Intersections of Cantor Sets Derived from Complex Radix Expansions

Neil MacVicar
Queen's University

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One World Numeration

Middle Third Cantor Set

Let C be the middle third Cantor set. The set C is the attractor of

$$f_0(x) = x/3 \text{ and } f_2(x) = (x+2)/3.$$

- C satisfies $f_0(C) \cup f_2(C) = C$.
- C is compact and non-empty.
- For every x, y, $|f_j(x) - f_j(y)| = 3^{-1}|x - y|$ for $j = 0, 2$.

C is self-similar.

Intersections with Translates

Definition

A subset of a complete metric space is *self-similar* if it is the attractor of a finite collection of similarities.

Consider $\alpha \in (0,1)$, such that $C \cap (C+\alpha) \neq \emptyset$.

For example, $\alpha = 2/3$. Then

$$C \cap (C + 2/3) = C/3 + 2/3.$$

Question

For which α is $C \cap (C+\alpha)$ self-similar?

Strong Eventual Periodicity

Definition

A sequence $(a_j)_{j=1}^{\infty}$ of integers is *strongly eventually periodic* (SEP) if there exists a finite sequence $(b_\ell)_{\ell=1}^{p}$ and a nonnegative sequence $(c_\ell)_{\ell=1}^{p}$, where p is a positive integer, such that

$$(a_j)_{j=1}^{\infty} = (b_\ell)_{\ell=1}^{p}\overline{(b_\ell + c_\ell)_{\ell=1}^{p}},$$

where $\overline{(d_\ell)_{\ell=1}^{p}}$ denotes the infinite repetition of the finite sequence $(d_\ell)_{\ell=1}^{p}$.

Example

Let $(a_j) = (3, 4, 7, 9, 7, 9, \ldots)$. Here $p = 2$, $(b_\ell) = (3, 4)$, and $(c_\ell) = (4, 5)$.

Self-Sim./SEP Relationship

Observation

$C \cap (C + \alpha)$ is non-empty iff $\alpha = \sum_{j=1}^{\infty} \alpha_j 3^{-j}$, with $\alpha_j \in \{0, \pm 2\}$.

Theorem (G-T. Deng, X-G. He, Z-X. Wen, 2008)

Suppose $\alpha \in (0, 1)$. The set $C \cap (C + \alpha)$ is self-similar iff $(2 - |\alpha_j|)_{j=1}^{\infty}$ is SEP.

Remark

Those α which have multiple expansions can be shown to result in non-self-similar intersections.

Example

If $\alpha = 2/3 = 2/3 + 0/9 + 0/27 + \cdots$, then $(2 - |\alpha_j|) = (0, 2, 2, \ldots)$.

This sequence is SEP with $p = 1, b_1 = 0, c_1 = 2$.

Restricted Digit Sets

C can also be viewed as $\left\{ \sum_{j=1}^{\infty} d_j r^{-j} : d \in D \right\}$, where $r = 3$ and $D = \{0, 2\}$.

In greater generality, $r \in \mathbb{N}_{\geq 3}$ and $D \subset \{0, 1, \ldots, r-1\}$.

Extensions of "self-similar iff SEP" for these and other subsets of $[0, 1]$ can be found in

- D. Kong, W. Li, M. Dekking. Intersections of homogeneous Cantor sets and β-expansions. *Nonlinearity*, 23:2815-2834, 2010.
- W. Li, Y. Yao, Y. Hang. Self-similar structure on intersection of homogeneous symmetric Cantor sets. *Math. Nachr.*, 284:298-316, 2011.
- S. Pedersen, J. D. Philips. On intersections of Cantor sets: self-similarity. *Commun. Math. Anal.*, 16:1-30, 2014.

Example in the Plane

Question

What about higher dimensions?

Example

Consider the pair $b = -3 + i$, $D = \{0, 4, 8\}$. We can define

$$T_{b,D} := \left\{ \sum_{j=1}^{\infty} d_j b^{-j} : d_j \in D \right\}.$$

This is the attractor of the similarities $f_0(z) = b^{-1}z$, $f_4(z) = b^{-1}(z+4)$, and $f_8(z) = b^{-1}(z+8)$.

When is $T_{b,D} \cap (T_{b,D} + \alpha)$ self-similar for $\alpha \in \mathbb{C}$?

