- (Hensley, 2006) Exact, with a unique piecewise-analytic invariant measure; using the transfer operator.



Iwasawa CFs in $\mathbb{R}^d$



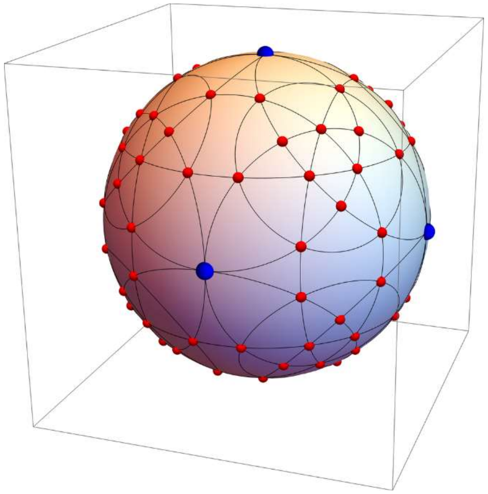
Iwasawa CFs generalize the above examples, with the following data:

- A space is $X = \mathbb{R}^d$, viewing $\mathbb{C}$ as $\mathbb{R}^2$, quaternions as $\mathbb{R}^4$, or octonions as $\mathbb{R}^8$.
- A discrete group of digits $\mathcal{Z} \subset \mathrm{Isom}\left(\mathbb{R}^d\right)$. Often, we will have $\mathcal{Z} \subset \mathbb{R}^d$.
- An inversion of the form $\iota(x) = \frac{\mathcal{O}(x)}{|x|^2}$, with $\mathcal{O} \in O(d)$ of order-two. Includes $z \mapsto \frac{1}{z}$.

An **Iwasawa continued fraction** is then an expression of the form

$$\mathcal{K}_{i=0}^{n} a_i = (a_0 \circ \iota \circ a_1 \circ \cdots \circ \iota \circ a_n)(0),$$

where each $a_i \in \mathcal{Z} \setminus \{id\}$, or a (possibly formal) limit of such expressions.

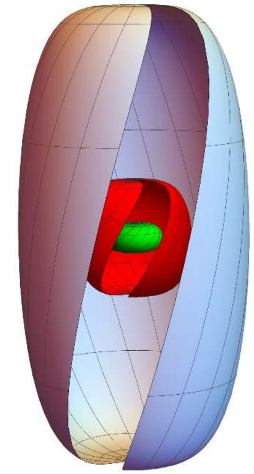


An **Iwasawa CF *algorithm*** then consists of

- a fundamental domain $K \subset B(0,1)$ for $\mathcal{Z}$,
- induced nearest-integer mapping $[\cdot]_K \colon \mathbb{R}^d \to \mathcal{Z}$ characterized by $x \in [x]_K(K)$,
- Induced CF mapping $T(x) = [\iota(x)]_K^{-1}\left(\iota(x)\right)$ and CF digits $a_i = \left[\iota\left(T^{i-1}x\right)\right]_K$.

Theorem (LV 2022 arXiv) Under mild assumptions, $x = \mathcal{K}_{i=1}^{\infty} a_i$.

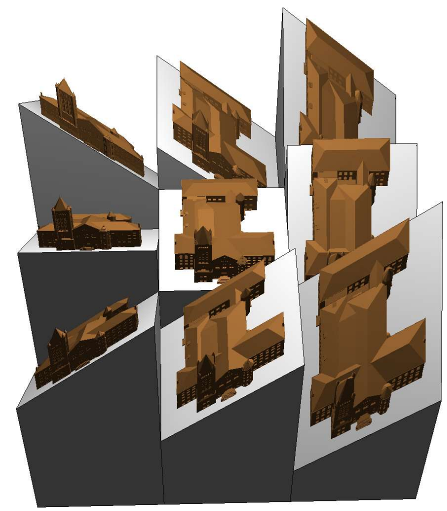
Non-Euclidean Iwasawa CFs



More generally, Iwasawa CFs also include CFs on boundaries of hyperbolic spaces (that is, rank-one symmetric spaces of non-compact type) whose isometry groups $SU(n+1,1,\mathscr{k})$ have an *Iwasawa decomposition* KAN and whose parabolic boundary at infinity is the *Iwasawa group* N.

