Multiplicative Diophantine approximation in the parameter space of beta-dynamical system

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Classic Diophantine approximation

A dynamical interpretation

For any $\theta \in \mathbb{Q}^c$, let $R_\theta \colon x \mapsto x + \theta$ be the irrational rotation on the unit circle. Then the set

$$\{\theta \in \mathbb{Q}^c \colon \|n\theta\| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

can be viewed as a set of parameters $\{\theta : \theta \in \mathbb{Q}^c\}$ of the family of dynamical systems R_θ:

$$\{\theta \in \mathbb{Q}^c \colon \|R_\theta^n(0)\| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where $\varphi \colon \mathbb{N} \to (0, 1]$ is a positive function.

Multiplicative case

The set studied in classic multiplicative Diophantine approximation can be rewritten as the following set

$$\Big\{(\theta, \vartheta) \in \mathbb{Q}^c \times \mathbb{Q}^c : \|R_\theta^n(0)\| \cdot \|R_\vartheta^n(0)\| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

Beta-dynamical system

Beta-transformation

Given $\beta > 1$, the β-transformation $T_\beta\colon [0,1] \to [0,1]$ is defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \quad x \in [0,1].$$

Rényi (1957): integer base $\longrightarrow$ non-integer base.

Every $x \in [0,1]$ can be represented as

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \frac{\varepsilon_2(x,\beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \cdots,$$

where $\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1} x \rfloor \in \{0, 1, \cdots, \lfloor \beta \rfloor\}$.

Parry measure

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \quad x \in [0, 1].$$

Rényi (1957): There exists a unique probability measure ν_β on $[0, 1]$ satisfying

1. ν_β is T_β-invariant;
2. ν_β is equivalent to the Lebesgue measure $\mathcal{L}$.

Moreover, T_β is ergodic with respect to ν_β and

$$1 - \frac{1}{\beta} \leq \frac{d\nu_\beta}{d\mathcal{L}} \leq \frac{1}{1 - \frac{1}{\beta}}.$$

Parry measure

Gelfond (1959), Parry (1960): Its density is the jump function

$$h_\beta(x) = \Theta(\beta) \sum_{x < T_\beta^n 1} \frac{1}{\beta^n}, \quad x \in [0,1],$$

where

$$\Theta(\beta) = \int_0^1 \left(\sum_{x < T_\beta^n 1} \frac{1}{\beta^n} \right) dx$$

is the normalizing factor.

Hofbauer (1978): ν_β is the unique measure of maximal entropy.

Diophantine analysis for beta-transformation

Orbits under different β-transformations

Given a point $x \in (0, 1]$, its orbits under β-transformations may have completely different distributions on $[0, 1]$ when β varies.

Blanchard (1989): There is a kind of classification of β's according to the distribution of $\mathcal{O}_\beta = \{T_\beta^n 1 : n \in \mathbb{N}\}$:

1. $\mathcal{O}_\beta$ is ultimately zero;
2. $\mathcal{O}_\beta$ is ultimately non-zero periodic;
3. $\mathcal{O}_\beta$ is an infinite set but 0 is not an accumulation point of $\mathcal{O}_\beta$;
4. 0 is an accumulation point of $\mathcal{O}_\beta$ but $\mathcal{O}_\beta$ is not dense in $[0, 1]$;
5. $\mathcal{O}_\beta$ is dense in $[0, 1]$.

Corresponding β-shifts

Information of beta-transformation can be determined by the orbit of the critical point 1.

Blanchard (1989): the corresponding symbolic dynamical systems are as follows:

1. subshift of finite type;
2. sofic system;
3. specified system;
4. synchronizing system;
5. none of the above.

Diophantine analysis in the parameter space

Schmeling (1997): For any $x \in (0, 1]$,

$$\liminf_{n\to\infty} \|T_\beta^n x\| = 0 \quad \text{for } \mathcal{L}\text{-a.e. } \beta > 1. \tag{1}$$

Question. What are the quantitative properties of the convergence speed in (1)?

Let $\varphi \colon \mathbb{N} \to (0, 1]$ be a positive function. For any $x \in (0, 1]$, define

$$D_x(\varphi) = \left\{\beta \in (1, +\infty) \colon \|T_\beta^n(x)\| < \varphi(n), \text{ i.o. } n \in \mathbb{N}\right\}.$$

Metric theoretic results

Persson & Schmeling (2008), Li, Persson, Wang & Wu (2014), etc.

