Construction of absolutely normal numbers

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One world numeration seminar

Numeration systems

Let $b \geq 2$ be an integer. Every real $x \in [0,1]$ admits a representation of the form

$$x = \sum_{k \geq 1} a_k b^{-k} = 0.a_1 a_2 a_3 \dots$$

with $a_k \in \{0, 1, \dots, b-1\} =: \mathcal{N}_b$.

Let $x \in [0,1]$, $d \in \mathcal{N}_b$ and $n \in \mathbb{N}$. Then we define the frequency of the digit d among the first n digits by

$$\Pi(x; d, n) = \frac{1}{n} \#\{1 \leq k \leq n \colon a_k = d\}$$

Normality

We call $x \in [0,1]$

- *simply normal* to base b if for all $d \in \mathcal{N}_b$

$$\lim_{n\to\infty} \Pi(x; d, n) = b^{-1};$$

- *normal* to base b if it is simply normal to bases b, b^2, b^3, etc.;
- *absolutely normal* if it is normal to all bases $b \geq 2$.

Absolutely normal numbers

Theorem (Borel 1909)

Almost all real numbers with respect to the Lebesgue measure are absolutely normal.

Known constructions:

- Sierpinski (1917)
- Schmidt (1961/62)
- Levin (1979)
- Turing (*1992*)

Polynomial time construction

Theorem (Becher, Heiber, Slaman (2014))

We construct an absolutely normal number in polynomial time.

Main ingredients:

- It suffices to construct a simply normal number with respect to every base $b \geq 2$.
- In every step i we work with bases $b \in \{2, \ldots, t(i)\}$.
- Construction uses nested cylinder sets

$$\cdots \supset I_{i,2} \supset I_{i,3} \supset \cdots \supset I_{i,t(i)} \supset I_{i+1,2} \supset I_{i+1,3} \supset \cdots$$

- For each interval I and each base b there exists a cylinder set $I_b \subset I$ such that

$$|I_b| \geq |I| / 2b.$$

Dynamical point of view

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus (which we identify with $[0,1]$) and

$$\begin{aligned} T_b \colon \mathbb{T} &\to \mathbb{T} \\ x &\mapsto bx \bmod 1. \end{aligned}$$

For $d \in \mathcal{N}_b$ we define

$$P_d = \left] \frac{d}{b}, \frac{d+1}{b} \right[.$$

Furthermore for $k \geq 1$ we set $a_k = d$ if $T_b^{k-1} x \in P_d$. Then

$$x = \sum_{k \geq 1} a_k b^{-k}.$$

Topological dynamical system

Let M be a metric space and let $T\colon M \to M$ be a continuous map. Then we call (M, T) a topological dynamical system.

Let $\mathcal{P} = \{P_0, \ldots, P_{b-1}\}$ be a finite collection of disjoint open sets. Then $\mathcal{P}$ is called a topological partition if

$$M = \overline{P_1} \cup \overline{P_2} \cup \cdots \cup \overline{P_{b-1}}.$$

Symbolic dynamical system

Let $\Sigma = \{0, 1, \ldots, b-1\}$ be the alphabet corresponding to the topological partition $\mathcal{P}$. Furthermore we define

$$\Sigma^k = \{0, 1, \ldots, b{-}1\}^k, \quad \Sigma^* = \bigcup_{k \geq 1} \Sigma^k \cup \{\varepsilon\} \quad \text{and} \quad \Sigma^{\mathbb{N}} = \{0, 1, \ldots, b{-}1\}^{\mathbb{N}}$$

to be the set of words of length k, the set of finite and the set of infinite words over Σ, respectively, where ε is the empty word.

Symbolic dynamical system

We call $\omega = a_1 \dots a_n \in \Sigma^n$ allowed if

$$\bigcap_{k=0}^{n-1} T^{-k}(P_{a_k}) \neq \emptyset.$$

Let $\mathcal{L} = \mathcal{L}_{\mathcal{P},T}$ be the set of allowed words. Then there exists a unique shift space $X = X_{\mathcal{P},T} \subset \Sigma^{\mathbb{N}}$, whose language is $\mathcal{L}$. Furthermore let $\mathcal{L}_n = \mathcal{L} \cap \Sigma^n$ be the set of all words of length n in $\mathcal{L}$.

The symbolic expansion

We want to link the expansion with element $x \in M$. Clearly every $x \in M$ has an expansion. For the opposite direction we suppose that, for any $\omega = a_0 a_1 a_2 \ldots \in X$, the set $\bigcap_{n=0}^{\infty} \overline{D_n(\omega)}$ is a singleton set. This yields uniqueness in both directions and we define the map $\pi\colon X \to M$ by

$$\bigcap_{k=0}^{\infty} T^{-k} P_{a_k} = \{\pi(\omega)\}\,.$$

This makes the following diagram commute

$$\begin{array}{ccc} X & \overset{S}{\to} & X \\ \downarrow\pi & & \downarrow\pi \\ M & \overset{T}{\to} & M \end{array}$$

where S is the left-shift on X.