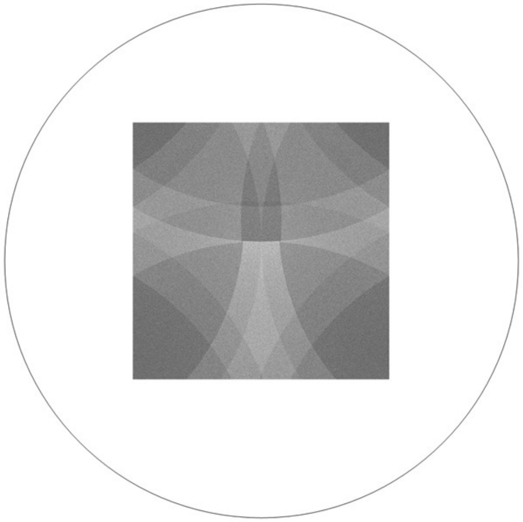
These include **Heisenberg CFs** defined with data:

- Space $\mathrm{Heis} = \mathbb{C}\times\mathbb{R}$ with group law $(z,t)*(z',t') = (z+z', t+t'+2\mathrm{Im}(\overline{z}\,z'))$.
- Heisenberg integers $\mathcal{Z} = \mathbb{Z}[\mathbb{i}]\times\mathbb{Z}$ used as digits.
- Koranyi inversion $\iota(z,t) = \left(\frac{-z}{|z|^2+\mathbb{i}\,t}, \frac{-t}{|z|^4+t^2}\right)$.
- Compatible metric induced by Koranyi gauge $|(z,t)|^4 = |z|^4 + t^2$.
- Left-invariant Haar measure is Lebesgue measure.



Heisenberg CFs are convergent (LV 2015) and have a Diophantine interpretation compatible with hyperbolic geometry (V 2016, LV 2020).

Ergodicity & invariant measures



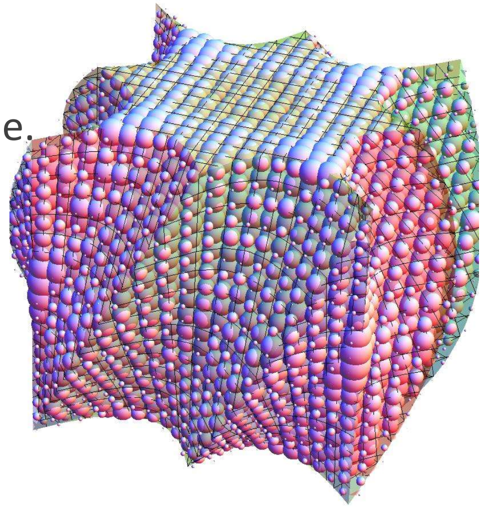
Proofs of ergodicity appear to fall into two approaches, which are easier in

- Cylinder analysis, followed by either measure theory or operator theory.
  In higher dimension, one needs to track more complicated shapes.
- Connection to geodesic flow (Series and others).
  The method was considered “intrinsically two-dimensional” (Adler-Flatto).

Theorem (LV 2022) Suppose an Iwasawa CF is discrete, proper and complete.
Then it is a factor of a speedup of geodesic flow on its modular manifold.
Thus, it is ergodic with an invariant measure absolutely continuous to Lebesgue.

Discreteness: the modular group $\langle \mathcal{Z}, \iota \rangle \subset SU(n+1, 1, \mathscr{k})$ is discrete.
Properness: the closure of the tile K is contained in the open ball B(0,1).
Completeness: $\mathrm{Stab}_{\langle \mathcal{Z}, \iota \rangle}(\infty) = \mathcal{Z}$, i.e. no hidden symmetries exist such as

$$\frac{1}{\lceil 1}\Big\rfloor + \frac{1}{\lceil -1}\Big\rfloor + \frac{1}{\lceil 1+x}\Big\rfloor = -x$$



Note: the measure is infinite for some Rosen CFs (Gröchenig-Haas 1996).