Lü & Wu (2016): $D_x(\varphi)$ is of zero or full Hausdorff dimension according to

$$\limsup_{n\to\infty} \frac{\log \varphi(n)}{n} = -\infty \text{ or not.}$$

Lü & Wu (2020): $D_1(\varphi)$ is of zero or full Lebesgue measure in $(1, +\infty)$ according to the series $\sum \varphi(n)$ is convergent or not.

Lü, Wang & Wu (arXiv): $D_x(\varphi)$ is of zero or full Lebesgue measure in $(1, +\infty)$ according to the series $\sum \varphi(n)$ is convergent or not.

Multiplicative Diophantine approximation

Multiplicative case

$$W(\varphi) := \Big\{(\theta, \vartheta) \in \mathbb{Q}^c \times \mathbb{Q}^c : \|R_\theta^n(0)\| \cdot \|R_\vartheta^n(0)\| < \varphi(n), \ \text{i.o. } n \in \mathbb{N}\Big\}.$$

Theorem (Gallagher-type theorem)

Let $\varphi\colon \mathbb{N} \to \mathbb{R}^+$ be a positive decreasing function. Then

$$\mathcal{L}^2(W(\varphi)) = \begin{cases} 0, & \text{if} \quad \sum_{n=1}^{\infty} \varphi(n) \log n < \infty, \\ \text{Full}, & \text{if} \quad \sum_{n=1}^{\infty} \varphi(n) \log n = \infty. \end{cases}$$

Multiplicative case

Theorem (Jarník-type theory)

Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a positive decreasing function. For any $1 < s < 2$,

$$\mathcal{H}^s(W(\varphi)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} n\left(\frac{\varphi(n)}{n}\right)^{s-1} < \infty, \\ \infty, & \text{if } \sum_{n=1}^{\infty} n\left(\frac{\varphi(n)}{n}\right)^{s-1} = \infty. \end{cases}$$

Multiplicative case

Let **Bad** be the set of badly approximable points and let

$$\left\{(\theta,\vartheta)\in \mathbf{Bad}\times\mathbf{Bad} : \liminf_{n\to\infty} n\|R_\theta^n(0)\|\cdot\|R_\vartheta^n(0)\|=0\right\}.$$

Theorem (Pollington & Velani, Einsiedler, Katok & Lindenstrauss)

- *For any $\theta\in$ **Bad**, there is a subset A of **Bad** such that for any $\vartheta\in A$, the pair (θ,ϑ) satisfies Littlewood's conjecture.*
- *The exceptional set of Littlewood's conjecture is of zero Hausdorff dimension.*

Multiplicative case for beta-transformation

Lebesgue measure

Let $\varphi\colon \mathbb{N} \to (0, e^{-1})$ be a positive function. For any $x \in (0,1]$, let

$$M_x(\varphi) = \Big\{(\alpha,\beta) \in (1,+\infty)^2 \colon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| < \varphi(n),\ \text{i.o. } n \in \mathbb{N}\Big\}.$$

Theorem 1 (Gallagher-type result)

For any $x \in (0,1]$,

$$\mathcal{L}^2(M_x(\varphi)) = \begin{cases} 0, & \text{if } \displaystyle\sum_{n=1}^{\infty} \varphi(n) \log \frac{1}{\varphi(n)} < \infty, \\ \text{Full}, & \text{if } \displaystyle\sum_{n=1}^{\infty} \varphi(n) \log \frac{1}{\varphi(n)} = \infty. \end{cases}$$

Hausdorff measure

Theorem 2 (Jarník-type result)

Let $1 \leq s < 2$ be a real number. For any $x \in (0, 1]$,

$$\mathcal{H}^s(M_x(\varphi)) = \begin{cases} 0, & \text{if} \quad \beta_s = \infty, \\ \infty, & \text{if} \quad \beta_s < \infty, \end{cases}$$

where

$$\beta_s = \sup\left\{\beta > 1 : \sum_{n=1}^{\infty} \beta^n \left(\frac{\varphi(n)}{\beta^n}\right)^{s-1} < \infty\right\}.$$

"Bad" set

For any $x \in (0, 1]$ and $\epsilon \geq 0$, let

$$\mathbf{Bad}_x^{\epsilon} = \left\{\beta > 1 : \liminf_{n\to\infty} n^{\epsilon} \|T_\beta^n(x)\| > 0\right\}.$$

For any $\epsilon > 0$, let

$$\mathbf{Mad}_x(\epsilon) := \left\{(\alpha,\beta) \in \left(\mathbf{Bad}_x^{\epsilon} \setminus \mathbf{Bad}_x^{0}\right)^2 : \liminf_{n\to\infty} n^{\epsilon} \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| = 0\right\}.$$

“Bad” set

Theorem 3

Let $x \in (0,1]$ and $\epsilon > 0$.