Cylinder sets

For each $\omega = a_1 a_2 a_3 \ldots \in X$ and integer $n \geq 1$ we denote by $D_n(\omega)$ the cylinder set of order n corresponding to ω in M, *i.e.*

$$D_n(\omega) = \bigcap_{k=0}^{n-1} T^{-k}(P_{a_k}) \subseteq M.$$

Similarly for $\mathbf{w} = w_1 \ldots w_n \in \mathcal{L}$ we denote by $[\mathbf{w}] \subseteq X$ the cylinder set of order n corresponding to $\mathbf{w}$ in X, *i.e.*

$$[\mathbf{w}] = \{\omega = a_1 a_2 a_3 \ldots \in X \colon a_1 = w_1, \ldots, a_n = w_n\}.$$

Generic points

Let μ be a probability measure on X. Then we call μ *shift invariant* if for each $A \subseteq X$ we have $\mu(S^{-1}A) = \mu(A)$.
Let $\omega \in X$ and $\mathbf{b} = b_1 \dots b_\ell \in \mathcal{L}$. We define the frequency of occurrences of $\mathbf{b}$ in the first n letters of ω by

$$\Pi(\omega, \mathbf{b}, n) = \frac{1}{n} \# \left\{ 0 \leq k < n \colon S^k \omega \in [\mathbf{b}] \right\}$$

Then we call ω *generic for* μ if for all $\mathbf{b} = b_1 \dots b_\ell \in \Sigma^\ell$ we have

$$\lim_{n \to \infty} \Pi(\omega, \mathbf{b}, n) = \mu([\mathbf{b}])$$

Specification property

We say that a language $\mathcal{L}$ has the *specification property* with gap $g \geq 0$ if for any $\mathbf{a}, \mathbf{b} \in \mathcal{L}$ there exists a $\mathbf{w} \in \mathcal{L}$ with $|\mathbf{w}| \leq g$ such that

$$\mathbf{awb} = \mathbf{a} \odot \mathbf{b} \in \mathcal{L}.$$

Theorem (M, Mance (2016))

Let X be a shift, whose language has the specification property, and let μ be a shift invariant probability measure on X. Then we construct an element $x \in X$, which is generic for μ.

Entropy

Let X be a shift and $\mathcal{L}$ its language. Then the *topological entropy* $h(X)$ of the shift X is defined as

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \ln |\mathcal{L}_n| .$$

Let μ be a shift invariant measure. Then *measure-theoretic entropy* of μ is defined as

$$h(\mu) = \lim_{n\to\infty} -\frac{1}{n} \sum_{\mathbf{a}\in L_n} \mu([\mathbf{a}]) \log(\mu([\mathbf{a}])) .$$

Maximum measure

We always have

$$h(\mu) \leq h(X)$$

and we call μ a measure of maximal entropy if $h(\mu) = h(X)$.

This maximal measure has to be ergodic. If there is a unique maximal measure, then we call X intrinsically ergodic.

Theorem (Birkhoff (1931))

Let X be a shift and μ be an ergodic probability measure on X. Then almost all $x \in X$ are generic for μ.

Specification and maximal measure

Theorem (Bowen (1971))

Let X be a shift. If X has the specification property then X is intrinsically ergodic.

Theorem (Pavlov (2016))

If X has only weaker forms of specification property, then X has two ergodic measures with maximal entropy.

Ambiguities

In the decimal system we have

$$0.99999\ldots = 1.00000\ldots$$

Similar things could happen in other shifts. Thus we define the sets

$$U = \bigcup_{d=0}^{b-1} P_d, \quad U_n = \bigcap_{k=0}^{n} T^{-k}(U) \quad \text{and} \quad U_\infty = \bigcap_{n=0}^{\infty} U_n.$$

We consider only numbers in U_∞ to be normal.

Normality

Let (M, T) be a topological dynamical system and $\mathcal{P}$ be a topological partition of M. Furthermore let X and $\mathcal{L}$ be the associated shift space and language, respectively. Suppose that X has the specification property and let μ be unique maximal measure on X. We call $x \in U_\infty$ normal if x is generic for μ, *i.e.* for each $\mathbf{b} = b_1 \dots b_\ell \in \mathcal{L}$ we have

$$\frac{1}{n} \# \left\{ 0 \leq k < n \colon T^k x \in D_\ell(\mathbf{b}) \right\} \xrightarrow[n \to +\infty]{} \mu([\mathbf{b}]).$$

β-shift

Let $\beta > 1$. Then we define the β-transformation by

$$\begin{aligned} T_\beta \colon \mathbb{T} &\to \mathbb{T} \\ x &\mapsto \beta x \bmod 1. \end{aligned}$$