Exactness and smoothness

Focusing on especially well-behaved systems, we now proved:

Theorem (LV 2023) The CF system associated to the Hurwitz integers within the quaternions is exact, CF-mixing, satisfies a Kuzmin-type theorem, and has a unique invariant measure equivalent to Lebesgue measure, whose density is bounded and piecewise-analytic with finitely many pieces.

More generally, the underlying theorem includes the following examples, with K the Dirichlet region at 0 and an inversion compatible with the lattice in that $\mathrm{x} \in \mathcal{Z} \rightarrow \mathcal{O}(x)$:

- Real CFs with integer digits
- Complex CFs with Gaussian integer digits $\mathbb{Z}[\mathbb{i}]$
- Complex CFs with Eisenstein integer digits $\mathbb{Z}\left[\frac{1+\sqrt{3}\mathbb{i}}{2}\right]$
- Quaternionic CFs with Hurwitz integers
- Quaternionic CFs with Gausenstein integers
- Octonionic CFs with Cayley integers
- 3D CFs with the cubic lattice
- 3D CFs with the hexagonal prism lattice
- 3D CFs with rhombic dodecahedral lattice

Proof methods

The proof combines three ingredients:

- Understand the cylinder structure, proving the finite-range property and providing a certain finite partition of K.

- Verify Nakada's black-box assumptions, making use of conformality of the inversion. Obtain exactness, unique Lipschitz-continuous invariant measure, CF-mixing, and a Kuzmin-type result.

- Extend Hensley's argument, filling in some details and showing that the transfer operator becomes compact when viewed on the appropriate product of complex Banach spaces. Conclude that the measure is piecewise-analytic.

(A) For each digit a with C_a non-empty, $T_a : C_a \to K$ is a one-to-one, continuous map with continuous first order partial derivatives and $\det DT_a \neq 0$.

(B) The finite range property is satisfied. That is, there exist a finite number of positive-measure subsets $U_1, U_2, \ldots, U_J$ of K such that for each nonempty C_s, we have that $T^{|s|}C_s = U_j$ for some j. This equality may hold up to measure zero.
We shall denote by $\mathcal{F}$ the partition of K generated by the U_j's, and refer to elements of $\mathcal{F}$ as *cells*.

(C) Rényi's condition is satisfied: there is a uniform constant $L > 1$ such that for all strings s, if $T^{|s|}C_s = U_j$ for some j, then

$$\sup_{x \in U_j} \omega_s(x) \leq L \inf_{x \in U_j} \omega_s(x). \tag{2.1}$$

(D) Cylinders uniformly shrink to 0 in diameter as the number of digits increases. That is, letting

$$\sigma(m) := \sup_{|s|=m} \operatorname{diam} C_s,$$

we have $\lim_{m\to\infty} \sigma(m) = 0$.

(E) Each U_j contains a full cylinder.

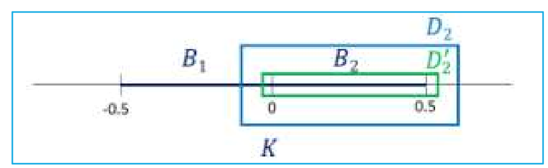
(F) There is a constant $R_1 > 0$ such that for every finite digit sequence s with $n = |s|$ and all $x, y \in C_s$ we have

$$|\omega_s(T^n x) - \omega_s(T^n y)| \leq R_1 \lambda(C_s) d(T^n x, T^n y).$$

(G) There is a constant $R_2 > 0$ such that for every s with $n = |s|$ and all $x, y \in C_s$ we have

$$d(x, y) \leq R_2 d(T^n x, T^n y).$$

(H) Let $\mathcal{L}_m = \{s : |s| = m \text{ and } C_s \text{ is not contained in a cell } F \in \mathcal{F}\}$ and $\gamma(m) = \sum_{s \in \mathcal{L}_m} \lambda(C_s)$. We have $\lim_{m\to\infty} \gamma(m) = 0$.



