Inflection points in the Lyapunov spectrum for IFS on intervals

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Cookie-cutters

For a countable index set I, let

$$X = \cup_{i \in I} X_i$$

with $X_i \subset [0, 1]$ being pairwise disjoint intervals.

Cookie-cutter maps

A cookie-cutter map is a smooth map

$$T : X \to X$$

with $\# I$ *branches* if $T|_{X_i}$ is a surjection from X_i to $[0, 1]$ for any $i \in I$.

Linear cookie-cutter: $|T'(x)| = x_i = |X_i|^{-1}$ for any $i \in I$.
Expanding (hyperbolic): $\inf_{x \in X}\{|T'(x)|\} > 1$.
Repeller: $\Lambda = \cap_{i=0}^{\infty} T^{-i}(X)$.

Lyapunov spectrum

Lyapunov exponent

The *Lyapunov exponent* of the map T at $x \in \Lambda$ is defined to be

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|,$$

in case the above limit exists.

Lyapunov spectrum(Eckmann-Procaccia)

The *Lyapunov spectrum* is defined as

$$L(\alpha) = \dim_H\{x \in \Lambda : \lambda(x) = \alpha\}$$

for values $\alpha \in [-\infty, \infty]$.

Inflections

Inflection

An *inflection point* of a real analytic map is a point where the second derivative of the map changes signs (or the map changes convexity).

Inflections

Regularity and range

1. $L(\alpha)$ depends smoothly on α and $\{\alpha : L(\alpha) \neq 0\}$ is bounded for a conformal expanding (hyperbolic) map with a compact repeller-Weiss.
2. $L(\alpha)$ may be not differentiable at some points and $\{\alpha : L(\alpha) \neq 0\}$ may not be bounded for non-hyperbolic maps or maps with non-compact repellers, see for example, Pollicott-Weiss, Nakaishi, Iommi, Fan-Liao-Wang-Wu, Kesseböhmer-Stratmann, Gelfert-Rams, $\cdots$.

Inflections of the spectrum in case of smoothness

1. Inflection points of the Lyapunov spectrum exist-Iommi-Kiwi.
2. The number of Lyapunov inflections for piecewise linear expanding maps with finitely many branches is even-Iommi-Kiwi.
3. Any 2-branch piecewise linear expanding map admits at most 2 Lyapunov inflections-Jenkinson-Pollicott-Vytnova.
4. The number of Lyapunov inflections is unbounded for the family of piecewise linear expanding maps, and there exists a countable-branch piecewise linear map with infinitely many points of Lyapunov inflections-Jenkinson-Pollicott-Vytnova.

Problems of Iommi-Kiwi and Jenkinson-Pollicott-Vytnova

Upper bound on the number of Lyapunov inflections(Iommi-Kiwi)

For an n-branch piecewise linear expanding map, what is the largest possible number P_n of Lyapunov inflections it can admits?

$P_2 = 2$.

The dual inverse problem (Jenkinson-Pollicott-Vytnova)

For any (even) natural number n, what is the smallest number Q_n such that there exists a Q_n-branch piecewise linear map whose Lyapunov spectrum has n Lyapunov inflections?

$Q_2 = 2$
$Q_n \leq 2^{(n/2+26)^2+1}$ for any n.

About P_3 and Q_4

Theorem 1

For any 3-branch piecewise linear expanding map, the number of its Lyapunov inflections is less than or equal to 2.

Theorem 2

There exists a 4-branch piecewise linear map, such that its Lyapunov spectrum has exactly 4 inflection points.

About P_n and Q_n

Theorem 3

For an n-branch piecewise linear expanding map with $n \geq 4$, the number of its Lyapunov inflections is less than or equal to $\frac{n(n-1)(n+4)}{6}$.

Theorem 4

For any $n \geq 5$, there exists an n-branch piecewise linear map, such that its Lyapunov spectrum has at least $2n-4$ inflections.