Radix Expansions

Theorem (I. Katai & J. Szabo, 1974)

Let $b = -n + i$ and $z \in \mathbb{C}$. Then

$$z = d_\ell b^\ell + d_{\ell-1} b^{\ell-1} + \cdots + d_0 + \sum_{k=1}^{\infty} d_{-k} b^{-k},$$

for some $d_j \in \{0, 1, \ldots, |b|^2 - 1\}$, $j \in \mathbb{Z}$.

Example

Let $b = -4 + i$.

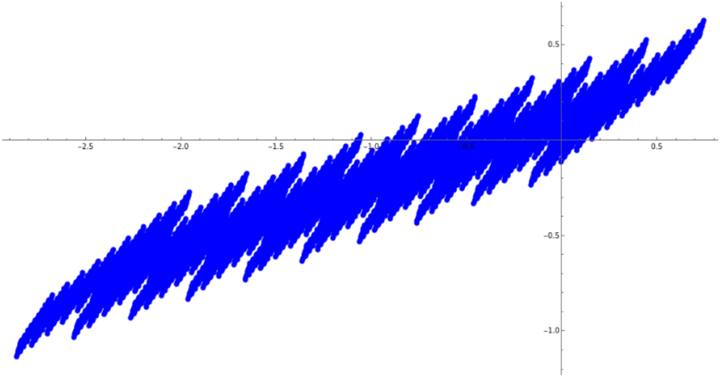
$$\frac{7+4i}{13} = \sum_{j=1}^{\infty} 10 b^{-2j}$$

AND

$$\frac{7+4i}{13} = b + 7 + \sum_{j=1}^{\infty} 10 b^{-2j+1}$$

Fundamental Tile

Let $b = -3 + i$ and $D = \{0, 1, \ldots, 8, 9\}$



Matrix Form

Definition

A matrix $B \in M_n(\mathbb{R})$ is called an *expanding matrix* if all its eigenvalues have modulus greater than 1.

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{Z}^2 \\ b = -3 + i & \to & B = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix} \\ \{0, 1, \ldots, 9\} & \to & \{0, e_1, 2e_1, \ldots, 9e_1\}, e_1 = (1, 0)^T. \end{array}$$

Given an expanding matrix B, $f(x) = B^{-1}(x + d)$ is a contraction on $\mathbb{R}^n$.

Generalized SEP

Definition

A sequence $(A_j)_{j=1}^{\infty}$ of nonempty subsets of $\mathbb{Z}^n$ is called *strongly eventually periodic* (SEP) if there exist two finite sequences of sets $(B_\ell)_{\ell=1}^{p}$ and $(C_\ell)_{\ell=1}^{p}$, where p is a positive integer, such that

$$(A_j)_{j=1}^{\infty} = (B_\ell)_{\ell=1}^{p}\overline{(B_\ell + C_\ell)_{\ell=1}^{p}},$$

where $B + C = \{b + c : b \in B, c \in C\}$.

Example

Let $(A_j) = (\{0,4\},\{1,5\},\overline{\{0,4,8\},\{1,3,5,7\}})$. Here $p = 2$, $B_\ell = (\{0,4\},\{1,5\})$, and $C_\ell = (\{0,4\},\{0,2\})$.

Higher Dimensional Self-Sim./SEP

Theorem (N. M., 2024)

Suppose an expanding matrix $B \in M_n(\mathbb{Z})$ and finite $D \subset \mathbb{Z}^n$ are chosen such that expansions $\sum_{j=1}^{\infty} B^{-j} e_j$ with $e_j \in (D - D)$ are unique.

The set $T_{B,D} \cap (T_{B,D} + \alpha)$ is the attractor of an IFS of the form $\{B^{-p}x + v_i\}_{i=1}^{N}$, where v_i is an element of $\mathbb{R}^n$ for $i = 1, 2, \ldots, N$ and p is a positive integer, if and only if the sequence

$$(((D \cap (D + \alpha_j)) - \beta_j)_{j=1}^{\infty}$$

is SEP for some sequence $(\beta_j)_{j=1}^{\infty} \in (D - D)^{\mathbb{N}}$.

Illustrative Example

Example

Let $b = -3 + i$ and $D = \{0, 4, 8\}$.