- *For any $\beta \in \mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$, there is a subset $A(\beta)$ of $\mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$ with Hausdorff dimension 1 such that for any $\alpha \in A(\beta)$,*

$$\liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| = 0.$$

- *For any $\beta \in \mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$, there is a subset $A'(\beta)$ of $\mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$ with Hausdorff dimension 1 such that for any $\alpha \in A'(\beta)$,*

$$\liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| > 0.$$

Lebesgue measure

Lebesgue measure

$$M_x(\varphi) = \Big\{(\alpha,\beta) \in (1,+\infty)^2 : \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| < \varphi(n), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

Theorem 1 (Gallagher-type result)

For any $x \in (0,1]$,

$$\mathcal{L}^2(M_x(\varphi)) = \begin{cases} 0, & \text{if } \displaystyle\sum_{n=1}^{\infty} \varphi(n) \log \frac{1}{\varphi(n)} < \infty, \\ Full, & \text{if } \displaystyle\sum_{n=1}^{\infty} \varphi(n) \log \frac{1}{\varphi(n)} = \infty. \end{cases}$$

Decomposition

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \frac{\varepsilon_2(x,\beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \cdots .$$

Let

$$\varepsilon(x,\beta) := \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots .$$

For any $n \in \mathbb{N}$, let

$$\Omega_n(x) = \{\varepsilon_1(x,\beta)\cdots\varepsilon_n(x,\beta) : \beta > 1\} .$$

For any $u, v \in \Omega_n(x)$, let

$$I(u) = \{\beta > 1 : \varepsilon_1(x,\beta)\cdots\varepsilon_n(x,\beta) = u\} \text{ and } I(u,v) = I(u) \times I(v).$$

Decomposition

For any $N \in \mathbb{N}$, let β_N be the unique positive solution of the equation

$$x = \frac{\delta_x}{\beta} + \frac{1}{\beta^{N+2}},$$

where

$$\delta_x = \begin{cases} 0, & \text{if } x \in (0,1); \\ 1, & \text{if } x = 1. \end{cases}$$

It is clear that $\beta_N \searrow 1$ as $N \to \infty$ and

$$\varepsilon(x, \beta_N) = \delta_x 0^N 10^\infty.$$

Decomposition

Let a_N be a positive integer large enough. Let

$$\mathbb{U}_{a_N} = \{u \in \Omega_{a_N}(x) \colon u \geq \delta_x 0^N 1 0^{a_N - N - 2}\}.$$

It is easy to check that

$$(1, +\infty)^2 = \bigcup_{N=1}^{\infty} [\beta_N, +\infty)^2 = \bigcup_{N=1}^{\infty} \bigcup_{u,v \in \mathbb{U}_{a_N}} I(u, v).$$

Fix $N \in \mathbb{N}$ and $u, v \in \mathbb{U}_{a_N}$. We will estimate the measure

$$\mathcal{L}^2(I(u, v) \cap M_x(\varphi)).$$

Convergent part

For any $n \in \mathbb{N}$, let

$$\mathbf{M}_n(\varphi) = \left\{(\alpha, \beta) \in (1, +\infty)^2 : \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| < \varphi(n)\right\}$$

Then

$$M_x(\varphi) \cap I(u, v) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} (\mathbf{M}_n(\varphi) \cap I(u, v)).$$

Lemma 1

Let $N \in \mathbb{N}$ and $u, v \in \mathbb{U}_{a_N}$. For any $n \in \mathbb{N}$ large enough, we have

$$\mathcal{L}^2(\mathbf{M}_n(\varphi) \cap I(u, v)) \ll \left(\varphi(n) \log \frac{1}{\varphi(n)} + \frac{18}{n^2}\right) \cdot \mathcal{L}^2(I(u, v)).$$

Convergent part

Lemma 2

For any $n \ge a_N$ and $\varrho, \varsigma \in \Omega_n(x)$ with $\varrho|_{a_N} = u, \varsigma|_{a_N} = v$, the measure of the set

$$I(\varrho,\varsigma;\varphi) = \Big\{(\alpha,\beta) \in I(\varrho,\varsigma) : \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| < \varphi(n)\Big\}$$

satisfies

$$\mathcal{L}^2(I(\varrho,\varsigma;\varphi)) \ll \varphi(n) \log \frac{1}{\varphi(n)} \cdot \Big(\underline{\beta}(\varrho)\Big)^{-n} \Big(\underline{\beta}(\varsigma)\Big)^{-n},$$

where $\underline{\beta}(\varrho)$ is the left end point of $I(\varrho)$.