About the number of Lyapunov inflections with respect to the essential branches and distribution of the inflections

Essential branch number-number of branches with different slopes. There are some results on estimating the number of Lyapunov inflections with respect to the essential branch number.

Example 1

A 150-branch piecewise linear expanding map with 50 branches of slope 1000 and the other 100 branches of slope 2000 admits at most 2 Lyapunov inflections.

The logarithmic slopes can be used to describe the positions of the Lyapunov inflections.

Example 2

A 2-branch piecewise linear expanding map with one branch of slope 10 and another branch of slope 1000, if it admits two Lyapunov inflections, then $(\log 10 + \log 1000)/2$ must sit between the two inflections.

Proof of Theorem 3

The concavity-convexity characteristic function for piecewise linear maps (Jenkinson-Pollicott-Vytnova)

Let $F(t) = \sum_{i=1}^{n} x_i^t$ for $t \in [-\infty, \infty]$. Let

$$G(t) = 2 \log F(t) - \frac{(F'(t))^2}{F''(t)F - (F'(t))^2}$$

for $t \in [-\infty, \infty]$. Then $\alpha(t)$ is an inflection point of $L(\alpha(t))$ if and only if $G(t) = 0$.

Proof of Theorem 3

The proof of Theorem 3 takes the following steps:

1. The zeros of $G'(t)$ is the same as zeros of
$$H(t) = F^2(t)F'''(t) + 2(F'(t))^3 - 3F(t)F'(t)F''(t).$$
2. $H(t) = \sum_{1\le i<j\le n} \left((x_i^2 x_j)^t(\log x_j - \log x_i)^3 + (x_i x_j^2)^t(\log x_i - \log x_j)^3\right) + \sum_{1\le i<j<k\le n}(x_i x_j x_k)^t Q(i,j,k) = 0$ belongs to a class of well-known equation-*equation of exponential sums*, refer to Moreno and Tossavainen's work.
3. Then the function $f : \mathbb{R} \to \mathbb{R}$
$$f(t) = \sum_{j=0}^{n} a_j p_j^t$$
has at most n zeros for $a_0 \neq 0$ and distinct positive $\{p_j\}_{0\le j\le n}$. (Tossavainen)
4. Apply the above theorem to estimate the number of roots of $H(t)$.

Proof of Theorem 1

By Theorem 3,

$$P_3 \leq \frac{n(n-1)(n+4)}{6}\Big|_{n=3} = 7.$$

Through the following steps we can sharpen the bound to 2.

1. Differentiate $H(t)$ to get $H'(t)$.
2. Show that $H'(t)$ has at most 1 zero, so $g(t)$ has at most 2 zeros.

Proof of Theorem 2

The proof is realised by a key technique called root-surgery on some proto-type 3-branch piecewise linear expanding map, through the following steps.

1. Find some proto-type 3-branch piecewise linear map with 2 inflections.
2. Adding a fourth branch into the proto-type 3-branch piecewise linear map, as long as the slope of the fourth branch is large enough, we can guarantee the 2 inflections of the 3-branch map survive in the spectrum of the new 4-branch map, while two new inflections appear in the spectrum.

Proof of Theorem 4

The proof of Theorem 4 goes the similar way of proof of Theorem 2.

For $n \geq 4$, find some n-branch piecewise linear map with $2(n-2)$ Lyapunov inflections by doing root-surgeries repeatedly on some proto-type $(n-1)$-branch piecewise linear map with $2(n-3)$ Lyapunov inflections. Then the conclusion follows by mathematical induction.

Some numeration systems

- The Gauss map: $1/x - \lfloor 1/x \rfloor$.
- The Lüroth map: $\lfloor 1/x \rfloor (\lfloor 1/x \rfloor + 1)x - \lfloor 1/x \rfloor$.
- $\cdots$

These and their restrictions on branches can be understood as piecewise non-linear maps.

Problems

- What are P_n and Q_n for the continued fraction system?
- What are P_n and Q_n for the Lüroth system?
- $\cdots$