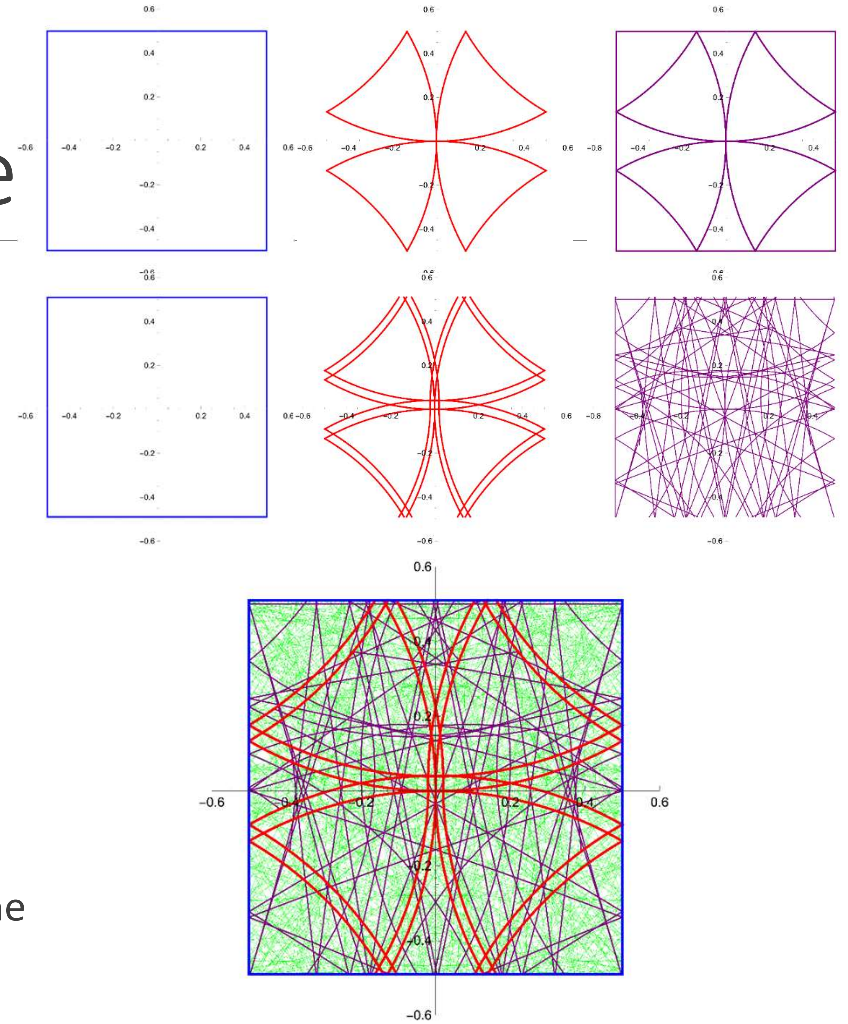
Serendipity & finite range

A fibered system is *finite range* if the collection of normalized cylinders $\{T^{|w|}C_w\}$ is finite.

Fix an Iwasawa CF algorithm and take $\mathrm{E} = \bigcup_n T^n \partial K$. The algorithm is *serendipitous* if the union stabilizes after *finitely many* iterations and furthermore the complement $K \setminus E$ has finitely many connected components.

Lemma: If K is bounded by finitely many hyperplanes and spheres, then finite range is equivalent to serendipity.

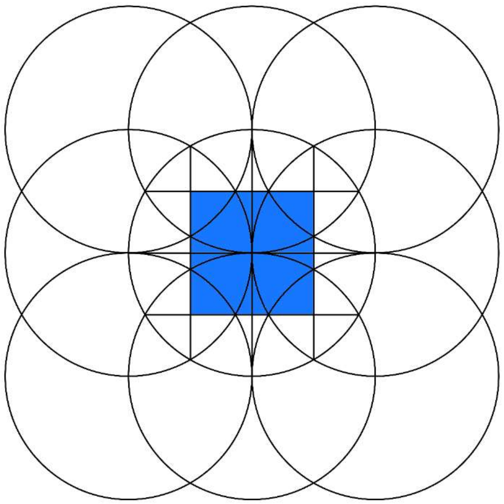
Corollary (serendipity is fragile): Unless α is a root of a quadratic equation, α-CFs in $\mathbb{R}$ and $\mathbb{C}$ are not finite-range.
Proof: In $\mathbb{R}$, serendipity is equivalent to ∂K having a finite orbit. In $\mathbb{C}$, we can view real α-CFs as a subsystem along the imaginary axis, and make an argument about arcs.