Choose $\alpha = -4b^{-1} - 8b^{-2} + \sum_{j=1}^{\infty} 8b^{-(2j+2)}$. This yields

$$(D \cap (D + \alpha_j)) = (\{0,4\}, \{0\}, \overline{\{0,4,8\}, \{8\}}).$$

Note that $\{0,4,8\} = \{0,4\} + \{0,4\}$, $\{8\} = \{0\} + \{8\}$.

$$T_{b,D} \cap (T_{b,D} + \alpha) = \{0,4\}b^{-1} + \{0\}b^{-2} + \{0,4,8\}b^{-3} + \{8\}b^{-4} + \cdots$$

$$b^{-2}(T_{b,D} \cap (T_{b,D} + \alpha)) = \{0\}b^{-1} + \{0\}b^{-2} + \{0,4\}b^{-3} + \{0\}b^{-4} + \cdots$$

The similarities are $f(z) = b^{-2}(z + d)$ where
$d \in \{4b + 4b^{-1} + 8b^{-2}, 4b^{-1} + 8b^{-2}, 4b + 8b^{-2}, 8b^{-2}\}$.

Neighbours of Tiles

Definition

We call a nonzero Gaussian integer g *a neighbour of* $T_{b,D}$ if

$$T_{b,D} \cap (T_{b,D} + g) \neq \emptyset.$$

Suppose

$$\sum_{j=1}^{\infty} e_j b^{-j} = \sum_{j=1}^{\infty} e_j' b^{-j}.$$

Then

$$(e_1 - e_1') + \sum_{j=1}^{\infty} e_{j+1} b^{-j} = \sum_{j=1}^{\infty} e_{j+1}' b^{-j}.$$

If $e_j \in (D - D)$, then $(e_1 - e_1')$ is a neighbour of $T_{b,(D-D)}$.

Neighbours of Tiles Generated by $-n+i$

Conclusion

If D is chosen such that the difference of differences of D are not neighbours of $T_{b,(D-D)}$, then expansions in $(b, D-D)$ are unique.

For $(-3+i)$, the real neighbours of the tile generated by $(-3+i, \{0, \pm 1, \ldots, \pm 9\})$ (checked via algorithm) are $\{\pm 1, \pm 2, \pm 3\}$.

Theorem (N. M., 2024)

For integers $n \geq 5$, a neighbour s of the tile generated by $(-n+i, \{0, \pm 1, \ldots, \pm n^2\})$ satisfies $|\operatorname{Re}(s) - n \operatorname{Im}(s)| < 3$.

- K. Scheicher, J. M. Thuswaldner. Neighbours of self-affine tiles in lattice tilings. In *Fractals In Graz 2001*, pages 241-262. Birkhäuser Basel, 2003.

Special Case

Theorem (N. M., 2024)

Fix an integer $n \geq 2$ and let $b := -n + i$. Suppose $D = \{0, m\}$ where $2 \leq m \leq n^2$ and that α has a unique radix expansion in base $(b, D - D)$.
Let $\gamma := \sum_{j=1}^{\infty} \gamma_j b^{-j}$ where $\gamma_j := \min(D \cap (D + \alpha_j))$.

If $C(\alpha)$ is self-similar and is the attractor of an IFS containing a similarity of the form $f(x) = \lambda x + (1 - \lambda)\gamma$, $0 < |\lambda| < 1$, then $(m - |\alpha_j|)_{j=1}^{\infty}$ is SEP.

Conversely, if $(m - |\alpha_j|)_{j=1}^{\infty}$ is SEP then $C(\alpha)$ is self-similar.

Hausdorff Dimension

Let (b, m, α) satisfy the previous theorem and let $(m - |\alpha_j|)_{j=1}^{\infty} = (a_\ell)_{\ell=1}^{p}\overline{(a_\ell + u_\ell)_{\ell=1}^{p}}$.

The set $T_{b,\{0,m\}} \cap (T_{b,\{0,m\}} + \alpha)$ is the attractor of the collection of maps

$$f(x) = b^{-p}\left(x + \sum_{\ell=1}^{p}(y_\ell b^{p-\ell} + z_\ell b^{-\ell}) - \gamma\right) + \gamma$$

where for each ℓ, $y_\ell, z_\ell \in \{0, m\}$ such that $y_\ell \leq a_\ell$ and $z_\ell \leq u_\ell$.

Those maps satisfy the strong separation condition, $f_i(A) \cap f_j(A) = \emptyset$ for $i \neq j$. Therefore

$$\dim_H(T_{b,\{0,m\}} \cap (T_{b,\{0,m\}} + \alpha)) = \frac{\log(2)\sum_{\ell=1}^{p}(a_\ell + u_\ell)}{mp\log(|b|)}.$$

Thank you for your attention!