Divergent part

Let

$$\widetilde{M}_x(\varphi) = \left\{(\alpha,\beta) \in (1,+\infty)^2 : \ T_\alpha^n(x) \cdot T_\beta^n(x) < \varphi(n), \ \text{i.o. } n \in \mathbb{N}\right\}.$$

It suffices to show that for any $u, v \in \mathbb{U}_{a_N}$ with N large,

$$\mathcal{L}^2\left(\widetilde{M}_x(\varphi) \cap I(u,v)\right) = \mathcal{L}^2(I(u,v)).$$

This will be achieved by Knopp's lemma if for any $m \geq a_N$ and $w, \tau \in \Omega_m(x)$ with $w|_{a_N} = u, \tau|_{a_N} = v$,

$$\mathcal{L}^2\left(\widetilde{M}_x(\varphi) \cap I(w,\tau)\right) \geq \rho\mathcal{L}^2(I(w,\tau)),$$

where ρ is an absolute constant depending only on u, v and x.

Divergent part

Fix $m \geq a_N$ and $w, \tau \in \Omega_m(x)$ with $w|_{a_N} = u$, $\tau|_{a_N} = v$.

For every large l, let

$$\mathbb{B}_l(w,\tau) = \left\{(\varrho,\varsigma) \in \Lambda_l(x)\times\Lambda_l(x) \colon \varrho|_m = w, \varsigma|_m = \tau, \overleftarrow{\varrho}|_K \neq 0^K, \overleftarrow{\varsigma}|_K \neq 0^K\right\},$$

and

$$\widetilde{F}_l(w,\tau;\varphi) = \bigcup_{(\varrho,\varsigma)\in\mathbb{B}_l(w,\tau)} \left\{(\alpha,\beta) \in I(\varrho,\varsigma) : T^l_\alpha(x)\cdot T^l_\beta(x) < \varphi(l)\right\},$$

where

$$\Lambda_l(x) = \left\{\varrho \in \Omega_l(x) \colon \left\{T^l_\beta x \colon \beta \in I(\varrho)\right\} = [0,1)\right\}.$$

Divergent part

Proposition 1

For every large l, we have

$$\mathcal{L}^2\left(\widetilde{F}_l(w,\tau;\varphi)\right) \gg \varphi(l)\log\frac{1}{\varphi(l)}\cdot\mathcal{L}^2(I(w,\tau)),$$

Proposition 2

For every large n and $l \geq n+K$, we have

$$\begin{aligned}&\mathcal{L}^2\left(\widetilde{F}_n(w,\tau;\varphi)\cap\widetilde{F}_l(w,\tau;\varphi)\right)\\&\ll\frac{1}{\mathcal{L}^2(I(w,\tau))}\mathcal{L}^2\left(\widetilde{F}_n(w,\tau;\varphi)\right)\mathcal{L}^2\left(\widetilde{F}_l(w,\tau;\varphi)\right),\end{aligned}$$

Divergent part

Lemma 3 (Chung-Erdös inequality)

Let $(X, \mathcal{F}, \mu)$ *be a probability space and* $\{E_n\}_{n\geq 1}$ *be a sequence of measurable sets. If* $\sum_{n\geq 1}\mu(E_n) = +\infty$, *then*

$$\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \frac{(\sum_{1\leq i\leq n}\mu(E_i))^2}{\sum_{1\leq i,j\leq n}\mu(E_i\cap E_j)}.$$

$$\mathcal{L}^2\left(\widetilde{M}_x(\varphi)\cap I(w,\tau)\right) \geq \mathcal{L}^2\left(\limsup_{l\to\infty}\widetilde{F}_l(w,\tau;\varphi)\right) \geq \rho\mathcal{L}^2(I(w,\tau)).$$

Hausdorff measure

Hausdorff measure

Theorem 2 (Jarník-type result)