We use the topological partition $\mathcal{P} = \bigcup_{d=0}^{\lceil\beta\rceil-1} P_d$ with

$$P_d = \left]\frac{d}{\beta}; \frac{d+1}{\beta}\right[\text{ for } d = 0, \dots, \lceil\beta\rceil - 2 \quad \text{and} \quad P_{\lceil\beta\rceil-1} = \left]\frac{\lceil\beta\rceil-1}{\beta}; 1\right[.$$

If we set

$$a_k = d \text{ if } T_\beta^{k-1} x \in P_d,$$

then

$$x = \sum_{k\geq 1} a_k \beta^{-k}.$$

Ergodicity

Renyi (1957), Gelfond (1959) and Parry (1960) constructed an ergodic measure μ_β and Hofbauer (1979) and Walters (1978) showed that the β-shift is intrinsically ergodic.

Birkhoff's theorem implies that almost all numbers are normal (with respect to μ_β).

Absolutely Pisot normal number

An algebraic number β is a Pisot number, if all of its complex conjugates lie inside the unit circle.

Theorem (M, Scheerer, Tichy (2018))

Let $(\beta_n)_{n\geq 1}$ be a sequence (finite or infinite) of Pisot numbers. Then we construct a number which is normal with respect to all bases β_n in polynomial time.

Idea:

$$\cdots \supset I_{i,\beta_1} \supset \cdots \supset I_{i,\beta_{t(i)}} \supset I_{i+1,\beta_1} \supset \cdots$$

Problems:

- simply normal for β_n^k with $k \geq 1$ does not imply normal to base β_n;
- the ergodic theorems give qualitative results but no quantitative ones;
- the β_n-adic intervals have different sizes.

Continued fraction expansion

Let T_G be the Gauss map defined by

$$\begin{aligned} T_G \colon \mathbb{T} &\to \mathbb{T} \\ x &\mapsto \begin{cases} \frac{1}{x} \bmod 1 & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This time we have an infinite topological partition $\mathcal{P} = \bigcup_{d=1}^{\infty} P_d$ with

$$P_d = \left] \frac{1}{d+1}; \frac{1}{d} \right[\quad \text{for } d = 1, 2, \ldots$$

If we set

$$a_k = d \text{ if } T_G^{k-1} x \in P_d,$$

then

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Ergodicity

The associated shift space X is ergodic with respect to the Gauss-Kuzmin measure:

$$\mu_{GK}(A) = \frac{1}{\ln(2)} \int_A \frac{\mathrm{d}x}{1+x}.$$

Birkhoff's theorem implies that almost all numbers are normal (with respect to μ_{GK}).

Absolutely normal

Theorem (Scheerer (2017))

Construction of an absolutely normal number that is also normal with respect to the continued fraction expansion.

Theorem (Becher, Yuhjtman (2019))

Construction of an absolutely normal number that is also normal with respect to the continued fraction expansion in polynomial time.

Theorem (Laureti (2023+))

Let $(\beta_n)_{n\geq 1}$ be a sequence of Pisot numbers. Then we construction a number which is normal with respect to all Pisot bases β_n and also continued fraction normal.

A question of Mendès France

Does there exist a real x that is normal in base 2, such that also its inverse $1/x$ is normal in base 2?

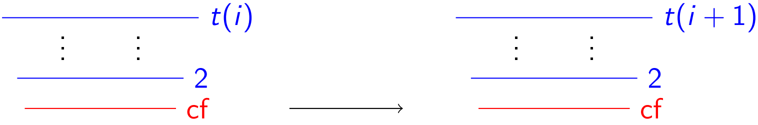
Theorem (Becher, M (2022))

Construction of an absolutely normal number x such that $1/x$ is also absolutely normal in polynomial time.

Idea of proof

$$\text{If } x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \text{ then } \frac{1}{x} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}},$$

The construction



Central limit theorem

- Central limit theorem of Morita and Vallée: There exist constants K, c and a positive integer n_1 such that for each cf-interval I and for every integer $n \geq n_1$ there exist a union of cf-intervals $J \subset I$ of relative order n such that

$$\frac{|I|}{4} e^{-2nL-2c} \leq |J| \leq 2\,|I|\, e^{-2nL+2c}.$$

The union is larger than

$$\frac{K\,|I|}{\sqrt{n}}.$$

Bad zones

Bad zones are cylinder sets with large discrepancy.
By Kiefer, Peres and Weiss (2001) we have

$$\frac{B_{cf}}{|\sigma_{cf}|} \leq c_1 e^{-c_2 n}$$

Bernstein's inequality states for integer $b \geq 2$

$$\frac{B_b}{|\sigma_b|} \leq c_3 e^{-c_4 n}$$