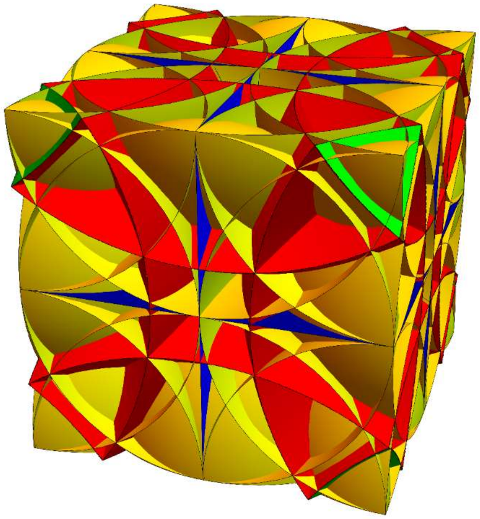


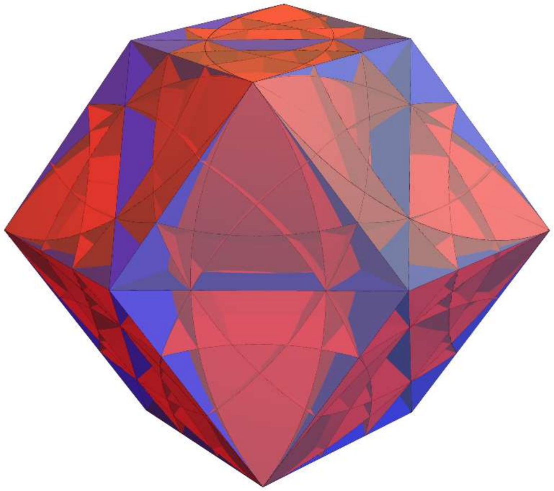
Proving serendipity and finite range

To prove serendipity, we show that certain families of hyperplanes and spheres are invariant under relevant transformations: inversions and translations that revisit K.

The invariant measure is then piecewise-analytic on the complement of the system!

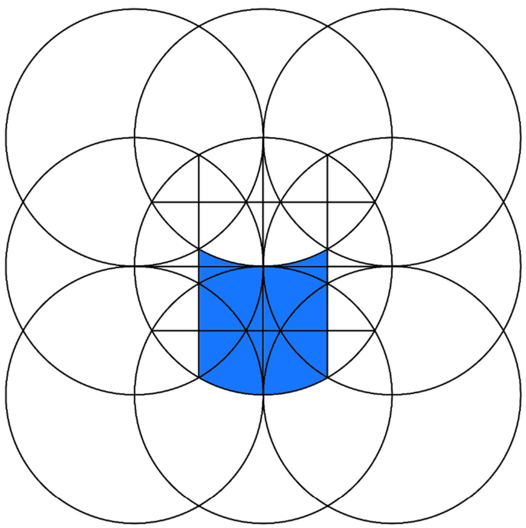


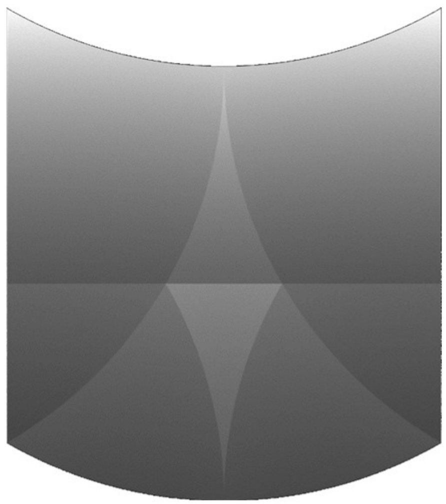




A couple of open questions

- Classify serendipitous systems; are there any non-commutative serendipitous systems?
- Study exactness in improper serendipitous systems.
- Can any of the invariant measures be computed explicitly?





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