Let $1 \le s < 2$ be a real number. For any $x \in (0,1]$,

$$\mathcal{H}^s(M_x(\varphi)) = \begin{cases} 0, & \text{if} \quad \beta_s = \infty, \\ \infty, & \text{if} \quad \beta_s < \infty, \end{cases}$$

where

$$\beta_s = \sup\left\{\beta > 1 : \sum_{n=1}^{\infty} \beta^n \left(\frac{\varphi(n)}{\beta^n}\right)^{s-1} < \infty\right\}.$$

The first part: $\beta_s = +\infty$

We show that for any N large and $u, v \in \mathbb{U}_{a_N}$,

$$\mathcal{H}^s(M_x(\varphi) \cap I(u,v)) = 0.$$

Note that

$$M_x(\varphi) \cap I(u,v) = \bigcap_{m=a_N}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{w\in\Omega_n(x), w|_{a_N}=u} \bigcup_{\tau\in\Omega_n(x), \tau|_{a_N}=v} I(w,\tau;\varphi).$$

We only consider the cover of one of its subset

$$\widetilde{I}(w,\tau;\varphi) = \left\{(\alpha,\beta) \in I(w,\tau) : T_\alpha^n(x) \cdot T_\beta^n(x) < \varphi(n)\right\}.$$

The first part: $\beta_s = +\infty$

Denote

$$\Upsilon_n \asymp \sqrt{\varphi(n)}(\underline{\beta}(u))^{-n/2}(\underline{\beta}(v))^{-n/2}.$$

Divide the set $\widetilde{I}(w,\tau;\varphi)$ into several parts as follows:

$$I_0 = \left\{(\alpha,\beta) \in \widetilde{I}(w,\tau;\varphi) \colon \alpha - \underline{\beta}(w) \le \Upsilon_n,\ \beta - \underline{\beta}(\tau) \le \Upsilon_n\right\},$$
$$I_i = \left\{(\alpha,\beta) \in \widetilde{I}(w,\tau;\varphi) \colon 2^{i-1}\Upsilon_n < \alpha - \underline{\beta}(w) \le 2^i\Upsilon_n\right\},$$
$$I'_j = \left\{(\alpha,\beta) \in \widetilde{I}(w,\tau;\varphi) \colon 2^{j-1}\Upsilon_n < \beta - \underline{\beta}(\tau) \le 2^j\Upsilon_n\right\}.$$

I_i can be covered by $2^{2(i-1)}$ squares with side length $2^{1-i}\Upsilon_n$.
I'_j can be covered by $2^{2(j-1)}$ squares with side length $2^{1-j}\Upsilon_n$.

The second part: $\beta_s < +\infty$

Proposition 3

If $\beta_s < \infty$, *then* $\mathcal{H}^{s-1}\left(\widetilde{E}_x(\varphi)\right) = \infty$, *where*

$$\widetilde{E}_x(\varphi) = \left\{\beta \in (1, +\infty) \colon\ T_\beta^n(x) < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

$$(1, \infty) \times \widetilde{E}_x(\varphi) \subset M_x(\varphi).$$

Hausdorff dimension

Corollary 1

Let $\varphi\colon \mathbb{N} \to (0, e^{-1})$ *be a positive function. For any* $x \in (0, 1]$,

$$\dim_H M_x(\varphi) = \begin{cases} 1, & \text{if} \quad \kappa(\varphi) = -\infty, \\ 2, & \text{if} \quad \kappa(\varphi) > -\infty, \end{cases}$$

where $\dim_H$ *denotes the Hausdorff dimension and*

$$\kappa(\varphi) = \limsup_{n\to\infty} n^{-1} \log \varphi(n).$$

“**Bad**” set

“Bad” set

Theorem 3

Let $x \in (0, 1]$ and $\epsilon > 0$.

- *For any $\beta \in \mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$, there is a subset $A(\beta)$ of $\mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$ with Hausdorff dimension 1 such that for any $\alpha \in A(\beta)$,*
$$\liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| = 0.$$
- *For any $\beta \in \mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$, there is a subset $A'(\beta)$ of $\mathbf{Bad}_x^\epsilon \backslash \mathbf{Bad}_x^0$ with Hausdorff dimension 1 such that for any $\alpha \in A'(\beta)$,*
$$\liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| > 0.$$

Characterization of “Bad” set

Let $x \in (0, 1]$ be fixed. For any $\beta > 1$ and $n \in \mathbb{N}$, let

$$l_n(x,\beta) = \max\Big\{k \geq 0 \colon \varepsilon_{n+1}(x,\beta) = 0, \cdots, \varepsilon_{n+k}(x,\beta) = 0\Big\},$$

and

$$l_n^*(x,\beta) = \max\Big\{k \geq 0 \colon \varepsilon_{n+1}(x,\beta) = \varepsilon_1^*(1,\beta), \cdots, \varepsilon_{n+k}(x,\beta) = \varepsilon_k^*(1,\beta)\Big\},$$

where $\varepsilon^*(1,\beta) := \varepsilon_1^*(1,\beta)\varepsilon_2^*(1,\beta)\cdots$ is the quasi-greedy expansion of 1 in base β. It is clear that

$$\beta^{-l_n(x,\beta)-1} \leq T_\beta^n(x) < \beta^{-l_n(x,\beta)} \text{ and } 0 < 1 - T_\beta^n(x) \leq \beta^{-l_n^*(x,\beta)}.$$

Characterization of "Bad" set

Proposition 4

Let $\epsilon \geq 0$ be a real number. For any $\beta \in (1, +\infty)$,

$$\liminf_{n\to\infty} n^{\epsilon} T_{\beta}^{n}(x) > 0 \Longleftrightarrow \limsup_{n\to\infty} (l_n(x,\beta) - \epsilon \log_{\beta} n) < +\infty;$$

$$\liminf_{n\to\infty} \left(1 - T_{\beta}^{n}(x)\right) > 0 \Longleftrightarrow \limsup_{n\to\infty} l_n^{*}(x,\beta) < +\infty.$$

As a direct consequence, we have

(1) $\beta \in \boldsymbol{Bad}_x^0$ *if and only if*

$$\limsup_{n\to\infty} l_n(x,\beta) < +\infty \quad \text{and} \quad \limsup_{n\to\infty} l_n^{*}(x,\beta) < +\infty;$$

(2) $\beta \in \boldsymbol{Bad}_x^{\epsilon}$ *if*

$$\limsup_{n\to\infty} (l_n(x,\beta) - \epsilon \log_{\beta} n) < +\infty \quad \text{and} \quad \limsup_{n\to\infty} l_n^{*}(x,\beta) < +\infty.$$

Strategy for the first part

- Since $\beta \notin \mathbf{Bad}_x^0$, we can choose a largely sparse sequence of positive integers $\{m_j\}_{j\geq 0}$ such that

$$\lim_{j\to\infty} \|T_\beta^{m_j}(x)\| = 0.$$

- Then, we collect those parameters $\alpha > 1$ such that among the digit sequence $\varepsilon(x,\alpha)$, a zero block of length $\epsilon \log_\alpha m_j$ follows at the position m_j for all $j \geq 0$, which implies that

$$m_j^\epsilon \cdot T_\alpha^{m_j}(x) \asymp 1.$$

If furthermore, $\varepsilon^*(1,\alpha)$ can be avoided for a long run in $\varepsilon(x,\alpha)$, we will have

$$\alpha \in \mathbf{Bad}_x^\epsilon \setminus \mathbf{Bad}_x^0 \ \text{ and } \ \liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| = 0.$$

Strategy for the second part

- Choose a largely sparse sequence of positive integers $\{m_j\}_{j\geq 0}$ satisfying for all $j \geq 1$ and $m_j \leq n < m_j + j$

$$\max\{l_n(x,\beta), l_n^*(x,\beta)\} \leq \frac{\epsilon}{2}\log_\beta m_j$$

so that

$$n^{\epsilon/2}\|T_\beta^n(x)\| \gg 1.$$

- Collect those parameters $\alpha > 1$ such that among $\varepsilon(x,\alpha)$, a zero block of length j ($\leq \frac{\epsilon}{2}\log_\alpha m_j$) follows at the position m_j for all $j \geq 1$, which implies that for all $m_j \leq n < m_j + j$,

$$n^\epsilon \cdot T_\alpha^n(x) \cdot \|T_\beta^n(x)\| = n^{\epsilon/2} T_\alpha^n(x) \cdot n^{\epsilon/2}\|T_\beta^n(x)\| \gg 1.$$

If furthermore, $T_\alpha^n(x)$ is large for all other integers $n \in \mathbb{N}$ and $\varepsilon^*(1,\alpha)$ can be avoided for a long run in $\varepsilon(x,\alpha)$, we will have

$$\alpha \in \mathbf{Bad}_x^\epsilon \setminus \mathbf{Bad}_x^0 \quad \text{and} \quad \liminf_{n\to\infty} n^\epsilon \|T_\alpha^n(x)\| \cdot \|T_\beta^n(x)\| > 0.$$

Thanks for your attention!